PREPARATION FOR GRADUATE STUDY

IN MATHEMATICS

A Report of

The Panel on Pregraduate Training

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FOREWORD: THE GOAL

The task of the Panel on Pregraduate Training is to recommend programs for undergraduates who intend to go on to graduate work in mathematics. The first stage of the Panel's work, to consider an ideal undergraduate program for the future research mathematician, was completed in 1963, and the conclusions have been published in the report Pregraduate Preparation of Research Mathematicians [page 369].

The recommendations in the 1963 booklet are for the first four years of a seven-year program leading to a solid Ph.D. and a career in mathematical research. The booklet contains guidelines and several detailed sample course outlines. Note, however, that in forming these recommendations the Panel made no allowance for the student's possible deficiencies in preparation or tardiness in selecting a goal, for inadequacies of staff, for lack of suitable textbooks, etc. Furthermore, the program was designed for the very gifted undergraduate working under ideal circumstances. Some schools already offer programs compatible with the 1963 recommendations. At many others, however, the recommendations cannot be put into effect very soon. For them, the 1963 booklet provides a goal. The present booklet provides interim guidance.

THE PRESENT RECOMMENDATIONS

The Panel on Pregraduate Training, in considering the CUPM reports Pregraduate Preparation of Research Mathematicians [page 369] and A General Curriculum in Mathematics for Colleges*, makes the following observations:

1. The first two years of the GCMC program include a thoughtful study of calculus through partial derivatives and multiple integrals, and of linear algebra through the elementary theory of vector spaces and linear transformations.

2. If all mathematics students follow the first two years of the GCMC program, those who decide relatively late on graduate work in mathematics will not lose very much, even though early work in the spirit of the 1963 recommendations (Pregraduate Preparation of Research Mathematicians) is very desirable for the future professional mathematician.

* The original GCMC report is not included in this COMPENDIUM. However, the 1972 Commentary on a General Curriculum in Mathematics for Colleges appears on page 33.

3. The first two years of the GCMC program are within the reach of all schools, even if they have to offer a precalculus course as preparation.

The lower division of the GCMC program and its relation to pregraduate training are discussed in more detail later on.

The upper division (last two years). Because of the heterogeneity of the mathematical world, the Panel recognizes that no single curriculum will work for all schools. We therefore recommend the following priorities.

1. A minimal upper-division program for mathematics majors who intend to continue the study of mathematics in graduate school appears in the 1965 GCMC report and is reprinted below:

> A mathematics major program for students bound toward graduate mathematics: Mathematics 1, 2, 2P, 3, 4, 5, 6, 10, 11, 12, 13. A stronger major would be desirable, but this is adequate to enter good graduate schools at the present time [1965]. With two semesters' advanced placement the student still has Mathematics 7, 8, 9 from which to complete a better major.

Any college not already offering a comparable program should take immediate steps to do so.

2. If, however, the college can also supply courses designed especially for the pregraduate student, then it should provide one, or if possible both, of the one-year sequences (analysis and algebra) described below. The choice, if only one can be given, should be determined by the staff's capabilities. The one-semester analysis course (Mathematics 11) of the GCMC recommendations is itself aimed primarily at the pregraduate student; it could be replaced by the one-year course (Mathematics 11-12). On the other hand, the one-semester algebra course (Mathematics 6M) of GCMC serves many purposes; it should be retained for these purposes and the one-year algebra course described below should be added for the pregraduate student. [See page 93 for outlines of Mathematics 11-12 and of Mathematics 11. See page 68 for an outline of Mathematics 6M.]

The Panel has consulted many outstanding graduate mathematics departments; a solid grounding in algebra and analysis is what they most want from incoming students. If time and resources permit, it is, of course, desirable to introduce the pregraduate student to a broader range of material, but not at the expense of depth in algebra and analysis.

Introducing the program. Many departments of mathematics can adopt the present recommendations now. Many departments, indeed, are already offering even more substantial programs; they are referred to the 1963 recommendations and urged to proceed as far and as fast as they can in the directions suggested there. We have no universal advice for departments that feel unable to go as far as the present recommendations now.

Objectives of the program. Our concern is with all prospective graduate mathematics students regardless of their destination in today's diversified mathematical profession. The mathematical involvement of the professional mathematician, heavily influenced by the rapid developments in computer science, continues to broaden and deepen in education, in industry, and in government.

The subject matter recommended for the pregraduate program is discussed briefly above and in more detail below. This subject matter is, of course, very important; but equally important are the spirit and tone of the teaching, not only because they are reflected in the ultimate quality of the student's performance but also because they can influence the student to decide for or against a career in mathematics.

The student should be introduced to the language of mathematics in both its rigorous and idiomatic forms. He should learn to give clear explanations of some fundamental concepts, statements, and notations. He should develop facility with selected mathematical techniques, know proofs of a collection of basic theorems, and acquire experience in constructing proofs. He should appreciate the power of abstraction and of the axiomatic method. He should be aware of the applicability of mathematics and of the constructive interplay between mathematics and other disciplines. He should begin to read mathematical literature with understanding and enjoyment. He should learn from illustration and experience to cultivate curiosity and the habit of experimentation, to look beyond immediate objectives, and to make and test conjectures. In short, the student must be helped to mature mathematically as well as to acquire mathematical information.

There are many ways in which the student can be helped to mature mathematically. He can be taught in special "honors" classes for superior students. In the earlier stages he can be given independent reading assignments in textbooks, and later he can be assigned the more difficult task of reading papers in journals. He can be taught through "reading courses." He can make reports in seminars and colloquia. He can prepare an undergraduate thesis containing work original for him although not necessarily original in the stricter sense. He can be taught through the "developmental course," in which he is led to develop a body of mathematical material under the guidance of the professor. In general, the Panel feels very strongly that every pregraduate curriculum should include work to develop mathematical self-reliance, initiative, and confidence.

Identifying students. Far too few college students successfully complete a graduate program in mathematics. The Panel recommends strongly that every effort should be made to identify pregraduate mathematicians as early as possible, preferably when they enter college (a task often complicated by the students' own incorrect notions about their mathematical capabilities).

Lower-division courses. As we have already said, the Panel regards the basic sequence Mathematics 1, 2, 3, 4, 5 of the GCMC recommendations as essential for the pregraduate student. We comment briefly on this program.

For the first semester of college-level mathematics, the GCMC presents a course dealing with the integral and differential calculus of the elementary functions, together with the associated analytic geometry. To meet the needs of the future graduate student most effectively, this course should be designed with three specific objectives in mind. First, the course should build a strong intuitive concept of limits based on concrete examples. These examples can be drawn from geometry, physics, biology, etc. This may be followed by setting down a strong enough axiom system about limits to encompass their elementary properties obtained intuitively. The student should be told which of his current axioms will be future theorems. The formal definition of a limit is too difficult to be swallowed whole by the student at this point; our greatest service to the student would be to give him a firm intuitive grasp of the concept.

The second objective of this course should be to improve the student's ability to handle mathematical rigor. This can be done, for example, by using the axioms about limits in a rigorous development of the calculus.

The third and final purpose of this course should be to teach the student to calculate--for, certainly, it is the ability to calculate with the calculus that makes the calculus the powerful tool that it is.

In the next calculus courses, Mathematics 2 and 4, the GCMC is concerned, in part, with the possible introduction of a fair amount of multivariable calculus much earlier than is customary. This is less important for pregraduate mathematics students than for some other students, since pregraduate students will take all the courses.

The attitude in presenting the material is more important. After courses 1 or 3 there should be a gradual but considerable increase of mathematical maturity. Course 11-12 treats continuity, differentiation, and integration at the level of sophistication required in the theory of "real variables." Consequently, the study of these concepts in courses 2 and 4 should bring the student to an insight which makes the transition easier. There will be little in Mathematics 5 to help in this direction. Thus, the student must be led in Mathematics 2 and 4 to a considerable appreciation of rigor and to the effective personal use of mathematical language.

The 1965 GCMC recommendations included an introductory course in probability, Mathematics 2P, for all students in their first two years. The present Panel feels that the student preparing for graduate work in mathematics might better be making faster progress toward upper-division courses, deferring his work in probability.

Mathematics 3 is a short course in linear algebra. The Pregraduate Panel concurs with the GCMC committee in recommending that this course should come no later than the beginning of the second year.

Upper-division courses. Mathematics 5 contains material frequently presented as the second half of a course called "advanced calculus." Certainly the pregraduate student, whatever his branch of mathematical study, needs to acquire skill in the techniques and understanding of the concepts of mappings between Euclidean spaces of dimension at least 2 (i.e., systems of several functions of several variables).

After Mathematics 1, 2, 3, 4, 5 the Panel recommends for pregraduate students, for reasons discussed earlier, a year course in abstract algebra instead of GCMC Mathematics 6M and a year course in real analysis instead of GCMC Mathematics 11. Possible outlines for such an algebra course are presented below to indicate the flavor and scope that the Panel considers desirable for the pregraduate student; an outline for the analysis course, which is the same as GCMC Mathematics 11-12, can be found on page 93.

We repeat that any department which can offer more than these two courses should turn to the 1963 recommendations (Pregraduate Preparation of Research Mathematicians, page 369) for further suggestions.

COURSE OUTLINES

Abstract Algebra

The purpose of this year course is to introduce the student to the basic structures of abstract algebra and also to deepen and strengthen his knowledge of linear algebra. It provides an introduction to the applications of these concepts to various branches of mathematics. [Prerequisite: Mathematics 3]

OUTLINE A

1. Groups. (10 lessons) Definition. Examples: vector spaces, linear groups, additive group of reals, symmetric groups, cyclic groups, etc. Subgroups. Order of an element. Theorem: Every subgroup of a cyclic group is cyclic. Coset decomposition.

Lagrange theorem on the order of a subgroup. Normal subgroups. Homomorphism and isomorphism. Linear transformations as examples. Determinant as homomorphism of GL(n) to the nonzero reals. Quotient groups. The first two isomorphism theorems. Linear algebra provides examples throughout this unit.

2. Further group theory. (10 lessons) The third isomorphism theorem. Definition of simple groups and composition series for finite groups. The Jordan-Hölder theorem. Definition of solvable groups. Simplicity of the alternating group for $n > 4$. Elements of theory of p-groups. Theorems: A p-group has nontrivial center; a p-group is solvable. Sylow theory. Sylow theorem on the existence of p-Sylow subgroups. Theorems: Every p-subgroup is contained in a p-Sylow subgroup; all p-Sylow subgroups are conjugate and their number is congruent to 1 modulo p.

3. Rings. (10 lessons) Definition. Examples: the integers, polynomials over the reals, the rationals, the Gaussian integers, all linear transformations of a vector space, continuous functions on spaces. Zero divisors and inverses. Division rings and fields. Domains and their quotient fields. Examples: construction of field of four elements, embedding of complex numbers in 2×2 real matrices, quaternions. Homomorphism and isomorphism of rings. Ideals. Congruences in the ring of integers. Tests for divisibility by 3, 11, etc., leading up to Fermat's little theorem, $a^{p-1} \equiv 1 \pmod{p}$, and such problems as showing that $2^{32} + 1 \equiv 0 \pmod{641}$. Residue class rings. The homomorphism theorems for rings.

4. Further linear algebra (continuing Mathematics 3). (12 lessons) Definition of vector space over an arbitrary field. (Point out that the first part of Mathematics 3 carries over verbatim and use the opportunity for some review of Mathematics 3.) Review of spectral theorem from Mathematics 3 stated in a more sophisticated form. Dual-space adjoint of a linear transformation, dual bases, transpose of a matrix. Theorem: Finite-dimensional vector spaces are reflexive. Equivalence of bilinear forms and homomorphism of a space into its dual. General theory of quadratic and skew-symmetric forms over fields of characteristic different from 2. The canonical forms. (Emphasize the connections with corresponding material in

Mathematics 3.) The exterior algebra defined in terms of a basis--2- and 3-dimensional cases first. The transformation of the p-vectors induced by a linear transformation of the vector space. Determinants redone this way.

5. Unique factorization domains. (12 lessons) Primes in a commutative ring. Examples where unique factorization fails, e.g., in $Z[\sqrt{-5}]$. Definition of Euclidean ring, regarded as a device to unify the discussion for Z and $F[x]$, F a field. Division algorithm and Euclidean algorithm in a Euclidean ring; greatest common divisor. Theorem: If a prime divides a product, then it divides at least one factor; unique factorization in a Euclidean ring. Theorem: A Euclidean ring is a principal ideal domain. Theorem: A principal ideal domain is a unique factorization domain. Gauss' lemma on the product of two primitive polynomials over a unique factorization domain. Theorem: If R is a unique factorization domain, then $R[x]$ is a unique factorization domain.

6. Modules over Euclidean rings. (14 lessons) Definition of module over an arbitrary ring viewed as a generalization of vector space. Example: vector space as a module over $F[x]$ with x acting like a linear transformation. Module homomorphism. Cyclic and free modules. Theorem: Any module is a homomorphic image of a free module. Theorem: If R is Euclidean, A an $n \times n$ matrix over R, then by elementary row and column transformations A can be diagonalized so that diagonal elements divide properly. Theorem: Every finitely generated module over a Euclidean ring is the direct sum of cyclic modules. Uniqueness of this decomposition, decomposition into primary components, invariant factors, and elementary divisors. Application to the module of a linear transformation, leading to the rational and Jordan canonical forms of the matrix. Several examples worked in detail. Similarity invariants of matrices. Characteristic and minimal polynomials. Hamilton-Cayley theorem: A square matrix satisfies its characteristic equation. Application of module theorem to the integers to obtain the fundamental theorem of finitely generated abelian groups.

7. Fields. (10 lessons) Prime fields and characteristic. Extension fields. Algebraic extensions. Structure of $F(a)$, F a

field, a an algebraic element of some extension field. Direct proof that if a has degree n, then the set of polynomials of degree $n-1$ in a is a field; demonstration that $F(a) \cong F[x]/(f(x))$, where f is the minimum polynomial of a. Definition of $(K:F)$, where K is an extension field of F. Theorem: If $F \subset K \subset L$ and $(L:F)$ is finite, then $(L:F) = (L:K)(K:F)$. Ruler-and-compass constructions. Impossibility of trisecting the angle, duplicating the cube, squaring the circle (assuming π transcendental). Existence and uniqueness of splitting fields for equations. Theory of finite fields.

OUTLINE B (including Galois theory)

A course culminating in and climaxed by Galois theory can be constructed by compressing the topics in Outline A into somewhat less time and adding material at the end. Outline B suggests such a course. The appeal of Galois theory as a part of a year course in algebra is obvious. The material ties together practically all the algebraic concepts studied earlier and establishes a clear connection between modern abstraction and a very concrete classical problem. The cost is equally obvious; the depth of much of the earlier material must be reduced, or several topics eliminated. Whether the advantages justify the cost is debatable.

Recognizing that there is merit on both sides, the Panel offers Outline B as an alternative to Outline A with the following words of caution and explanation:

(a) For most pregraduate students today, Outline A probably represents the better balance between coverage and pace.

(b) Outline B contains all the material in Outline A plus Galois theory. Thus Outline B should be attempted only when more time is available or the students are <u>clearly</u> capable of an accelerated pace. Accordingly, each unit in this outline is assigned a range of suggested times, the extremes representing these two alternatives.

(c) In order to break up the rather substantial concentration on group theory at the beginning of Outline A, some of this material has been moved to a position near the end of Outline B, where it fits naturally with Galois theory. This same shift may be made in Outline A by taking the units in the order 1, 3, 4, 5, 6, 7, 2.

1. <u>Groups</u>. (6-10 lessons) Outline A, unit 1.
2. <u>Rings</u>. (7-10 lessons) Outline A, unit 3.

3. Further linear algebra. (10-12 lessons) Outline A, unit 4.

4. Unique factorization domains. (10-12 lessons) Outline A, unit 5.

5. Modules over Euclidean rings. (12-14 lessons) Outline A, unit 6.

6. Fields. (7-10 lessons) Outline A, unit 7.

7. Galois theory. (8-10 lessons) Automorphisms of fields. Fixed fields. Definition of Galois group. Definition of Galois extension. Fundamental Theorem of Galois Theory. Separability. Equivalence of Galois extension and normal separable extension. Computation of Galois groups of equations. Existence of Galois extensions with the symmetric group as Galois group. Theorem on the primitive element: If $(L:F)$ is finite and there exist only a finite number of intermediate fields, then $L = f(a)$ for some $a \in F$.

8. Further group theory. (9-10 lessons) Outline A, unit 2.

9. Galois theory continued. (8-10 lessons) Hilbert's theorem 90: If L is finite and cyclic over F, g is a generator of the Galois group, and x is an element of L of norm 1 over F, then $x = y(yg)^{-1}$ for some $y \in L$. Also the additive form of Hilbert's theorem 90. Galois groups of $x^n - a$. Definition of solvability by radicals. Theorem: An equation is solvable by radicals if and only if its Galois group is solvable. The unsolvability of the general equation of degree n, $n \geqq 5$. Other examples. Roots of unity and cyclotomic fields.