A TRANSFER CURRICULUM IN MATHEMATICS

FOR TWO-YEAR COLLEGES

A Report of

The Panel on Mathematics in Two-Year Colleges

January 1969

I. BACKGROUND

1.   Two-Year Colleges.[1]

It is impossible not to be impressed, perhaps a little over-
whelmed, by the growth and diversity of the two-year college sector
of American higher education. To cite only a few facts indicating
the rate of growth: new institutions are being added at the rate of
one each week to the approximately 900 that now exist; the largest
student body in any Florida educational institution is that of Miami-
Dade Junior College; 86 per cent of all freshmen entering California
public colleges in 1966 enrolled in two-year colleges; Seattle Com-
munity College opened in 1967 with an initial enrollment of 12,688
students. The extraordinary rate of growth in number of institutions
and size of individual institutions is matched by their diversity, of
size, purpose, make-up of student body, type of institutional control,
variety of curricula, etc. A glance at a random selection of two-year
college catalogs reveals this diversity quite strikingly. Further
aspects of it can be noted in the annual directories of the American
Association of Junior Colleges, and others are described and docu-
mented in a 1967 report to Congress from the National Science Founda-
tion.[2]

Anyone interested in mathematical education who is accustomed
to the comparatively stable national scene in either the elementary
and secondary schools or in the four-year colleges and universities
may find the two-year college picture bewildering. There is an im-
mense variety of programs, including not only those comparable to the
first two years in a university, but also a large assortment of tech-
nical, occupational, and semiprofessional programs as well as pro-
grams for general education and remedial study. The programs offered
vary widely from school to school; the two-year college is often much
more responsive to the community it serves and somewhat less respon-
sive to tradition than the other institutions.

The student body attracted by these programs or by the conven-
ience and economy of two-year college education exhibits extremes of
age and maturity, background and preparation, and ability and moti-
vation. More than two thirds of these students regard themselves

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1. The name "two-year college" is intended to tie in with the
    "lower division" at a college or university. Students at any of
    these schools may spend more or less than two years on their
    "lower-division" work.

2. The Junior College and Education in the Sciences, U. S. Govern-
    ment Printing Office, Washington, D. C., 1967.

initially as transfer students,[1] but only about one third eventually proceed to a four-year college.[2] This phenomenon is reflected in the figures on two-year college course enrollments in mathematical science compiled by the Conference Board of the Mathematical Sciences:[3]

> "Of these course enrollments 324,000 were classified by the institutions themselves as being in freshman courses and only 24,000 as being in sophomore courses. This is a much sharper drop than in four-year colleges."

This survey and the NSF report contain revealing data on the faculty. They describe a faculty which includes many part-time teachers, has a broad range of academic preparation, is recruited from a variety of sources (graduate schools 24%, colleges and universities 17%, high schools 30%, other sources 29%)[4] and is highly mobile ("About 25 per cent of all junior college professors were new to their particular campuses in 1964-65.").[5]

The situation sketched here has attracted the attention of many organizations concerned with the improvement of American higher education, of which CUPM is but one.

2.   CUPM.

The Mathematical Association of America (MAA) is the national professional organization concerned with the teaching of mathematics on the college level. The Committee on the Undergraduate Program in Mathematics (CUPM) is one of the standing committees of the MAA. Simultaneously, CUPM is one of the eight college commissions in the sciences, supported by the National Science Foundation, "to serve as instruments through which leading scientists can provide stimulation, guidance, and direction to the academic community in the improvement of undergraduate instruction."

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1. NSF Report, p. 92.

2. NSF Report, p. 5.

3. Report of the Survey Committee, Volume I. Aspects of Undergraduate Training in the Mathematical Sciences. [For more recent figures, see Volume IV. Undergraduate Education in the Mathematical Sciences, 1970-71.] Available from Conference Board of the Mathematical Sciences, Joseph Henry Building, 2100 Pennsylvania Avenue, N.W., Suite 834, Washington, D.C. 20037.

4. NEA figures for 1963-64 and 1964-65 in all fields (see CBMS survey, p. 76).

5. NSF Report, p. 71.

The early curriculum recommendations published by CUPM dealt with specific aspects of education in mathematics, such as the training of physical science and engineering students, the training of teachers of elementary and high school mathematics, and the undergraduate preparation of graduate students of mathematics. It eventually became clear that CUPM had a responsibility to show how an overall curriculum could be constructed which was within the capabilities of a fairly small college and allowed for the implementation of its various special programs. A study of this problem led to the 1965 report A General Curriculum in Mathematics for Colleges (GCMC).* This report has received wide publicity through CUPM regional conferences and Section meetings of the MAA. Its major features have met with general approval and are having a growing influence on college textbook and curriculum reforms in tangible as well as in many intangible ways. Regarding this influence, it must be stressed that CUPM does not write curricula in order to prescribe what courses a department should teach, but rather to offer generalized models for discussion and to provide the framework for meaningful dialogue (within schools, between schools, and on a broader scale) on serious curricular problems. These models are meant to be both realistic and forward looking.

The present report is a natural extension of these efforts. In addition it provides, together with GCMC, an aid to articulation of two- and four-year programs.

3.   The Present Report.

This initial CUPM report on two-year colleges is aimed at the transfer programs only. Some history of the study and reasons for this choice are outlined in the introduction to this section of the COMPENDIUM. Here we mention the following reasons:

Transfer programs are offered at almost all two-year colleges, and they determine the basic mathematics offerings. Local variations are least in this area.

A workable, imaginative solution to the problem of a transfer curriculum would provide the natural first step and could go far towards solution of the problems in the other two areas. It would answer the needs of the great majority of the students (two thirds intend to transfer), especially if it kept open a variety of options for these students, at least through the first year.

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* The report presents a curriculum which can be taught by a staff of four or five and which includes courses for the various special programs. (An exception is the special course sequence for elementary teacher training; staff for these courses is not included in the estimate.)

The transfer curriculum lends itself more readily to a curricu-
lum study with follow-up conferences. The other two problems (mathe-
matics for general education and mathematics for technical-occupa-
tional programs) involve additional considerations, such as peda-
gogical techniques and the special needs of students with very
specific goals, making this general approach less effective. CUPM's
wide experience with the GCMC report and conferences (frequently
involving two-year college teachers) had clearly indicated a need and
a demand for similar efforts suitably adapted to the realities of
two-year colleges.

The present report differs from GCMC in several major respects.
Among these is the fact that it includes more detailed course descrip-
tions, discussions of the rationale for choices that were made, and
frequent comments on how topics might be taught. It also includes
comments on implementation; the use and influence of the computer;
articulation; etc. (see Chapter 5). Finally, it includes explicit
recommendations on teacher training (see Chapter 3).

4.   Staff.

The CBMS Survey mentioned earlier reports the following data on
staff. The 167 two-year colleges having enrollments of more than
2,000 employed 44 full-time mathematics staff members with a doctor-
ate in some field (not necessarily a mathematical science); 451 with
a master's degree plus one additional year of graduate study in some
field; and 439 with a master's degree in mathematics. The corre-
sponding figures for the 543 schools of under 2,000 enrollment are
66, 306, and 513 respectively. There are, of course, many additional
staff members with less academic preparation or who teach part time.

From these figures we can compute roughly the average number of
staff members with academic preparation in the above range: five or
six for the larger schools and one or two for the smaller schools.
The corresponding figures when part-time faculty are included are
eight and two, roughly.

The present report takes account of these facts by outlining a
group of basic university-parallel courses that provide the necessary
offerings for normal transfer programs (including elementary teacher
training). It was prepared with the small department in mind. A
larger department would have the potential for offering some addi-
tional courses to supplement these. Several possibilities for such
courses are suggested in the report.

The important thing is that we believe our basic courses can be
taught by the equivalent of approximately two full-time staff members.
In Chapter 5, Section 1, we give an illustration of how this can be
done. The actual number of teachers required in a particular case
will depend, of course, on class size, teaching load, and other such
factors.

It is perhaps the most striking common feature of two-year colleges that their faculties are highly student-oriented, much more so than the more discipline-oriented faculties at the four-year colleges and universities.[1] Nevertheless, it is important not to overlook problems of academic qualifications. In particular, a report such as this or the GCMC report would exist in a partial vacuum if there were no accompanying considered statements concerning the academic qualifications desired of the teachers of the programs. Such a report has been written to accompany the GCMC[2] and another report outlining a graduate program to achieve the proposed training has been prepared.[3] Simultaneously with its decision to begin with the university-parallel study, CUPM organized an ad hoc Committee on Qualifications for a Two-Year College Faculty in Mathematics. The report of this Committee[4] will be a companion to this report.

5.   The Proposed Programs.

     The courses which CUPM proposes are described in detail in Chapters 2, 3, and 4. We summarize them here under the broad classification of Basic Offerings and Additional Offerings, and discuss some of the programs which such offerings permit. Where appropriate, we draw attention to the comparable course in the report A General Curriculum in Mathematics for Colleges. The latter, of course, serves as an alternative to be considered by those two-year colleges that are structured on purely university-parallel lines.

<div align="center">BASIC OFFERINGS</div>

I.   Calculus Preparatory

     (a) Mathematics 0. Elementary Functions and Coordinate
           Geometry.

     (b) Mathematics A. Elementary Functions and Coordinate
           Geometry, with Algebra and Trigonometry.

     One or both of these courses should be offered by every two-
     year college.

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1. See, for example, R. H. Garrison. Junior College Faculty: Issues
     and Answers. American Association of Junior Colleges, 1967.

2. Qualifications for a College Faculty in Mathematics (1967).

3. A Beginning Graduate Program in Mathematics for Prospective
     Teachers of Undergraduates (1969).

4. Qualifications for Teaching University-Parallel Mathematics
     Courses in Two-Year Colleges (1969).

II. Calculus and Linear Algebra

   (a) Mathematics B. Introductory Calculus. (An intuitive
        course covering the basic concepts of single-variable
        calculus. Similar to Mathematics 1 [page 44].)

   (b) Mathematics C. Mathematical Analysis. (A more rigorous
        course completing the standard calculus topics, as in
        Mathematics 2, 4 [page 51].)

   (c) Mathematics L. Linear Algebra. (An elementary treatment
        similar to Mathematics 3 [page 55], but parallel to,
        rather than preceding, the last analysis course.)

Categories I and II constitute the basic pre-science offerings and
should be offered by every two-year college with a transfer program.

III. Business and Social Science

   Mathematics PS. Probability and Statistics. (An introductory
   course stressing basic statistical concepts.)

IV. Teacher Training

   Mathematics NS. Structure of the Number System. (A year
   course as recommended by the Panel on Teacher Training for the
   preparation of elementary school (Level I) teachers. The
   second year of preparation, algebra and geometry, should also
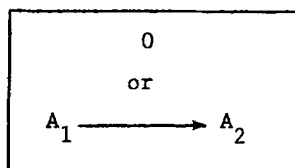   be offered whenever possible (see Chapter 3).)*

This completes the minimal set of offerings envisioned for a two-
year college that embraces the usual range of university-parallel
programs.

   The courses A, C, and NS are full year courses. If we consider
a semester system and use the obvious notation, the basic program is
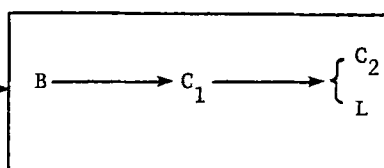represented by the following diagram.

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* Since this report was written, the recommendations of the Panel
  on Teacher Training have been revised. See the 1971 report
  Recommendations on Course Content for the Training of Teachers
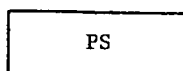 of Mathematics, page 158.
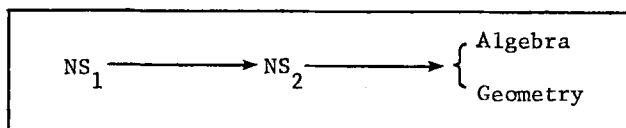
BASIC OFFERINGS

Calculus Preparatory   Calculus-Linear Algebra

$$
\boxed{\begin{array}{c} 0 \\ \text{or} \\ A_1 \longrightarrow A_2 \end{array}} \longrightarrow \boxed{\begin{array}{c} B \longrightarrow C_1 \longrightarrow \left\{ \begin{array}{c} C_2 \\ L \end{array} \right. \end{array}}
$$

Business and Social Science

$$
\boxed{\quad PS \quad}
$$

Elementary Teacher Training

$$
\boxed{\; NS_1 \longrightarrow NS_2 \longrightarrow \left\{ \begin{array}{l} \text{Algebra} \\ \text{Geometry} \end{array} \right. \;}
$$

ADDITIONAL COURSE OFFERINGS (Optional)

1. Mathematics FM. Finite Mathematics. (A course of consider-
    able interest and utility, especially for students of non-
    physical sciences.)

2. Mathematics DE. Intermediate Differential Equations.

3. Mathematics DA. Differential Equations and Advanced Calculus.

4. Mathematics PR. Probability Theory. (A calculus-based course
    adapted from Mathematics 2P [page 76].)

5. Mathematics NA. Numerical Analysis.

    Selections from this list are suggested to round out a depart-
ment's offerings according to its special needs and interests.

PROGRAMS*

    We see this combination of courses as providing a variety of
tracks to meet the needs of students with different educational goals
and mathematical backgrounds and abilities. Some of the possibilities
are described here.

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* See page 224, Pace and manner of presentation.

(i) For the student in physical science, engineering, and mathematics (including future secondary school teachers) the program should include B, C, and L. A sequence for well-prepared students is provided by the semester courses

$$B \longrightarrow C_1 \longrightarrow C_2 \longrightarrow L,$$

plus possible additional courses. A slightly less well-prepared student can achieve this by taking the sequence of courses

$$0 \longrightarrow B \longrightarrow C_1 \longrightarrow (C_2 \text{ and } L).$$

The student who is not prepared for Mathematics 0 at the outset requires an extra semester or summer session to complete calculus at the two-year college, as indicated by the program

$$A_1 \longrightarrow A_2 \longrightarrow B \longrightarrow C_1 \longrightarrow (C_2 \text{ and } L).$$

(ii) Students of the biological, management, and social sciences probably will take Mathematics PS. Many of them will want a calculus course. Assuming adequate preparation for Mathematics 0, a reasonable sequence for such students is

$$PS \longrightarrow 0 \longrightarrow B,$$

to which Mathematics FM may be added in the third or fourth semester.

Although Mathematics L is thought of as following half of Mathematics C, it is possible for a bright student to take it after Mathematics B. The resulting sequence

$$0 \longrightarrow B \longrightarrow L$$

would be of great value as a mathematically stronger alternative to the more standard

$$0 \longrightarrow B \longrightarrow FM,$$

or $\qquad 0 \longrightarrow FM \longrightarrow B.$

(iii) If the primary interest of a student is the inclusion of some mathematics as a part of his general education, then any of the courses PS, A, 0, FM or the sequences already mentioned will, depending on his previous preparation, serve the purpose effectively.

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II. THE BASIC UNIVERSITY-PARALLEL COURSES

In this chapter we describe in some detail the courses we propose for the basic offerings and discuss reasons for the choices that have been made.

1.   Calculus Preparatory: Mathematics 0 and Mathematics A.

CUPM has taken the position that precalculus mathematics properly belongs in the high school but that many colleges may need to continue teaching courses at this level. In view of this need, a course, Mathematics 0, is described in <u>A General Curriculum in Mathematics for Colleges</u>. The same course is proposed for the two-year college.

In addition, recognizing the presence in two-year colleges of many students whose high school preparation is in need of reinforcement, the Panel suggests another course, Mathematics A. The subject matter of Mathematics A includes that of Mathematics 0, but the intended pace is much slower so that reviews of topics from arithmetic, algebra, and geometry can be introduced and pursued at appropriate stages of the Mathematics 0 outline.

It is hoped that each two-year college will be able to adapt the basic idea of this course to its own needs, perhaps offering both Mathematics 0 for its well-prepared students and a version of Mathematics A to meet the needs of the rest of its students who hope to complete some calculus. The courses, although designed for students who plan to take calculus, should carry credit for general education requirements. Most schools, however, have special general education courses and would advise their use in preference to Mathematics 0 or A for the purpose.

Mathematics 0. <u>Elementary Functions and Coordinate Geometry</u>.

Discussion: The prerequisites for Mathematics B (Introductory Calculus) include the following two components, (a) and (b):

(a) Three years of secondary school mathematics. The usual beginning courses in algebra (perhaps begun in eighth grade) and geometry account for two of these years. The remaining year should include: quadratic equations; systems of linear and quadratic equations and inequalities; algebra of complex numbers; exponents and logarithms; the rudiments of numerical trigonometry; the rudiments of plane analytic geometry, including locus problems, polar coordinates, and geometry of complex numbers; and arithmetic and geometric sequences.

(b) A course such as Mathematics 0 (or Mathematics A below): A course outline for Mathematics 0 is given on page 75.

Note: The proposed first course in calculus (see page 226) treats topics in analytic geometry only incidentally, as the calculus throws new light on the subject. Thus, rather than an integrated analytic geometry and calculus, it is more in the nature of a straight calculus course. It assumes as a prerequisite sufficient command of coordinate geometry for the study of single-variable calculus and some familiarity with the elementary functions and general concepts of function as outlined above.

Mathematics A.   Elementary Functions and Coordinate Geometry, with Algebra and Trigonometry.

Discussion: This special course is designed for those students who, because of a weak mathematics background, are not prepared to begin an intense calculus preparatory course such as Mathematics 0. Many two-year colleges are currently meeting the needs of these students with courses in intermediate algebra, trigonometry, college algebra, and analytic geometry. It is felt that the contributions of all such courses to the overall two-year college offerings would be achieved better through the medium of this single course, adapted to local conditions.

A two-year college might choose to offer only Mathematics A or only Mathematics 0 or both, depending on the needs of the student population.

When teaching courses of high school level in a two-year college, it is customary to repeat the material in essentially the same form as it was presented in high school. For students who were not successful in high school, this approach is often no more fruitful the second time than the first. The virtue of Mathematics A is its fresh approach to old topics. Instead of requiring courses that repeat early high school mathematics and are then followed by a course on elementary functions, it is suggested that a repetition of the high school material be interwoven with the topics of the latter course.

The notion of function is given a central and unifying role in Mathematics A. Functions and their graphs serve as a peg on which to hang the review of elementary material. They also provide a new perspective, as well as a method of illustrating the old material. In this way, it is thought that the necessary review can be presented in a manner sufficiently fresh and interesting to overcome much of the resistance that students may carry with them from high school. (Moreover, this new approach should make Mathematics A more pleasant for the faculty to teach.) The development of a sound understanding of the function concept provides a solid cornerstone on which to build additional mathematical concepts in later courses.

This course should serve at least the following three purposes:

(i) To prepare the student for calculus and other advanced mathematics by including the material of Mathematics 0. Rather than being concerned with precisely which topics should be included, however, the emphasis should be on developing the ability to understand and use mathematical methods at least at the level of Mathematics B.

(ii) To review and remedy deficiencies in arithmetic, algebra, geometry, and basic logic:

> (a) by means of assignments and class drill associated with Mathematics 0 topics. (Every opportunity should be seized to expose and stamp out abuses of logic and notation, and such common atrocities as $1/2 + 1/3 = 1/5$, $1/2 + 1/3 = 2/5$, $(a + b)^2 = a^2 + b^2$, and $\sqrt{a + b} = \sqrt{a} + \sqrt{b}$, familiar to all teachers.)

> (b) by interjecting discussions of review topics as the occasion arises in Mathematics 0 topics (see illustrations, page 221).

(iii) To develop mathematical literacy. By this we mean the ability to read and understand mathematical statements and the ability to translate into mathematical language (making proper use of logical connectives) statements and problems expressed in ordinary English. Continual practice should be given in solving "word problems" and in analyzing mathematical statements, with particular emphasis on developing the ability to understand and to use deductive reasoning.

Because of the limited time available, some material must be slighted. In this instance classical synthetic plane geometry is not covered in the detail or to the extent that is common in high school. However, we compensate for this by including enough analytic geometry to provide sufficient geometric preparation for calculus. Care has been used in choosing the topics in Mathematics A so as to include those topics which the student needs in order to progress successfully to calculus or to calculus-related courses. It nevertheless seems likely that five hours a week will be needed for Mathematics A during the first semester and possibly also during the second semester. It should be a slow course that includes a great deal of problem solving and also lengthy excursions as suggested by student interests and needs. Although Mathematics A serves to prepare students for calculus, it should also make students aware of the power and beauty of mathematics. The teacher should exploit every opportunity to provide examples and applications appropriate to the age and maturity of the students. Extensive use of "word problems" can be very effective in developing the ability to think mathematically and to use mathematics. We have attempted to give some indications along these lines in the course outline. In our view it is far more important to stir and develop the interest of the students than to cover each suggested topic.

It is essential that the students be forced throughout the course to translate English sentences into mathematical ones and vice versa, in order to improve their mathematical literacy.

Unfortunately, there exists at this time no single textbook completely appropriate for a course such as Mathematics A. However, by drawing upon several existing texts, the teacher should be able to find suitable textual material, illustrative examples, and exercises with which to build the course and achieve the remedial objectives.

Finally, we believe that a student who is not capable of handling Mathematics A upon entering a two-year college will in all likelihood be unable to pursue mathematical subjects in a four-year college with profit. Such a student should not be considered a transfer student in mathematical subject fields.

COURSE OUTLINE FOR MATHEMATICS A

Concept of function. Introduction as a rule associating to each element of a set a unique element of another set. (There shouldn't be very much worry about the definition of the word "set." Many examples should be given, with special attention devoted to functions that cannot be easily represented by formulas: for instance, the number of birds in a given locality or on a given tree as a function of the time or of the individual tree; the number of female students at the college as a function of the year; a function whose domain is the set {A, B, C} and whose range is the set {D, E}; etc.) Cartesian coordinate systems and graphs of functions. Review of relevant geometry, such as basic facts concerning perpendiculars and directed line segments. Review of negative numbers, decimals, and the arithmetic of fractions, keyed to the problem of representing functions graphically. Decimals reviewed in this vein. Relative sizes of numbers illustrated on the x-axis. Motivation by means of examples of the definitions of sum, product, difference, and quotient of functions. Review of rational operations with algebraic expressions. Graphing of inequalities. Review of fractions and decimals, with special emphasis on comparison of sizes. Simple probability as a set function, i.e., a brief discussion of possibility sets, truth sets, and the assignment of measures to sets and their subsets. Conditional probability.

Polynomials of one variable. Linear functions of one variable and their graphs. Slope of a straight line. Various forms of the equation of a straight line. The graph of a linear equation as a straight line. Quadratic functions of one variable and their graphs. Completing the square. Use of completion of the square to solve some simple maximum and minimum problems. (This provides an opportunity to introduce some interesting "word problems.") Definition of polynomial functions of one variable. Sum and product of polynomials. Laws of exponents for integer powers. Division of polynomials, with the process first taught purely as an algorithm (stress should be given to the similar process for integers, whereby one integer is divided by another and a quotient and remainder obtained; the formula $m = qn + r$). Examples, pointing out the desirability of finding zeros of polynomial functions and ranges where the functions are positive or negative. Review of factoring: common factor, quadratics and the quadratic formula, difference of squares, sum and difference of cubes. The factor and remainder theorems. Theorems on rational roots. The need for complex numbers. Complex numbers and their algebra (introduced as a natural extension of the real number system). Geometric interpretation of addition of complex numbers, multiplication by real numbers (i.e., the vector operations) and of multiplication (see illustrations, page 221).

Arithmetic and geometric sequences. Definitions and sum formulas. Approximation (noting error term) of $\frac{1}{1-x}$ by $\sum_{k=0}^{n} x^k$ for $-1 < x < 1$ (both by using the sum formula and by repeating the division algorithm). Intuitive discussion of the meaning of limit. (Application can be made to the fractional representation of an infinite repeating decimal. The limit concept can be illustrated by using both repeating and nonrepeating infinite decimals.)

Combinations and the binomial theorem. Finite sets, their unions and intersections. Combinations and permutations of members of a set. The binomial coefficients $\binom{n}{k}$ as the number of combinations (subsets) with k members chosen from a set with n members. Additional simple probability, using the formulas for numbers of permutations and combinations. More applications and "word problems" using poker, dice, baseball lineups, etc. Review of the manipulation

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of integer exponents. Direct evaluation of $(x + y)^n$ for
n = 2, 3, 4. The binomial theorem for natural number exponents.
Proof using the formula for number of combinations.

Rational functions and polynomials of more than one variable.
Rational functions and their graphs, with particular attention given
to factoring the denominators and to the concept of asymptote. Use
of asymptotes to strengthen further the intuitive notion of limit.
Linear inequalities. Simultaneous linear equations in two or three
unknowns. Applications to elementary linear programming. Polynomial
functions of two variables. Quadratic polynomials of two variables.
Definitions and resulting equations of conics, but only for the axes
parallel to the coordinate axes (no discussion of rotation of axes).
Simultaneous solution of a linear and a quadratic equation in two
variables, of two quadratics, and geometric interpretation of the
solutions. (The student should be made aware of the importance of
functions of several variables; he should know that the initial re-
striction to one variable is only for the purpose of handling the
simpler problem first.)

Exponential functions. Review of the laws of exponents, with
explanations of why, with $a > 0$, a^0 is defined to be 1, a^{-x} to
be $1/a^x$, and $a^{p/q}$ to be $\sqrt[q]{a^p}$ or $(\sqrt[q]{a})^p$. Real numbers reviewed
and presented as infinite decimals. Discussion of the need to give
meaning to arbitrary real power. Discussion of approximations to
specific numbers such as $10^{\sqrt{2}}$ or $(1.2)^\pi$ and how this concept
might be used along with the limit concept to define such exponen-
tials. (This provides a good opportunity to contribute another step
toward the understanding of the limit concept. If a computer is
available, convergence can be made to seem plausible by actually
computing approximations to a number such as $(1.2)^\pi$, using the
first few places in the decimal expansion of π.) General exponen-
tial functions. Sketch of the theory of binomial series for nega-
tive and fractional exponents, with more discussion of the limit
concept. Use of these in approximations of roots and powers. Appli-
cations of exponential functions to selected topics such as growth,
interest, or electricity.

Logarithmic functions. Discussion of the general notion of the

inverse of a function. Application to the definition of the loga-
rithm to base a as the inverse of the exponential to base a.
Review of the use of common logarithms in calculation. The slide
rule (only a few lessons; see illustrations, page 221). Some discus-
sion of the idea of approximating exponentials and logarithms by
polynomials; no proofs given. (If a computer is available, use it
to verify that the approximating polynomials give values of the ex-
ponential and logarithm agreeing with those in tables. If time and
interests permit, consider $\left(1 + \frac{x}{n}\right)^n$ as n increases, to relate
compound interest and exponentials and to motivate polynomial
approximations of e^x.)

Trigonometric functions. Review of simplest geometric proper-
ties of circles and triangles, especially right triangles (very few
formal geometric proofs). Definition of trigonometric functions as
ratios (illustrate, but do not belabor, problems of triangle solv-
ing). Trigonometric functions defined on the unit circle as func-
tions of a real variable. Graphs of trigonometric functions. Sym-
metry; even and odd functions. Review of ratios, fractions, and
decimals once again. The following trigonometric identities: the
elementary identities $\sin x/\cos x = \tan x$, $\sin (x + \frac{\pi}{2}) = \cos x$,
etc.; the Pythagorean identities $\cos^2 x + \sin^2 x = 1$, $\sec^2 x$
$= 1 + \tan^2 x$, $\csc^2 x = 1 + \cot^2 x$; formulas for $\cos (x \pm y)$,
$\sin (x \pm y)$, $\cos x \pm \cos y$, and $\sin x \pm \sin y$. Law of sines and
law of cosines. Trigonometric (polar) representation of complex
numbers. Roots of complex numbers. Graphs of sums and products of
trigonometric and exponential functions. Periodicity. Applications
to periodic motion and other periodic phenomena. Inverse trigono-
metric functions using the general notion of inverse function. Their
graphs.

Functions of two variables. Introduction to 3-dimensional co-
ordinate systems, functions of two variables, and graphs of functions
of two variables for a few simple cases.

Some illustrations. 1. Elementary probability theory pro-
vides an excellent opportunity to develop both the ability to reason
deductively and the ability to translate "word problems" into mathe-
matical language. Examples can be chosen from horse racing, tossing

of dice or coins, drawing colored balls from bags, etc. The following is one of an endless variety of problems that are useful for developing the ability to analyze a problem and to assign probability measures to sets:

A man tosses two coins and then informs you that the outcome includes at least one head. Determine the probability that the outcome consisted of two heads, given that the rule determining the man's statement is the following (two separate cases are considered):

Case (a). He says, "There is at least one head," precisely when this is true; otherwise he says nothing.

(The fact that he spoke means the outcome TT is eliminated from the four possibilities, leaving three equally likely cases and an answer of 1/3.)
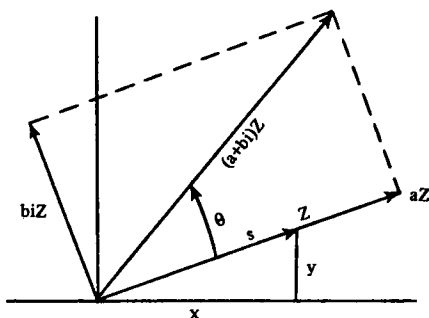
Case (b). He says, "There is at least one head," when the outcome is HH. He says, "There is at least one tail," when the outcome is TT. In the remaining cases, he chooses which of these two statements to make on the basis of a coin toss.

(Under these rules, the outcome TT has probability zero in view of the statement actually made. The remaining elements of the sample space have probabilities 1/2 for HH and 1/4 each for HT and TH. The answer is 1/2.)

2. The student will easily follow the interpretation of complex numbers as vectors and their addition as vector addition. He should also be taught to interpret this addition as a transformation of the plane. Thus $(a + bi) + (x + iy)$, where x and y are variables, defines a _translation_ of the entire plane in the fixed direction of $a + bi$. This remark is preliminary to the more difficult step (for the student) of interpreting $(a + bi)(x + iy)$ as an "operator" on the plane, a point of view which is of prime importance in many technologies.

This is broken down into steps: $a(x + iy)$ extends the vector $(x + iy)$ in the ratio $a:1$ [might be called an "extensor" or "amplifier" (one considers also the case that a is negative)]. Next, $i[x + iy] = ix - y$ is a rotation (of the variable vector and the entire plane) through $+90^{\circ}$ (i.e., counterclockwise), and $bi[x + iy]$ extends this new vector in the ratio $b:1$. From this

the geometric meaning of
(a + bi)(x + iy)   appears as
a vector sum. The length of
this sum vector is $s\sqrt{a^2 + b^2}$,
where s is the length of
Z = (x + iy),   its polar angle
is as shown in the figure
(where θ, the polar angle of
a + bi,   occurs as shown by
reason of similar triangles).
Hence, multiplication by a + bi
is an operator which rotates the plane through the angle θ and
stretches all vectors in the ratio $\sqrt{a^2 + b^2}$ to 1. This gives the
geometric interpretation of a product of complex numbers (multiply
lengths and add angles); later in the course it gives the addition
formulas for sine and cosine (simultaneously). Of course, this
geometric interpretation could also follow a discussion of trigo-
nometry, as an application.

    3. Word problems involving conjunctions and disjunctions are
useful in sharpening logical thinking. For example:

    Thirty students are given grades 3, 2, or 1. If at least
    10 got 3's and at most 5 got 1's, what can be said about
    the average score S for that class? $[3 \geqq S \geqq 65/30]$. What
    can be said about the number N of students who scored 2?
    If at most 10 got 3's and at least 5 got 1's, what can be
    said about S? $[1 \leqq S \leqq 65/30]$. What can be said about the
    number N of students who scored 2?

Other (similar) problems can be introduced in a variety of contexts.

    4. A brief discussion of the slide rule is of value in rein-
forcing the student's understanding of logarithms. Conversely, such
an understanding can be the basis for learning the principles upon
which the instrument is designed and can give the student an ade-
quate foundation for self-taught skill in its use.

    Also, the student's interest in the slide rule can be exploited
to great advantage in order to develop through practice his skill at

estimating the magnitude of the result of an arithmetic computation (the problem of placing the decimal point).

2.   Calculus and Linear Algebra: Mathematics B, C, and L.

This group of courses constitutes the proposed standard offerings for students expecting to major in mathematics, engineering, or the physical sciences, but it is structured in such a way as to serve a variety of other needs. In particular, this is achieved by the course Mathematics B (similar to Mathematics 1 [page 44]). The main feature of this course is the inclusion of the main concepts of calculus (limit, derivative, integral, and the fundamental theorem) in the setting of a single variable, dealing with the elementary functions studied in the earlier course and including computational techniques and applications of these ideas and methods. Thus the course provides a meaningful and usable study of the calculus for students who are unable to pursue the full sequence, as well as a desirable introduction for those who are.

Mathematics B can also be regarded as the calculus of elementary functions and, as such, forms a natural unit with Mathematics 0 or A: a thorough study of the elementary functions.

The remainder of the usual calculus topics, including series, functions of several variables, and elementary differential equations, are described here as a single one-year course, Mathematics C.

Finally, we describe a course in linear algebra, Mathematics L, which is similar to Mathematics 3 [page 55], only slightly less ambitious and employing a strong geometric flavor.

We consider it very important that the two-year college student who starts Mathematics C should complete it at the same school, rather than risk the probable discontinuity of study and loss of time attendant upon transfer before completion of this material. For this reason we have abandoned the feature of GCMC which proposes that linear algebra precede the study of functions of several variables and which allows a somewhat deeper study of the latter. This feature constituted only one of the reasons for the inclusion of linear algebra in the lower-division offerings. The remaining arguments apply as well to the two-year college situation, where the course can be taken simultaneously with the last half of Mathematics C in a two-year sequence starting with Mathematics 0, and so will serve an important group of students.

Pace and manner of presentation.

Without intending to prescribe the exact format in which the proposed courses are to be offered, we nevertheless find it convenient to assume occasionally a system of semesters (roughly 14 weeks

of classes each) in order to state relative intensities of presenta-
tion in the familiar terms of 3-, 4-, and 5-hour[1] semester courses.
In these terms we imagine the basic sequence of courses as embodying
an increase of pace, the pace of the first courses taking realistic
account of the wide range of student abilities and the last courses
matching in pace the comparable courses at the transfer institutions.[2]

To illustrate, a distribution which fits our image of the aver-
age, multi-purpose community college is:

Mathematics A_1, A_2, 0, and B--5 hours each

Mathematics C_1--4 hours

Mathematics C_2 and L--3 hours each

Such a progression of pace will, it is hoped, allow for enough
additional drill and reinforcement of topics in the earlier courses
and for enough attention to mathematical literacy to provide for the
student's growth in these areas to a degree of maturity which, at the
end of the two-year college experience, puts him on equal standing
with his fellow students at the best transfer school, where more
severe screening may have taken place.

It is essential that this nurturing and growth process be
achieved without weakening the content of the courses to the detri-
ment of the better students. Indeed, for the better of the science-
oriented students, it is important that these courses at the two-year
college achieve as deep a penetration of the subject matter as com-
parable university courses so as to prepare the student for more
advanced courses in mathematics and related disciplines. A calculus
course that fails to teach students to understand and analyze mathe-
matics, that gives too few and too careless demonstrations of mathe-
matical facts, will make later work very difficult for the student
who attends a four-year college whose upper-division curriculum pre-
supposes more: students who have learned in high school to think
mathematically may become disillusioned; students who have passed
such a course may have false impressions of their mathematical knowl-
edge; for all students, such courses mean the postponement of in-
volvement with the true meaning of mathematics.

This, however, most emphatically does not mean that the course
should strive for complete "rigor" as exemplified by, say, a detailed

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1. These refer to the number of class meetings, not to the amount of
   credit granted, which we consider a local problem.

2. To be sure, four-year colleges have students of varying abilities
   also, but there is more of a sink-or-swim philosophy than at many
   two-year colleges which regard the careful nurturing of students
   as a greater part of their function.

development of the real number system from axioms, preceded by formal discussions of logic and followed by careful epsilon-delta foundations for the calculus. It is possible to be intuitive and still be correct, simply by following intuitive motivations and plausibility arguments with precise statements and proofs where possible, and with honest omissions or postponements of these when necessary.

It is in this sense that Mathematics B is suggested to be an intuitive course (a certain amount of such omission is certainly indicated for the suggested amount of material to be covered). The gaps left in Mathematics B, the promises for more sound foundations, are intended to be partially filled in Mathematics C where, for example, questions of limit, continuity, and integrability reappear (in the series and several variables settings).

Mathematics C, therefore, is intended to be a more sophisticated course, both in selection and treatment of subject matter, than Mathematics B. For example, we feel that the inclusion of a study of sequences and series in Mathematics C not only is important in itself and for its applications but also is particularly effective for developing mathematical maturity. With the least upper bound property of the real numbers as the principal tool it is possible to give careful proofs of the convergence theorems for monotone bounded sequences and for series of nonnegative numbers with bounded partial sums. Moreover, the study of sequences and series is a mathematical discipline that shows the need for precise definitions and demonstrates the dependence of mathematical understanding and mastery of formal techniques on basic theory. Also, the subject dramatizes the inadequacies of many of the student's prior intuitive ideas. It provides the student with perhaps his first major experience with a mathematical situation for which precise methods are indisputably needed. Indeed, it is necessary that some of the concepts introduced in Mathematics B, for example "limits," be further developed here.

Mathematics B. Introductory Calculus.

Discussion: Mathematics B is designed to follow Mathematics 0 or Mathematics A. The purpose is to introduce the ideas of derivative and integral, motivated by geometrical and physical interpretations, and to study their interrelations through the Fundamental Theorem of Calculus; to develop techniques of differentiation and integration; and to demonstrate the power and utility of the subject matter through frequent and varied applications.

Skills in manipulation need to be stressed in order that the student may learn to solve effectively problems in the applications of calculus, and to facilitate his study of later topics.

Continuing emphasis should be placed upon developing mathematical literacy; the lessons should stress correct interpretations of the written statements that set forth theorems, problem conditions, and proofs.

COURSE OUTLINE FOR MATHEMATICS B

1. Slope of the secant line to a curve. Geometric discussion
of the limiting position. Derivative as slope of tangent line. Aver-
age rates and instantaneous rates of change as intuitive motivation
for derivative. Limit of difference quotient as special case of
limit of function. Intuitive discussion of $\lim_{x \to a} f(x)$ including
algebra of limits (describe the definition briefly using pictures
and mention that the limit theorems hold for this definition). Defi-
nition of derivative based on intuitive notion of limit; algebraic
properties: sum, product, and quotient formulas with proofs.

2. Derivative of x^n, polynomials, and rational functions.
Point out continuity of polynomials (limit obtained by direct substi-
tution). Examples of discontinuities, definition of continuity
(illustrate with pictures). Derivatives and continuity properties of
trigonometic functions reduced to showing $\lim_{h \to o} \dfrac{\sin h}{h} = 1$. The exist-
ence of $f'(a)$ implies that f is continuous at a.

3. Interpretations and applications: slope, velocity, rates
of change, curve sketching--including increasing, decreasing, maxima,
and minima (confined to rational functions and simple combinations of
trigonometric functions). Need for more differentiation techniques.

4. Chain rule. Its importance to building techniques of dif-
ferentiation. The obvious "proof"

$$\frac{df(g(x))}{dx} = \lim_{h \to o} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h \to o} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h}$$

$$= f'(g(x)) \cdot g'(x)$$

breaks down because $g(x+h) - g(x)$ might be zero for arbitrarily
small h. Difficulty is avoided either by a trick or by showing that
in this case both sides are zero, yielding a proof (either do it or
give a reference). Comments on the use of limit theorems and dif-
ferentiability hypotheses in this argument.

General differentiation formulas for rational and trigonometric
functions. Drill. Implicit differentiation. Derivatives of algebraic

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functions (general power formula) and inverse trigonometric functions.

5. More applications: word problems, related rates, curve sketching; need for more information on max, min, etc.

6. Intuitive discussion of Rolle's theorem, Mean Value Theorem. Proofs, assuming existence of maximum for continuous function on closed, bounded interval. Intermediate Value Theorem.

7. Higher derivatives and the differential as best linear approximation.

8. More applied problems. Curve sketching, rate problems and maximum and minimum problems with the more complete methods now available, acceleration, concavity.

9. Integration. (Motivate the integral as a limit of sums by observing how the latter arise naturally in several specific practical problems, e.g., area, work, moments, volume, mass, pressure. The integral is an abstract entity which embraces these and other quantities.) Discussion, with reliance on intuitive understanding of the limit involved. Antiderivative and Fundamental Theorem of Calculus. Plausibility arguments for its validity and stress on its meaning for computation. Practice in finding antiderivatives, need for further techniques.

10. Exponential and logarithm. Definition of $\log x$ as $\int_1^x \frac{1}{t} dt$. Derivation of properties of $\log x$ from this formula. Exponential function as inverse of $\log x$. Properties. Drill on complete list of differentiation formulas. Problems on applications involving these functions.

11. Applications of integration to finding areas, volumes of revolution, arc lengths, work done in emptying tanks.

12. If time permits: A more careful development of the definition of integral, discussion of properties of continuous functions, and proof of the Fundamental Theorem.

Mathematics C. <u>Mathematical Analysis</u>.

Discussion: Completing the traditional subject matter of calculus and giving an introduction to differential equations, this

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course serves also as the vehicle for an increasing sophistication, in both topics treated and the manner of their presentation.

Except for the obvious desirability of including the material on vectors prior to the beginning of Mathematics L (where, in any case, this material is briefly reviewed), no attempt has been made to divide the subject matter to fit quarters or semesters. There are two principal reasons for this. First, it seems important that students who intend to complete the entire calculus sequence (Mathematics B and C) should do it all at the same institution. Otherwise, wide variations in the order of topics can make transfer very difficult. Second, we leave to the particular college the problem of determining how many quarters or semesters and how many meetings per week are needed to present the material.

COURSE OUTLINE FOR MATHEMATICS C

Techniques of integration. Integration by parts. Integration by substitution, including use of the inverse trigonometric functions. Explanation of plausibility of general partial fraction expansion; use of partial fractions in integration. Emphasis on the use of tables of integrals. (Drill on techniques of integration provides useful practice in the manipulation of the elementary functions. Stress should be placed on this rather than the development of ingenuity in integration techniques for its own sake.)

Limits. Limit of $f(x)$ as x approaches a, where each deleted neighborhood of a contains a point of the domain of f. Right and left limits. Limits as x (or $-x$) increases without bound. Statement of basic theorems about limits with a proof of one or two of them and careful discussions of why all the theorems are reasonable and important. Continuity defined by the use of neighborhoods and then expressed in the language of limits. Continuity of sums, products, quotients, and composites of functions. Discussion of the Intermediate Value Theorem; zeros of a function; Newton's method. Discussion of the Maximum Value Theorem; proofs of Mean Value Theorems; l'Hôpital's rule. Integral as a limit; the Fundamental Theorem via the Mean Value Theorem; numerical integration and integrals.

Sequence and series. Definition of sequence as a function. Limits of sequences. Definition of series and sum of series. Relation to limit of sequence. Convergence of monotone bounded sequences

and of series of nonnegative numbers with bounded partial sums. Comparison, ratio, and integral tests for convergence of series. Conditional convergence. Power series and radii of convergence. Taylor's theorem with remainder. Taylor's series.

Vectors. 3-dimensional vectors: addition, multiplication by scalars, length, and inner (dot, or scalar) product. Parametric equations of lines and planes established by vector methods. Inner products used to find the distance between points and planes (or points and lines in a plane). Vector-valued functions and the position vector of a curve. Parametric equations of curves. Differentiation of vector-valued functions. Tangents to curves. Velocity as a tangent vector. Length of a curve. Polar coordinates. Curves in polar coordinates. Area in polar coordinates. Radial and transverse components of velocity for polar coordinates in the plane, with application to determining the length of a curve whose equation is expressed using polar coordinates. Area of surface of revolution. If time permits: vector products, acceleration, curvature, and the angle between tangential and radial lines.

Differentiation. Partial derivatives. Differentials of functions of several variables. The chain rule for functions of several variables. Directional derivatives, gradients, and tangent planes. Theorem on change of order of partial differentiation. Implicit and inverse functions. Taylor's theorem and maxima and minima of functions of several variables.

Integration. Further use of simple integrals for computing such things as work, force due to fluid pressure, and volume by the shell and disk methods. Multiple integrals. Volume. Iterated integrals. Change of variables for polar, cylindrical, and spherical coordinates.

Differential equations.* Discussion of what an ordinary differential equation is; of ways in which ordinary differential

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* Many current calculus texts include a chapter which provides a brief
  introduction to differential equations, and closely related topics
  are often interwoven through other chapters (e.g., simple growth and
  decay problems in connection with logarithmic and exponential func-
  tions). Such treatments indicate the time allowance, if not neces-
  sarily the spirit, of the treatment which we have in mind.

equations arise; and of what it means for a function to be a solution of a differential equation, with verification of solutions in specific cases. Use of tangent fields for equations of type $y' = f(x,y)$ to build more understanding of the meaning of a differential equation and of its solution curves. Integration of first-order equations with variables separable that arise in studying problems of growth and decay, fluid flow and diffusion, heat flow, etc. Integrating factors for first-order linear equations, with more applications. Second-order homogeneous equations with constant coefficients. Use of undetermined coefficients to solve the initial value problems of undamped and damped simple harmonic motion with forced vibrations. If time permits: more study of second-order nonhomogeneous equations; series solutions.

Mathematics L. Linear Algebra.

Discussion: There are perhaps three main reasons why a course in linear algebra should be offered by a two-year college that is able to do so. Such a course will contain the first introduction both to a kind of mathematical abstraction essential to the mathematical maturity of the student and, of course, also to serious mathematical ideas other than those of calculus. The subject is becoming more and more useful in the physical, biological, management, and social sciences. Finally, it provides the proper setting for a deeper understanding of the basic notions of analytic geometry that the student sees in Mathematics 0, A, B, and C.

In this program the linear algebra course is designed to be a course parallel to the last part of Mathematics C. Of course, vector ideas and methods should still be introduced as much as possible in the calculus sequence since they make possible a more meaningful development of that subject. Even though some basic vector ideas will be introduced in Mathematics C, the proposed linear algebra course can still be taken by capable students who have had only Mathematics B. To this end it begins with an introduction to geometric vectors, i.e., the course below has been designed in such a manner as to be accessible to a student who has not worked with vectors before.

In its 1965 publication A General Curriculum in Mathematics for Colleges, CUPM recommended that the student take linear algebra before the last term of calculus. This recommendation is gaining very widespread acceptance in the four-year college. However, the situation in two-year colleges is quite different; it is probably better there to allow the last part of calculus and the linear algebra to be taught in parallel, rather than to require one before the

other. There is little question that most four-year colleges and universities would rather see the transfer student come in with the calculus completed than with some linear algebra at the price of having to complete the calculus. To finish calculus before transferring, the student may not have time to take linear algebra first, although he may well be able to take it simultaneously with the last part of calculus. Moreover, those transfer students who will major in the social or management sciences may take only one term of calculus but almost certainly will find linear algebra more useful than any other kind of mathematics. Here again, it is important that good students be able to take both the last part of calculus and the linear algebra.

The course outline below emphasizes a geometrical outlook and adheres closely to the guidelines for linear algebra suggested by the following quotation from A General Curriculum in Mathematics for Colleges (1965):

> When linear algebra is taught as early as the first
> term of the second year of college one should examine care-
> fully the choice of topics. We need something intermediate
> between the simple matrix algebra which has been suggested
> for high schools and the more sophisticated Finite-Dimensional
> Vector Spaces of Halmos. The pieces of linear algebra which
> have demonstrated survival at this curriculum level for many
> years are: systems of linear equations and determinants from
> college algebra, uses of vectors in analytic geometry, and
> the calculus of inner products and vector cross products.
> More modern ideas call for the introduction of matrices as
> rectangular arrays with elementary row operations, Gaussian
> elimination, and matrix products; then abstract vector spaces,
> linear dependence, dimension, and linear transformations with
> matrices reappearing as their representations. Finally, of
> all subjects in undergraduate mathematics after elementary
> differential equations the one which has the widest useful-
> ness in both science and mathematics is the circle of ideas
> in unitary geometry: orthogonality, orthogonal bases, orthog-
> onal expansions, characteristic numbers, and characteristic
> vectors.

COURSE OUTLINE FOR MATHEMATICS L

Geometrical vectors. 2- and 3-dimensional vectors, sums, differences, scalar multiples, associative and commutative laws for these. Dot and cross products. Equations of lines in two and three dimensions and of planes, by vector methods. Normals and orthogonality, direction cosines. Components of vectors and vector operations in terms of them. Mention correspondence between plane (space) vectors at the origin and R^2 (R^3).

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Matrices and linear equations. Matrices, row and column vectors defined as rectangular arrays and n-tuples of real numbers. Sums of matrices and vectors, products of matrices. Examples and applications to elementary economics, biology, analytic geometry. Linear equations written in matrix form. Gaussian elimination. Row echelon form of matrix. Existence and uniqueness theorems for solutions of homogeneous and nonhomogeneous systems of linear equations. Numerical examples for 2, 3, 4 variables.

Vector spaces. Need for unifying concept to subsume n-tuples and geometric vectors of the previous paragraphs. Definition of abstract vector space over the reals. Examples of vector spaces from the above, including the null space of a matrix (solution space of a system of linear equations), function spaces, spaces of solutions of linear differential equations, space of polynomials of degree $\leq$ n, of all polynomials. Linear combinations of vectors. Geometric applications. Linear dependence and independence. Theorem: If $m \geq n$, then m linear combinations of n vectors are dependent. Definition of dimension and proof of uniqueness (finite-dimensional case only). Subspace. Geometric examples and function space examples. Dimension of subspaces: $\dim (S+T) = \dim S + \dim T - \dim (S \cap T)$, with appropriate definitions and examples.

Linear transformation. Definition of linear transformation. Matrices associated with linear transformations with respect to different bases. Sums and products of linear transformations defined. (Show how this leads naturally to definitions of sums and products of matrices given in the second paragraph above.) Range and kernel of linear transformation. $\dim (kernel) + \dim (range) = \dim (domain)$. Elementary matrices. Rank of linear transformation and matrix. Row rank = column rank. Linear equations re-examined from this point of view. Inverses of linear transformations and matrices. Calculation of inverse matrix by elimination. $PAQ = I_r$ if r = rank of A. Matrix of same linear transformation with respect to different bases. Similarity.

For the remainder of the course a choice may have to be made between the following two topics.

Determinants. The determinant as a function from matrices to the reals which is alternating and multilinear on the rows and columns, i.e., the determinant of a matrix is unchanged by adding a multiple of a row (column) to another row (column), the determinant of a diagonal matrix is the product of the diagonal entries, etc. (No attempt at proving existence in the general case.) Derivation of explicit formulas for 2×2 and 3×3 determinants. Calculation of several explicit $n \times n$ determinants for $n > 3$, for example, van der Monde 4×4 and 5×5 determinants. Expansion in terms of cofactors. If A, B are $n \times n$ matrices, then determinant (AB) = determinant A · determinant B. (Last two topics with only indications of proof.) A very brief discussion of the formula A adj (A) = I det (A). Cramer's rule.

Inner product spaces. Definition. Orthogonal and orthonormal bases. Gram-Schmidt process. Distances. Orthogonal complements. Orthogonal expansions. Examples in function and polynomial spaces where inner product is defined via an integral. Applications to 3-dimensional analytic geometry.

3.   Probability and Statistics: Mathematics PS.

The discussion and list of topics given here represents tentative recommendations arrived at jointly by some members of the CUPM Panels on Statistics and Two-Year Colleges, in order to elaborate the latter Panel's recommendation that a course of this kind be included in the basic offerings.

Historically, a proliferation of introductory statistics courses, differing not only in content but also in emphasis and method of approach, has developed. Some of the factors responsible for this are the desire by different disciplines to have courses tailored to their needs, the varied backgrounds of persons teaching statistics, and differences in objectives for the basic statistics courses. The Panel on Statistics of CUPM is undertaking a full study of the difficult problem of developing models for such courses. The study will look to the future. It will investigate the role of computers, teaching devices, and laboratories. It will consider special approaches, such as those stressing decision theory, nonparametric methods, Bayesian analysis, data analysis, and others. A major aspect of the study will be consultation with representatives of the various fields whose students are served by courses of this kind.

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The study may well indicate that no single statistics course is best suited to fill the variety of existing and anticipated needs; the Panel may eventually recommend a choice from among several different types of statistics courses (hopefully providing at the same time the impetus for the appearance of any new texts they may require).

The projected recommendations will, of course, supersede the suggestions given here in that they will be more explicit and will be based on a more extensive study of the problem.*

Discussion: An important element in the education of transfer students in many fields is a one-term introductory course in statistics, including an adequate treatment of the needed topics from probability. Such a course is often required of students majoring in particular fields: to cite a few examples, business administration, psychology, sociology, forestry, industrial engineering. In addition, it serves as an excellent elective subject for other students.

A course of this kind can be intellectually significant although based on minimal mathematical preparation. In what follows a prerequisite of two years of high school algebra is assumed.

The main function of a first course in statistics which will be terminal for many students and, for others, will provide a basis for studying specialized methods within their majors, is to introduce them to variability and uncertainty and to some common applications of statistical methods, that is, methods of drawing inferences and making decisions from observed data. It is also desirable, as a second objective, for the students to learn many of the most common formulas, terms, and methods.

Faculties of some departments served by this course stress the latter objective and, in addition, wish their students to be knowledgeable in certain specialized statistical techniques. We suggest that it is preferable for the students to learn basic concepts in the first course. (If a specialized technique is needed, it can be taught effectively in the subject matter course in the context in which it is used, at a small cost in time.)

It is quite plausible that an imaginative presentation which develops only a minority of the most common topics could be very effective in providing the students with an intelligent and flexible approach to statistical problems and methods through the development of basic concepts. Therefore, we regard the second objective of the course as secondary in importance.

Since the main objective of the course is understanding of the basic statistical concepts, proofs and extensive manipulations of

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* The report mentioned in these paragraphs appeared in 1972 and is entitled Introductory Statistics Without Calculus. See page 472.

formulas should be employed sparingly. While statistics utilizes these, its major focus is on inference from data. By the same token, the course should not dwell upon computational techniques. The amount of computation should be determined by how much it helps the student to understand the principles involved.

We urge the use of a variety of realistic problems and examples, to help motivate the student and to illustrate the approach to such problems by statistical methods. (The use of a laboratory in conjunction with the courses offers attractive possibilities for these purposes.) Examples drawn from disciplines of common interest to the class members should be sought. Also, the generation of data within the class (e.g., heights, birth dates) can be very useful.

Despite the diversity of basic courses, there has been a common set of topics included in most of them. The list below corresponds in an approximate fashion to this common set. A number of topics have been omitted from the list--some because we feel they would detract from more important topics: combinatorial aspects of probability, skewness, kurtosis, geometric mean, index numbers, and time series, for example. (These need not be avoided if they serve as pedagogical devices in attaining the major objectives of the course.) Other topics have been omitted because their incorporation would make it difficult to cover enough of the topics on the list. These include Bayes' theorem, Bayesian analysis, experimental design, sequential analysis, decision theory, and data analysis (insofar as this refers to an as yet emerging body of knowledge devoted to techniques for detecting regularities from data). These are potentially excellent topics; some of the possibilities for innovation in the basic statistics course are either (1) to stress some of these at the expense of omitting other (perhaps many) topics included in the list, or (2) to make one of these topics the basic focus for the entire course and discuss other topics in the framework thus established.

Finally, we urge that this course be a sound intellectual experience for the students. It should not be just a compendium of terms and techniques.

LIST OF TOPICS FOR MATHEMATICS PS

Note: This is not a course outline. It is not intended that this list of topics be necessarily covered in the order presented or in its entirety; an extremely skillful presentation would be required for a great majority of them to be combined satisfactorily in one course.

1. Introduction. Data: collection, analysis, inference. Examples of some uses of statistics emphasizing data collection, summarization, inference and decisions from data.

2. Probability: sample space, probability of an event, mutually exclusive events, independent and dependent events, conditional probability. (Keep enumeration to combinatorial notation at most. Use tables.) Random variable, expected value, mean and variance of a random variable.

3. Discrete random variable: binomial, use of binomial tables, Poisson (optional).

4. Summarization of data: grouping data. Central tendency: arithmetic mean, median, mode. Dispersion: range, standard deviation (others optional). Graphical analysis: histogram, frequency polygon, cumulative frequency polygon, percentiles.

5. Continuous random variables and distributions. Normal distributions. Use of tables. Normal approximation to binomial.

6. Sampling theory: random sampling, Central Limit Theorem, normal approximation to distribution of $\bar{x}$.

7. Point estimation: estimation of mean of the normal and mean of the binomial.

8. Interval estimation: confidence intervals on the mean of the binomial, confidence intervals on the mean of the normal using the t distribution. Large-sample approximate confidence intervals.

9. Hypothesis testing. Null hypothesis, alternate hypothesis, Type I and Type II errors, power of the test. Test of hypothesis on the mean of the normal using the t test. Test on the mean of the binomial. Large-sample approximate tests.

10. χ^2 tests: goodness of fit, contingency tables.

11. Comparison of two population means. Paired and unpaired cases using the t test (nonparametric or range tests may be substituted).

12. Nonparametric methods. Sign test, Mann-Whitney test, Wilcoxon test. Confidence interval on median.

13. Regression and correlation. Least squares, correlation coefficient. Estimation and tests.

14. One-way analysis of variance. F test, simultaneous confidence intervals.

III. TEACHER TRAINING AT TWO-YEAR COLLEGES

One of the major concerns of CUPM since its inception has been the proper mathematical training of prospective teachers of school mathematics. Although much progress has been made over the years on this serious problem, a great deal remains to be done, and the two-year colleges will play an increasingly important part in these efforts. For this reason we have included in this report a statement from the CUPM Panel on Teacher Training, together with some excerpts from their current recommendations.

[Editor's note: The original report contained a statement from CUPM's Panel on Teacher Training and a course outline for Mathematics NS (The Structure of the Number System). Rather than reprint these, we refer the reader to the latest report of the Panel on Teacher Training, Recommendations on Course Content for the Training of Teachers of Mathematics (1971), and to the course outlines contained therein. See page 177.]

IV. OPTIONAL ADDITIONAL OFFERINGS

Many two-year colleges offer courses for transfer students other than a basic set of courses such as those described in Chapters 2 and 3. With the growing trend for well-qualified and highly motivated students to attend two-year colleges, the demand for such additional courses will increase. The opportunity to teach such courses will certainly be attractive to faculty members and will offer them stimulating challenges. In this chapter we offer some suggestions for the kind of additional courses which we think deserve prime consideration, including sample course outlines and discussions of the circumstances under which the course is deemed desirable.

1.   Finite Mathematics: Mathematics FM.

This course is designed primarily for students interested in business, management, social, and biological sciences. Its purpose is to introduce a variety of mathematical topics, showing how these are related to problems in the areas mentioned above. It would be a valuable component of the curriculum only if a substantial amount of time, at least 20% of the course, were devoted to applications of the material developed. Simply taking up mathematical topics of potential relevance is not enough; indeed, unless it includes a substantial unit on applications it would be better not to offer the course.

In finite mathematics the student is exposed to mathematical topics of a different nature from those he has previously studied-- topics that apply to new types of problems and topics that are perhaps more closely related to his special interests. It is expected that some students, even if a small proportion, will be attracted to further study of mathematics by this course, either because they find these new topics interesting in themselves or because they recognize the need to learn more for further applications.

The course described, while not requiring all of the topics in Mathematics 0, can profitably build on the mathematical sophistication and maturity of that level of mathematical education in order to develop the ideas needed for the applications. A course certainly can be designed for students who have completed less, although it may then be necessary to have the course meet an extra hour per week in order to have enough time at the end of the course for the desired substantial unit on applications. A course assuming Mathematics PS could proceed somewhat further with applications, either in greater depth or in greater variety. However, we believe finite mathematics will serve a greater need and a larger audience if statistics is not a prerequisite.

The topics suggested are chosen because of their applicability; they include finite sets, probability, and linear algebra. The elements of logic may be, but need not be, treated as a separate topic. All that is needed is an ability to analyze and interpret compound statements formed by the usual connectives "and," "or," "not," "if and only if," "if ... then ...," and familiarity with what is meant by a logical argument. These ideas can be woven into the course. We prefer the latter procedure since it will help to insure time for applications.

Although it is not indicated in the outline, an attractive way of beginning is to present one or two applied problems that will be solved by the end of the course (e.g., one of the several Markov chain examples). As the course progresses and partial solutions are obtainable, the student should be apprised of the progress being made toward the ultimate solution to be given at the end of the course.

COURSE OUTLINE FOR MATHEMATICS FM

Sets. Subsets and set inclusion. The algebra of sets: union, intersection, complementation, difference. The relation with logic of compound sentences constructed from the connectives "and," "or," "not," "if and only if," "if ... then ... ." Tree diagrams as a systematic way of analyzing a set of logical possibilities. The number of subsets of a set. Cardinality of finite sets; card $(A \cup B)$ = card A + card B - card $(A \cap B)$. Partitions, binomial and multinomial theorems and related counting problems.

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Probability. The concept of a probability measure: the axioms, and the realization of these axioms for finite sets. (The intuitive basis for this model discussed and illustrated by several examples.) The Law of Large Numbers. Equiprobable measure. Probabilistic independence. Repeated occurrences of the same event and binomial distribution. Approximation of the binomial distribution by the normal distribution as a means of making predictions about the likelihood of outcomes of repeated independent trials. (If time permits, also discuss Poisson distribution as approximation of the binomial.) Introduction to Markov chains (particularly if the course is begun by a discussion of a problem involving Markov chains; otherwise this topic may be delayed).

Linear algebra. Vectors and matrices introduced as arrays of numbers. Addition of vectors and matrices. Products (motivated by such examples as: given a vector of prices of certain commodities and a matrix giving the quantity purchased on each of a set of days, find the total expense on each day). Solution of linear equations, including the cases where the number of equations and number of indeterminates differ, and including the cases of multiple solution and no solution. The definition of the inverse of a matrix. Its computation without using determinants. Applications to Markov chains; classification of chains, with the emphasis on absorbing chains and regular chains. Fixed vectors and eigenvectors. Examples such as: for regular chains, analyze a problem in social mobility, probability of change of job classification, change of residence, change in brand preference among choices for a particular article, probability of change of political party allegiance, genetic heredity; for absorbing chains, many of these same examples with a change in the probabilities, e.g., brand preferences, genetic heredity, as well as matching pennies, learning processes.

Applications. One of the largest classes of applications involves Markov chains, including the long-term development of proportions of dominant and recessive genes, learning models, models on the judgment of the lengths of lines, the spread of a rumor in a housing development, or an extension of any of the items mentioned above.

Another attractive example is linear programming, including: an introduction to convex sets; the fact that a linear function defined on a convex set assumes its maximum and minimum at a vertex; some simple examples that can easily be solved by hand, with indication of the magnitude of the task in case the convex set has many faces and vertices. If time permits: a discussion of means of solving such problems, including the simplex method.

Still another example is the theory of games. Here one can introduce the idea of a zero-sum two-person game, especially a 2 X 2 matrix game, the idea of pure and mixed strategies, pure strategy solutions both for strictly determined games and for those that are not strictly determined. Mixed strategies and the min-max theorem (proved for 2 X 2 matrix games). Examples: one of the simplified poker games; competition between two businesses.

2.   Differential Equations and Advanced Calculus: Mathematics
      DE and DA

Many two-year colleges now offer courses in differential equations. This practice is expected to continue because of the large number of students specializing in fields for which some familiarity with differential equations is important. Some students who take such a course do not transfer to four-year colleges and some of those who do transfer plan to major in fields which do not require many upper-division courses in mathematics at the four-year college. A two-year college might well offer a course in differential equations as an elective for such students, with Mathematics C and L as prerequisites. However, such a course has limited value if it trains students only in formal techniques for special cases without developing understanding of the general nature and meaning of differential equations and their solutions.

We regard linear algebra as being a more basic and important tool than differential equations for most students. For students who will take only one of the courses we strongly recommend linear algebra. Since the concept of vector space is very important when discussing general linear equations, linear algebra should be a prerequisite for differential equations. If students do not have this preparation, the treatment of systems of differential equations will, of necessity, be quite minimal.

The content of a differential equations course should be planned to meet the needs and interests of the students. Therefore, we have outlined a course, Mathematics DE (Intermediate Differential Equations), that has considerable flexibility and can be varied to

meet such needs and interests. Certain important topics have been listed first. Other topics are listed from which choices can be made, either as additional topics or as substitutions. For example, the study of existence and uniqueness theorems might be substituted for the study of systems of equations, or numerical and operator methods might be introduced throughout the course.

For some students, further study of calculus is more important than the study of differential equations. This is true for many students planning to take mathematics after transfer. For such students we recommend a course which develops some concepts of advanced calculus and also provides an extension of the study of differential equations begun in Mathematics C. This course, Mathematics DA (Differential Equations and Advanced Calculus), would follow Mathematics C and L. Recognizing varied student and faculty interests and time limitations, we describe the core material and some additional topics from which choices should be made as time permits.

The two courses whose outlines follow are designed to develop good understanding of the mathematics involved and to improve the mathematical maturity and ability of the students. These courses are suitable only for students who have gained from courses such as Mathematics C and L the ability to understand and to readily assimilate mathematics. In particular, it is assumed that the students have been given an introduction to differential equations such as that described in the outline of Mathematics C.

COURSE OUTLINE FOR MATHEMATICS DE

Review. First-order equations with variables separable; integrating factors for first-order equations. Second-order linear homogeneous equations with constant coefficients. The method of undetermined coefficients, with applications to undamped and damped simple harmonic motion with forced vibrations.

Linear differential equations. Superposition of solutions. Homogeneous equations of nth order with constant coefficients; existence and uniqueness theorems for initial value problems. Linear dependence and independence; vector spaces of solutions. Relations between solutions of a nonhomogeneous equation and the corresponding homogeneous equation. Methods of undetermined coefficients and variation of parameters. Applications to initial value problems.

Series solutions. Discussion of term-by-term differentiation and integration of a power series within the interval of convergence; proof that the interval of convergence does not change, with at least heuristic justification of the procedure. Use of series for solving

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some first-order and other simple cases for which convergence can be
verified easily and the procedure justified. Discussion of both the
method of undetermined coefficients and the use of Taylor series for
determining series of solutions. General theory of series solutions
about regular points. Discussion of the indicial equation and of
the nature of series solutions about regular singular points, with
most results stated and only heuristic justification given. Applica-
tions to classical equations such as those of Bessel and Legendre.
Some simple nonlinear equations, such as $y' = x^2 + xy^2$.

Systems of equations. Equivalence of general systems to sys-
tems of first-order equations. Vector representation of a system of
first-order equations. Use of matrices to solve systems with con-
stant coefficients. Description of the nature of the solution when
the coefficient matrix can be reduced to diagonal form, e.g., when
the eigenvalues are distinct or the matrix is symmetric; description
and application of the Jordan canonical form. Discussion of applica-
tions to mechanical and electrical problems.

Additional or alternative topics chosen from among the following:

1. Numerical methods. Difference equations and interpolation.
Runge-Kutta method. Elementary considerations of stability and error
analysis.

2. General linear equations. Wronskians; linear dependence
and independence; number of linearly independent solutions of an
ordinary linear differential equation.

3. Existence and uniqueness theorems. Convergence of power
series solutions. Existence and uniqueness proofs for first-order
equations using Picard's method. Generalization to systems of first-
order equations.

4. Operator methods. The operator D for linear equations
with constant coefficients; factoring and inversion of operators;
partial fraction techniques. The Laplace transform applied to linear
differential equations with constant coefficients and to simultaneous
linear first-order equations with constant coefficients.

5. Nonlinear differential equations. Special nonlinear equa-
tions which are reducible to linear equations; local stability; simple
phase-plane geometry of trajectories. Self-sustained oscillations of

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a nonlinear system. Forced oscillations of a nonlinear system (e.g., $x'' + k^2(x-cx^3) = A \cos ct$) and the corresponding resonance phenomenon.

COURSE OUTLINE FOR MATHEMATICS DA

Topology of the real line. Brief description of the construction of the real numbers from the rational numbers. Open and closed sets. Least upper bound property shown to imply Bolzano-Weierstrass and Heine-Borel properties.

Continuity. Preservation of connectedness and compactness by continuous maps; maximum value, mean value, and intermediate value theorems; continuity of inverses. Uniform continuity.

The above topics should be accompanied by numerous examples and by counterexamples showing the need for the hypotheses.

Linear differential equations. Extension of the methods developed for second-order linear equations in Mathematics C. Linear dependence and independence; number of linearly independent solutions of an ordinary linear differential equation; vector spaces of solutions. Relation between solutions of a nonhomogeneous equation and the corresponding homogeneous equation. Methods of undetermined coefficients and variation of parameters. Applications to initial value problems.

Convergence. Review of tests for convergence of series and sequences of constant terms. Algebraic operations with series and power series. Uniform convergence of sequences and series. Term-by-term integration and differentiation of sequences and series. Existence and uniqueness proofs for first-order differential equations using Picard's method.

Additional or alternative topics chosen from among the following:

1. Series solution of differential equations. Power series. Use of power series to obtain solutions of differential equations. General theory of series solution about regular points. Discussion of the indicial equation and of the nature of series solutions about regular singular points, with most results stated and only heuristic justification given. Applications to classical equations such as those of Bessel and Legendre.

2. <u>Systems of differential equations</u>. Equivalence of general systems to systems of first-order equations. Vector representation of a system of first-order equations. Use of matrices to solve systems with constant coefficients. Description of the nature of the solution when the coefficient matrix can be reduced to diagonal form, e.g., when the eigenvalues are distinct or the matrix is symmetric; description and application of the Jordan canonical form. Discussion of applications to mechanical and electrical problems.

3. <u>Riemann integration</u>. Area and integrals. Properties of definite integrals. Existence of integrals of continuous and monotone functions.

4. <u>Transformations</u>. Review of partial differentiation and of linear transformations and matrices. Jacobians. Inverse transformation and implicit function theorems. Change of variables in multiple integrals. Cylindrical and spherical coordinates.

3. Probability Theory: Mathematics PR.

The importance of probability theory makes it a prime candidate for inclusion early in the program of many transfer students. The usual treatment of this subject requires some background in calculus. Sufficient background for this course is provided by Mathematics B. The course itself will provide additional calculus material as the need arises.

This course should lay stress on problems and, in particular, on problems which provide motivation and develop interest in the conceptual aspects of probability theory. Problems of this kind can be found, for example, in <u>Fifty Challenging Problems in Probability</u> by F. R. Mosteller (Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1965).

The course outline below is an adaptation of Mathematics 2P [page 76]. The outline is brief. It is intended that the topics be treated in depth.

COURSE OUTLINE FOR MATHEMATICS PR

(a) <u>Probability as a mathematical system</u>. Sample spaces, events as subsets, probability axioms, simple theorems. Intuitively interesting examples. Special cases: finite sample spaces, equiprobable measure. Binomial coefficients and counting techniques

applied to probability problems. Conditional probability, independent events, Bayes' formula.

(b) <u>Random variables and their distributions</u>. Random variables (discrete and continuous), density and distribution functions, special distributions (binomial, hypergeometric, Poisson, uniform, exponential, normal, etc.), mean and variance, Chebychev inequality, independent random variables.

(c) <u>Limit theorems</u>. Poisson and normal approximation to the binomial, Central Limit Theorem, Law of Large Numbers, some applications.

4.   Numerical Analysis: Mathematics NA.

A two-year college should consider offering a course in numerical analysis provided it has

(a)   a curriculum of courses including all the "Basic Offerings";

(b)   someone on the staff qualified to teach the course;

(c)   access to computing facilities with reasonably short turn-around time. A time-sharing system is ideally suited to this purpose, as is a calculator-computer that students can use "hands on."

The first two provisions above are not likely to be challenged. To fulfill the second it may be possible for some institutions to find a part-time teacher from local industry who is interested in the mathematical issues raised. As for computing facilities, while numerical analysis can be studied effectively without a digital computer, the actual computational processes involved can be tedious, and the course would be much less likely to attract students. If the only accessible computing equipment involves long delays between starting a program and getting the answers, the result is frustration and a discontinuity of interest in the problem being solved.

This course has several purposes. It is intended to capitalize on the interest students have in computers in order to spur their interest in mathematics and to kindle in them a desire to study further topics in mathematics, for in it they should find that turning a problem over to a computer eliminates some mathematical problems while raising others. In addition, it provides an introduction to the important techniques of numerical solution to a variety of problems and introduces ways in which digital computers can be helpful in problem solution.

It should be recognized that this is not a course in computer science, but rather it is a mathematics course. It contains material closely related to the curriculum of any computer science program, however, and hence is appropriate for those who wish to pursue a major in computer science at a four-year college.

The course begins with a discussion of some of the problems that will be treated. Most of these will be problems with which the student is already acquainted in some form: definite integrals that cannot be expressed in closed form, differential equations that he has not learned how to solve, as well as large systems of linear equations with coefficients containing more than two significant figures, and nonlinear equations in a single variable which in theory he may know how to solve, but which in practice are quite impossible. The process of obtaining numerical solutions to these problems will stimulate the study of additional topics in analysis, linear algebra, probability, and statistics. The student will become concerned with approximations, rapidity of convergence, and error analysis.

Selection of topics from the outline will depend on how much, if any, attention to numerical techniques appeared in the calculus course, and on the tastes and interests of the instructor. It should in any case include examples of interpolation, approximation, and solutions of systems of equations.

COURSE OUTLINE FOR MATHEMATICS NA

Prerequisite: At least one year of calculus, including infinite series.

Introduction. Some typical numerical problems; the theoretical aspects of numerical analysis such as convergence criteria and error estimates, versus the algorithmic aspects such as efficiency of an algorithm and error control. (Most of these topics can be nicely introduced through examples, e.g., the solution of quadratic equations or the problem of summing an infinite series.)

Interpolation and quadrature. The linear and quadratic case of polynomial interpolation; basic quadrature formulas; numerical differentiation and its attendant error problems. If time permits, discussion of the general Lagrange and Newton interpolation formulas, the Aitken method, and Romberg integration.

Solution of nonlinear equations. Bisection method, successive approximations, including simple convergence proofs, Newton's method, method of false position. Application to polynomial and other equations and to special interesting cases such as square and cube roots

and reciprocals.

Linear systems of equations. The basic elimination step, Gauss and Gauss-Jordan elimination. Roundoff error, the ill-conditioning problem in the case $n = 2$. If time permits: brief introduction to matrix algebra and the inversion of matrices.

Solution of ordinary differential equations. Series solutions and their limitation. Euler's method, modified Euler's method, simplified Runge-Kutta.

V. IMPLEMENTATION

This final chapter is devoted to some comments on various aspects of the problem of implementing the proposals of the earlier chapters. It is never the intent of CUPM that its curriculum recommendations be "swallowed whole," but rather that they be used as a focus for the discussion of curricular problems on a national or regional scale and ultimately that they serve on the local level as a starting point for improvements tailored to local needs. It is in this sense that CUPM wants to see this program implemented. We point out below how it can be considered in small pieces; how it can be staffed economically; how problems of articulation with high schools, four-year colleges, and other two-year programs can be approached; and how such factors as the computer and the existing choice of textbooks may affect implementation.

We aim the following remarks at a generalized audience in full recognition of the fact that the actual readers represent diverse schools and that each will need to interpret them for his own needs.

1. Implementation by Stages.

CUPM does not, of course, recommend a total alteration of current practices, but rather an evolution in certain directions.

The curriculum seems to divide naturally into parts, reasonably independent from the viewpoint that changes in one part need not force major changes in the others. Thus the proposed structure can be approached in stages, beginning wherever a department recognizes the greatest need:

(1) The introduction of Mathematics 0 and (or) Mathematics A in place of current calculus-preparatory courses.

(2) The revision of the calculus courses along the lines of Mathematics B and C.

Remarks:

(a) If current practice includes highly integrated analytic geometry-calculus courses, these two stages are not quite independent. Nevertheless, the problem of proceeding with one of these stages and making appropriate adjustments in the other courses does not seem to be a major one.

(b) If stage 2 has been carried out successfully, the course Mathematics B should serve the students of nonphysical sciences who need an introduction to calculus. Thus, some need for special-purpose courses, such as business calculus, might be eliminated.
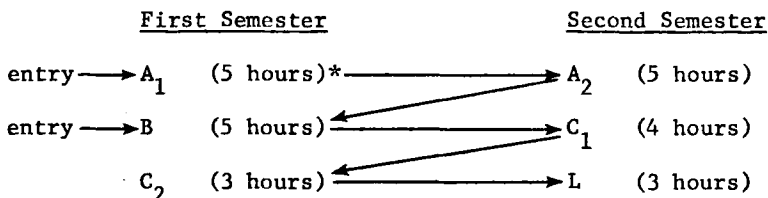
(3) Introduction of Mathematics L.

(4) Introduction of Mathematics PS.

Remark: Although courses of this kind are offered in most schools, frequently in departments other than mathematics, a well-designed course given in one department can help avoid duplication of effort within a school.

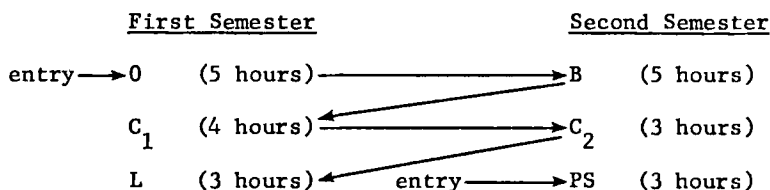(5) Introduction or extension of the offerings for elementary school teachers.

(6) Selection of appropriate additional courses.

We believe that none of these changes would create unusual demands in terms of the teaching staff they require. Indeed, that the entire basic program calls for only slightly more than the equivalent of two full-time staff members is indicated by the following sample patterns.

First Semester		Second Semester	
entry⟶A_1	(5 hours)*	A_2	(5 hours)
entry⟶B	(5 hours)	C_1	(4 hours)
C_2	(3 hours)	L	(3 hours)

This provides the full science sequence with two points of entry and uses approximately the equivalent of one full-time instructor.

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* See pages 224 and 225.

| First Semester | Second Semester |

entry ⟶ 0   (5 hours) ─────────────⟶ B   (5 hours)

C_1 (4 hours) ─────────────⟶ C_2 (3 hours)

L   (3 hours) ⟵ entry ────⟶ PS   (3 hours)

This offers the additional natural point of entry and Mathematics PS once each year, again using about one instructor. The addition of the course Mathematics NS (3 hours each semester) and perhaps the second year of teacher-training courses completes the basic offerings.

This simplified model provides considerable flexibility, perhaps even more than is needed by many of the smaller schools where the need to offer each course each semester may not be present. On the other hand, it does not take account of multiple-section courses, of the possibilities for more economy through such devices as a summer school entry to the science sequence, or of the additional flexibility that can perhaps be achieved with the quarter system. Nevertheless, it seems clear that departments working out more detailed models according to their own needs will find this program quite economical of manpower as compared to alternative curricula.

2.   Articulation.

Articulation efforts have as their primary goal establishing a procedure of transfer that is as smooth and convenient as possible for the student, thereby enabling him to be cognizant of the required courses in his field and to plan carefully whatever program he is pursuing. Although articulation between two-year colleges and four-year colleges encompasses many areas such as staffing, curricula, administration, and personnel, the main obstacle to be surmounted is that of "communication." Detailed information must be exchanged between the two, and that information must then be disseminated to all concerned.

As indicated in "Guidelines for Improving Articulation Between Junior and Senior Colleges,"* there are five main aspects of the transfer from the two-year college to the four-year college. These are: administration; evaluation of transfer courses; curriculum planning; advising, counseling, and other student personnel services; articulation programs. We have kept these in mind in formulating the present recommendations.

Many states, however, have large, geographically dispersed systems of two-year colleges and four-year colleges in which a slowdown or lack of communication causes some of the items listed in the

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   Avenue, N.W., Washington, D. C. 20036

"Guidelines" to become problems. Florida and California, for example, are states having geographically dispersed systems which have experienced or foreseen such problems. They have organized state-wide committees to discuss, study, and recommend solutions or guidelines for the solution of these problems.*

The Advisory Council on College Chemistry published a report in January, 1967, entitled "Problems in Two-Year College Chemistry, Supplement Number 20A." This report deals with curriculum and articulation problems in the area of chemistry that are in many ways identical to those in mathematics.

The counseling, guidance, and placement procedures employed by the two-year college serve to minimize the articulation difficulties both between the high school and two-year college and between the two- and four-year college. Complete information about the individual two-year college's mathematics program should be in the hands of local high school counseling staffs. In this manner the high school student, regardless of his background, can be informed as to where he should start his mathematics program in college. For example, a graduating student who suddenly realizes that he needs calculus for his major field, and who is not prepared, should be informed that the local two-year college offers courses such as Mathematics A and Mathematics 0 to correct the deficiency.

In general, the mathematics department is urged to exercise its responsibility for coordinating its work with high schools, general counselors, and transfer institutions. Two particular goals are to improve placement procedures and to establish sound criteria for transfer credit.

Articulation between institutions can cease to be a problem only when complete communication has been established.

3. General Education and Technical-Occupational Programs.

A further study of these topics will be a major part of the continuing work of the Panel. However, because of their great importance in two-year colleges a brief commentary on these subjects is included here.

General Education. A desirable way for a nonscience major to acquire some understanding and appreciation of mathematics may be to take a course or courses in general education mathematics. This is true if the course is carefully planned and taught by skilled and inspiring teachers. Above all else, the course should be so designed

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* See Report of the California Liaison Committee, November, 1967, Sacramento, and A Report on Articulation in Mathematics (Florida), June, 1966, Tallahassee.

and taught that the students discuss and do mathematics rather than only hear about mathematics.

No definite outline for such a course is suggested at this time. Each teacher should judge the best type of course to suit his class. CUPM hopes to offer him more specific assistance in this task soon. [See A Course in Basic Mathematics for Colleges, page 314.]

Whatever is done should be meaningful relative to the student's experience and studies; it should be taught in a new light relative to his past experiences with the study of mathematics. In many ways college students are becoming more sophisticated, and hence a new approach is needed to reinforce old mathematical experiences and to develop new ones. It is desirable that the course achieve the minimum levels of understanding, skills, and reasoning needed by all students regardless of their goals or preparation, and that it provide some base for continuation in mathematics for those students who can and should continue. The course should emphasize basic mathematical concepts of recognized importance for an educated person and should be oriented, where possible, toward meaningful applications. The material should cut across the traditional segmentation of arithmetic, algebra, and geometry. The solution of "word problems" should be emphasized throughout.

Mathematics 0, A, PS, and FM as described above can be very effective general education courses for students having the necessary preparation.

Technical-Occupational. Even though the majority of students who enter two-year colleges do so with the intention of transferring, there still exists a large group who seek training and instruction that will prepare them for employment at the end of two years. In order to accommodate the latter group of students, along with the portion of the first group who change their objectives, most two-year colleges provide a wide array of technical, semi-professional, and occupational programs.

The need for mathematics in these career programs varies widely with the program. Some require no more than simple arithmetic while others demand a working knowledge of mathematics ranging from elementary algebra through the calculus and differential equations. We feel that the course Mathematics A, with examples drawn from technical areas, has great potential as a service course for many such programs. If a program requires a more specialized course, it should be taught by a mathematics instructor who is familiar with the curriculum's objectives. The content and sequence in which the topics are taught should be determined by the coordinated efforts of the mathematics and career curriculum faculty members. The "integrated" or "related" mathematics of vocational programs may be taught more effectively by the vocational instructor.

Whenever possible, instruction and practice in the use of the computer for problem solving should be given in engineering- and science-oriented two-year occupational programs and use of the computer in data processing should be taught in business- and health-oriented two-year occupational programs.

4.   The Computer in Two-Year Colleges.

The two-year college must be prepared for, and articulate with, the changes taking place in both secondary schools and four-year colleges in the area of computing and in the use of computers in mathematics courses. Computers are rapidly being introduced into secondary schools both physically and as a part of the curriculum. Courses of three types are already being taught.

1) The first type of course serves as an introduction to and emphasizes "computer science" as a distinct discipline incorporating the many facets of modern computers. It is exemplified by the School Mathematics Study Group's course Algorithms, Computing, and Mathematics.

2) The second type of course emphasizes instruction in a computer language. It includes an exposition of the essentials of a programming language and many examples of programs for the solution of various categories of mathematical problems.

3) The third type of course emphasizes the integration of computing into the mathematics curriculum. It uses the computer as a part of the instructional process in selected subject matter areas within mathematics.

To indicate the trend further, we note that the School Mathematics Study Group is preparing a new curriculum intended to go from seventh through twelfth grades which emphasizes the use of algorithmic ideas and computation. Flowcharting, for example, is introduced in the seventh grade. Many colleges have introduced computing into their elementary mathematics courses--calculus, in particular.

The reasons for this far-reaching change appear to be pedagogical, substantive, and social.

Pedagogically, the computer is a mathematical instrument and a laboratory. Computing facilitates, extends, and enriches learning in mathematics; mathematics extends and facilitates and is essential to learning in computing. The mathematics class is the prime place in the two-year college for learning computing. Computing in the mathematics class can aid understanding, help to build concepts, and develop mathematical intuition.

Substantively, the student's education is broadened to include an introduction to computer science and numerical analysis.

Socially, the computer is becoming an essential part of many aspects of the student's life and work and is an excellent means of demonstrating the relevance of mathematics to the needs of society and the individual. In addition, training in the use of the computer is likely to provide the two-year college student with a skill which will open the doors to a variety of employment opportunities otherwise unavailable to him.

Every transfer student, with the possible exception of humanities majors, should at some time in his college career take a course in computer science such as the course C1 in <u>Recommendations for an Undergraduate Program in Computational Mathematics</u> [page 563]. We make this recommendation both because of the direct importance of the computer in many fields of knowledge and because we feel the students should be aware of the capabilities and limitations of the computer.

The two-year college should be prepared to introduce computing into mathematics courses wherever relevant and appropriate. All of our recommended courses can make good use of flowcharts, algorithms, and computer-programmed assignments.

Support for these recommendations can be found in the studies that have been and are being made of the role of the computer in higher education.

5.   <u>Implementation of Individual Courses</u>.

(a) Applications.

At all times in teaching these courses, every effort should be made to illustrate and motivate the material in them by showing how it is used in a large variety of applications. Furthermore, the basic orientation should be towards the solution of concrete problems; at all costs the deadly dry "definition, theorem, proof" approach should be avoided. Of course, there should be carefully stated definitions, theorems, and some proofs, especially in Mathematics C and L as well as in the postcalculus courses, but, especially in the earlier courses, PS, O, A, B, and FM, the problem-solving approach should predominate and new concepts should only be introduced after adequate motivation and examples.

One of the main aims of the teacher should be to get as active student participation in these courses as possible, by involving students in class discussions and a great deal of homework, which, of course, must be adequately criticized and discussed. One of the best ways to awaken and stimulate students' interest is constantly to demonstrate the many applications to a large variety of areas of

knowledge that their course material has. After all, a great deal
of the mathematics in this program originated in attempts at solving
very definite real-world problems.

Courses 0, A, and FM offer many opportunities to include appli-
cations to elementary probability and combinatorics. In the latter
course some mention of questions answered by linear programming tech-
niques can also be made. In the calculus courses the usual applica-
tions should, of course, be done but, at all times, the teacher must
be on the lookout for unusual applications as, for example, those
given in the last part of Calculus in the First Three Dimensions by
Sherman K. Stein (New York, McGraw-Hill Book Company, 1967). The
technical literature is full of applications of the calculus to sub-
jects such as biology and the social sciences and problems such as
traffic flow, pollution, etc. Every calculus teacher has an obliga-
tion to try to learn something about them. Mathematics L also has
many applications in these areas and the same general remarks apply.

(b) Rigor.

Statements (1) and (2), beginning on page 41 of Commentary
on A General Curriculum in Mathematics for Colleges, reflect our
views.

(c) Textbooks.

Obviously the textbooks for all the courses will have to be
chosen with a great deal of care. The teacher should make a contin-
uing effort to keep abreast of new books as they appear. Moreover,
while teaching any given course from a fixed book, the teacher should
be consulting several other books for additional problems, extra
illustrative material, and different slants on presentation. Clearly,
local conditions will often play a determining role in the choice of
books, but at all times the instructional staff should be aware of
other books existing for the same subjects and should make good use
of these additional sources.

The CUPM Basic Library List and Basic Library List for Two-
Year Colleges should prove particularly helpful in making choices
of texts and reference works for the courses described in this
report. Moreover, the book review sections of the American Mathe-
matical Monthly, Mathematical Gazette, Mathematics Magazine, Mathe-
matics Teacher, School Science and Mathematics, and SIAM Review
often have helpful comments about new books and should be consulted
regularly.