# A GENERAL CURRICULUM IN MATHEMATICS FOR COLLEGES

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**1. A report to the Association.** The Committee on the Undergraduate Program in Mathematics (CUPM) hereby presents to the Mathematical Association of America (MAA) its report on "A General Curriculum in Mathematics for Colleges" [1]. As a standing committee of the Association, CUPM requests that the Board of Governors receive this report and transmit it to the Sections of the Association for their study and comment. CUPM would like to see each Section of the Association draft a resolution, or committee report of its own, stating its evaluation of this General College Curriculum in Mathematics (CMC) which CUPM here proposes, and on the issues related to it.

CUPM has the authority under its charter from MAA to issue its own studies and recommendations; in the past we have used this authority to publish our curriculum recommendations on certain aspects of the college mathematics curriculum. This general college curriculum is so fundamental and so broad in its scope, however, that your Committee prefers to present this report to you for your study and comment before we do further work on it. We have endeavored to avoid being overly prescriptive of what colleges should do. Moreover, there are many issues involved in a general mathematics curriculum which are of such importance that CUPM should not attempt to resolve them for the college mathematics establishment. The "general curriculum" of this report consists essentially of an array of modular course units which goes as far as we feel we should at the present time towards offering a guide for curriculum construction in college mathematics. We think the proposed system is flexible enough to provide for local variation, for a variety of different individual curricula, and for enough dynamic progression in the college curriculum to permit these issues to be settled by the natural processes of work and discussion in the mathematical community.

In the beginning we set out to construct a recommended curriculum for "small" colleges but we soon found that this was not a significant specification. For one thing "small colleges" are no longer small. Moreover, it appeared that larger institutions might be able to use the same report as a minimal statement from which more advanced and complex department programs could be extended. So we changed our specification to prepare a list of course offerings few enough in number and conservative enough in content so that a staff of as few as four teachers can teach it. Thus the program we offer in this report is a minimal one which can serve as a basis for richer and more advanced programs wherever and whenever these are feasible. The various Panel recommendations [2] of CUPM represent what we have to say about richer programs in special areas of college mathematics. We hope that our basic program is so designed that the curriculum of a particular college can progress through it without the necessity of abolishing it to supplant it by an entirely new program five years from now. Indeed, a number of university and college departments already have curricula ahead of the general curriculum proposed in this report, at least for their better prepared students. Therefore we wish to emphasize that this severely economized program of 14 semester-courses is not to be interpreted either as an ultimate goal or a limitation on undergraduate mathematics.

**2. Secondary school prerequisites.** The general curriculum in mathematics, which CUPM proposes, assumes well-prepared students who offer upon entrance $3\frac{1}{2}$ or 4 years of high school mathematics including: A. Geometry and intermediate algebra, B. A study of the elementary functions, i.e. polynomials, rational and algebraic functions, exponential, logarithmic and trigonometric functions, including the more elementary parts of two- and three-dimensional analytic geometry.

Recognizing that in many situations some of these prerequisites must still be taught in college we describe the following one-semester remedial course.

MATH. 0. *Elementary Functions and Coordinate Geometry* (3 semester hours). A precalculus course covering topics B above.

**3. The proposed general curriculum in mathematics.** For students who can meet the prerequisites CUPM proposes that the following list of lower division courses be available each semester as needed. (See the report itself for details and sample course outlines.)

MATH. 1. *Introductory Calculus* (3 or 4 semester hours). A differential and integral calculus of polynomials and other elementary functions.

MATH. 2, 4. *Multivariate Calculus, Limits and Differential Equations* (3 or 4 semester hours each). Courses which we describe in two different versions. In the preferred version Math. 2 is a multivariate calculus, while Math. 4 strengthens limits and series and provides an introduction to linear differential equations. A more conventional form is also described in which Math. 2 continues the single variable calculus, while Math. 4 takes up multivariate calculus and a brief introduction to differential equations.

MATH. 2P. *Introductory Probability.* A course of first year level which can follow Math. 1 and may be substituted for Math. 2 when it is appropriate for particular students.

MATH. 3. *Linear Algebra* (3 or 4 semester hours). Linear systems, matrices, vectors, linear transformations, unitary geometry with characteristic values. Applications to geometry.

For the upper division, not necessarily offered each semester, our report proposes the following 9 semester courses.

MATH. 5. *Advanced Multivariable Calculus.* Primarily vector calculus, Stokes' and Green's theorems and a brief introduction to boundary value problems of partial differential equations.

MATH. 6. *Algebraic Structures.* Groups, rings and fields.

MATH 7. *Probability and Statistics.* Two versions are displayed, one emphasizing statistical inference and one going more deeply into probability.

MATH. 8. *Numerical Analysis.* Numerical methods for integration, differential equations, matrix inversion, estimation of characteristic roots. Oriented towards machine computation.

MATH. 9. *Geometry.* A number of types of courses might do here. We present outlines for a Euclidean geometry for prospective teachers and a differential geometry.

MATH. 10. *Applied Mathematics.* A course to illustrate the principles and basic styles of thought in solving physical or other scientific problems by mathematical methods. Specific content will depend on the persons involved.

MATH. 11 or 11–12. *Introductory Real Variable Theory.* Standard material on the real numbers, continuity, limits, differentiation and integration. The student will learn to make the proofs.

MATH. 13. *Complex Analysis.* Complex numbers, elementary functions, analytic functions, contour integration, Taylor and Laurent Series, conformal mapping, boundary value problems, integral transforms.

*Introduction to Computer Science.* The computer, algorithms, programming languages, problem solving. Many mathematics departments now offer such a course at first or second year level but for the general mathematics student and for his mathematics teacher CUPM feels that the question of how to begin computation and how to relate it to mathematics is not ready to be resolved by a recommendation.

**4. Some course sequences which this general curriculum provides.** In the major sequences below we adopt the limit of six semester courses in the upper division program of a student, although our program in many cases would provide more.

**a.** Two one-year sequences for liberal arts students, or majors in social and behavioral sciences, and business administration students are: Math. 1, 2 or Math. 1, 2P.

**b.** An advanced placement two-year sequence for mathematics or physical science students is: Math. 2, 3, 4, 5 or 2P, or 7 as appropriate.

**c.** A mathematics major program for students bound toward graduate mathematics is given by: Math. 1, 2, 3, 4, 5, 6, 10, 11, 12, 13. There are additional courses available to adapt this sequence for special interests.

**d.** A mathematics major program for applied mathematicians might be: Math. 1, 2, 2P, 3, 4, 5, 6, 8, 10, 11, 12 or 13. This exhausts the limit we assume but more courses are available. Computer Science, or equivalent introduction to computing, is a necessary supplement.

**e.** A program for prospective theoretically oriented graduate students in biological, management or social sciences substantially meeting the recommendation of that Panel is: Math. 1, 2P, 2, 3, 4, 7. Computer Science should be included.

**f.** The mathematics content of a program for prospective career computer scientists (not mathematics majors) is covered by: Math. 1, 2, 2P, 3, 4, 6, 7, 8. Obviously Computer Science and more in the area of computation is also needed.

**g.** A physics major bound for graduate work in physics would find good support in: Math. 1, 2, 3, 4, 5, 2P, 10, 13. Additional available courses: Math. 6, 7, 8, 11, 12.

**h.** The minimal program recommended for mathematics teachers of grades 9–12 by the Teacher Training Panel can be met by: Math. 1, 2, 3, 4, 6, 2P, 7, 9 and two semester electives, provided Math. 9 is a year-course of appropriate geometry.

On the other hand, this program does not provide for the training of elementary teachers. Colleges having this responsibility should offer the special course program recommended for this purpose by the Panel on Teacher Training [2]. The proposed program also does not provide for deep remedial instruction or for a noncalculus mathematics appreciation course to meet a liberal arts requirement.

We observe also that advanced high schools are already teaching courses for superior students covering the material of Math. 1, 2, 2P, and even 3. So advanced placement as high as Math. 4 is possible.

**5. Issues in the general mathematics curriculum.** We call attention to some issues which inevitably arise in planning a general curriculum in mathematics for colleges, one which will serve as many purposes as possible, and as economically as possible. College teachers generally and particularly those participating in MAA Section evaluations of this report may have ideas different from ours on the resolution of these problems. Naturally, we hope you will agree generally with the way in which we have tried to meet them. Here are some of the main ones.

1) What is the right normal starting point for college mathematics in 1965? Should Math. 0 have been eliminated? Or should we have provided for a year of precalculus mathematics as was customary some years ago?

2) What accomplishment should be expected from the pivotal Introductory Calculus, Math. 1?

3) To what extent can we count on high schools taking over the teaching of Math. 0, 1, 2, 2P, and 3? When?

4) Should Probability, Math. 2P, have been offered in noncalculus form? Should it include statistics? Combinatorial algebra?

5) Should we say that a three-hour Computer Science, which might be labelled Math. 2C, is the business and even responsibility of mathematics departments to teach? Alternatively can and should a shorter introduction to programming be adjoined to existing mathematics courses? Should mathematics courses be modified to include homework on the computer? These are hot issues. Our attitude was: Let us wait and see.

6) Do mathematics teachers share our view that multivariable calculus should begin in Math. 2 and continue with fortification from linear algebra

thereafter? Textbooks are a problem but not an insurmountable one. See the reasons for an affirmative answer in the report.

7) This report subsumes introductory differential equations into the lower division calculus sequence as some current texts now do it. Was this the right decision or should the first study of differential equations be held back for a separate course in the second or third year?

8) We wanted to include in the upper division courses an intermediate differential equations course of somewhat more theoretical nature than the conventional introduction. It was crowded out by the demand of chairmen of graduate departments for a year of real variables. Texts for such a course are now available. Should it be recognized in the general undergraduate curriculum?

9) There is a current issue on the proper place to introduce the technique of (exterior) differential forms. It appears now in some lower division calculus texts. We let it appear in our program for the first time in Math. 5, which is a third-year level advanced calculus. It also appears in the differential geometry version of Math. 9. What do mathematics teachers think about this?

10) What geometry should the general curriculum include? We believed that, within some bounds, it is better to let a teacher teach what he knows and likes instead of trying to specify some preferred geometry.

11) We omitted Number Theory from the list of 14 courses. Obviously this is a good course when there is someone who knows how to teach it and students to take it. Its omission here emphasizes the minimal character of this department program. Should number theory be included as a must in such a program?

12) Alternatively, or perhaps additionally, should the Algebraic Structures, Math. 6, have been extended to a year course where it could have included a little number theory and more advanced linear algebra than one can cover in the introductory and low-level Math. 3?

13) Should the general college curriculum include a terminal year-course in mathematics appreciation, or a cultural or philosophical course in mathematics for liberal arts students? Without recommending against it we did not include it in this report.

14) Can teacher education, including elementary teacher education, be absorbed into the general mathematics curriculum, or must it be, as we decided, essentially a special program?

15) Will the expansion in both the numbers and rate of students entering college force a return to deeper remedial instruction? Or can we safely predict that high school graduates will be better and better trained in mathematics each year?

16) Returning to advanced undergraduate mathematics, how much upper division mathematics can normally be required of a major? We used six semester courses as a norm.

17) Should there be separate *undergraduate* courses in applied mathematics? What is a course in applied mathematics? We assumed that all mathematics would be taught with applications in mind and in addition we thought that,

where it can be well done, at least one course in applied mathematics should be available.

18) Does the general college curriculum here described provide an adequate undergraduate education to serve as a foundation for undertaking contemporary graduate mathematics? If not, is it possible to do better within the limits of time normally allowed for undergraduate majors in a college? Also with the available human material, teacher and student?

19) We offered an outline for a one-semester course in real variables and another one for a year-course, as requested by chairmen of graduate departments. Is a year of real variables a really necessary offering for undergraduates or is it a luxury course which will be repeated in graduate school?

These are some of the questions which revolve around the general mathematics curriculum where a widespread study and discussion should be very fruitful for mathematics teaching. Some other perennial questions were omitted because their discussion is seldom fruitful. Some of these have to do with rigor in calculus (you cannot make yourself understood in what you say about it), with where we will get the teachers, and the textbooks.

**6. How this report was constructed.** CUPM has several Panels: the Panel on Mathematics for Physical Sciences and Engineering, the Panel on Pregraduate Training in Mathematics, the Panel on Teacher Training in Mathematics, the Panel on Mathematics for Biological, Management and Social Sciences. Also there is a standing Advisory Group on Communications (Library and publications). For several years the Panels have studied intensively the special needs of mathematical training in their areas of interest and have published their recommendations [2]. To prepare this general curriculum report CUPM formed an all-panel subcommittee which reported back to CUPM as a whole and now CUPM transmits it to the Association. Two members of the School Mathematics Study Group were included to insure a proper articulation with the new high school calculus course as planned by that group. This report was based not only on the panel reports of CUPM but also on the studies of leading departments of mathematics in the country.

The take-off point was the year-course on elementary functions, polynomials, rational and algebraic functions, logarithms, exponential, and trigonometric functions which have long occupied a dominant position in the mathematics curriculum in the last years of high school and first year of college. Usually it has been offered as precalculus mathematics, or with at most a smattering of calculus, under such names as "college algebra, trigonometry and analytic geometry," or "mathematical analysis—an integrated approach," or "fundamentals," or just "college mathematics." This material is recognizable in our proposed program as Math. 0, 1. What we did was to say that the latter half of it could and should become the calculus of the elementary functions, while the responsibility for teaching the first half should normally be assigned to the high schools.

Next we thought about flexibility and multiple tracks with many possible entrance points and many suitable exits from the program to accommodate to the diversity of students that colleges now serve. This led us to the semester building blocks, our modular units for many course sequences. While many colleges will offer separate "honors" tracks parallel to the main track, and that may be best where it can be done, our minimal approach attempts to take care of the spread in achievement, and to some extent the spread in ability, of entering students by advanced placement in the General College Curriculum. That is why it is important, wherever it is possible, to offer all of the lower division courses every semester.

In the report we offer some course outlines as samples, in some cases multiple samples, of possible interpretations of the courses which we list with deliberately bare, rather generic, "college catalogue" descriptions. These outlines were not all prepared in the Subcommittee. Some were prepared in Panels and for some we had the help of mathematicians outside CUPM. We assume that mathematicians who use this report will construct their own course outlines and hence do not regard our outlines as detailed recommendations of CUPM.

**References**

1. *A General Curriculum in Mathematics for Colleges*, available free of charge from CUPM, P. O. Box 1024, Berkeley, California 94701.

2. The following CUPM reports are available from CUPM: *Recommendations on the Undergraduate Mathematics Program for Engineers and Physicists, Tentative Recommendations for the Undergraduate Mathematics Program for Students in the Biological, Management and Social Sciences, The Pregraduate Preparation of Research Mathematicians.* (This is an idealized program. The Panel will shortly publish *Preparation for Graduate Study in Mathematics*, on how these idealized goals may be absorbed into regular curricula.) *Recommendations for the Training of Teachers of Mathematics, Recommendations for the Undergraduate Mathematics Program for Work in Computing*, and *Basic Library List, 1965*. Another report expected soon is one on *Applied Mathematics* in the undergraduate curriculum.

# ON ABSOLUTELY CONTINUOUS FUNCTIONS

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**1. Introduction.** The purpose of this note is to discuss the concept of absolute continuity for functions of a real variable. This topic has been well explored so most of our results will not be new. What is new is our approach. We begin with an elegant inequality which the author discovered lying buried as an innocent problem in Natanson's book [4]. From this inequality, which we dignify by calling the Fundamental Lemma, many well-known and some new results follow in an almost trivial fashion.

The notion of bounded variation is intimately connected with that of absolute continuity. We assume that the reader is familiar with this notion as well as the following standard properties of functions of bounded variation.