A COURSE IN BASIC MATHEMATICS FOR COLLEGES

A Report of

The Panel on Basic Mathematics

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I. INTRODUCTION

1. Background

There is a sizable student population in both two-year and four-year colleges taking courses with titles such as college arithmetic, elementary algebra, intermediate algebra, and introduction to college mathematics. In fact, according to a report issued by the Conference Board of the Mathematical Sciences, approximately 200,000 of the 1,068,000 students enrolled in mathematics courses in four-year institutions in the fall of 1965 were taking courses of this level. In 1966 approximately 150,000 out of 348,000 two-year college mathematics students were enrolled in such courses.* A large portion of these students do not take mathematics courses beyond this level or at best go on to courses of the college algebra and trigonometry type, after which their college-level mathematics training ends. These statements were supported by a survey of a representative sample of two-year colleges conducted by members of CUPM's Panel on Mathematics in Two-Year Colleges in the fall of 1969.

In January, 1970, CUPM formed a Panel on Basic Mathematics to consider the curricular problems involved in mathematics courses of this level. This Panel contained members of the Two-Year College Panel, which had already completed a great deal of preliminary work in this area, as well as additional representatives from two- and four-year colleges and universities. This Panel was charged with making curricular recommendations for the student population described above, whether enrolled in two- or four-year colleges. We will refer to the broad general area of courses below the level of college algebra and trigonometry as basic mathematics.

Many of the students in basic mathematics courses have seen this subject matter in elementary and high school without apparent success in learning it there. It is often the case that a second exposure to essentially the same material, similarly organized, is no more successful even though an attempt is sometimes made to present the subject matter in a more "modern" manner.

There are certainly many complex reasons for this state of affairs. Some of these may be psychological and sociological and may require the work of learning theorists and others trained in the

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* A more recent report of the Conference Board of the Mathematical Sciences (Report of the Survey Committee, Volume IV. Undergraduate Education in the Mathematical Sciences, 1970-71) indicates that approximately 193,000 of the 1,386,000 students enrolled in mathematics courses in four-year institutions in the fall of 1970 were taking courses at this level. In 1970 approximately 272,000 out of 584,000 two-year college mathematics students were enrolled in such courses.

social sciences in order to lessen their influence. However, it is our belief that the type of student described can also be greatly helped by reform in the mathematics curriculum. Evidently there are many ways of doing this; perhaps the most direct would be to allow these students to gain sufficient mastery of arithmetic in a mathematics laboratory by means of various learning aids and only then permit them to begin the elementary-intermediate-college algebra route. We believe that the curriculum described in what follows is another path that promises to be of greater interest and use to the present generation of college students. Of course there are many possible ways of attacking these problems, and we would not want our solution to be interpreted as the only possible one. Nonetheless, we believe that it deserves very serious consideration by the mathematical community and hope that many different kinds of institutions will find our suggestions, wholly or in part, of good use when dealing with the type of students described.

2. The Present Recommendations--Mathematics E

We propose the replacement of some of the currently existing basic mathematics courses by a single flexible one-year course, A Course in Basic Mathematics for Colleges (hereafter referred to as Mathematics E), together with an accompanying mathematics laboratory. <u>The main aim of this course will be to provide the students with enough mathematical literacy for adequate participation in the daily life of our present society</u>.

The adjunct laboratory, which we describe in detail in Section V, will serve to remedy the deficiencies in arithmetic that so many of these students possess. Moreover, the laboratory will offer opportunity for added drill in algebraic manipulation and for instruction in vocationally oriented topics of interest to particular groups of students. This learning should be tailored to each student's individual goals and needs but should not ordinarily require constant supervision by an instructor. It must also be carefully integrated with the material in the course proper.

As a subsidiary aim of the course we hope that the student will gain enough competence in algebraic manipulation, the translation of statements into algebraic formulae, and the careful use of language, to allow him to continue, if he wishes, with courses such as Mathematics A [page 216], intermediate or college algebra, or various vocational mathematics courses, including courses in computing. Thus the proposed course will include material on simple algebraic formulae, handling simple algebraic expressions, the distributive property (common factor), setting up and solving linear equations in one or two variables, the beginning of graphing, and the rudiments of plane geometry. However, we emphasize that the main thrust is to provide the basic literacy spoken of earlier; if there are pressures of time or lack of student interest, then topics should simply be omitted.

This course has been developed to meet a set of circumstances different from those which prevail in the secondary schools. Not only is there the increased maturity due to the higher age level of the students (many of whom may have been out of school for a number of years), but there is the significant fact that the students in question have elected to enter college. Thus, a new approach appropriate to these conditions is needed.

One device for meeting this need is the introduction of flowcharting and of algorithmic and computer-related ideas at the beginning of the course. These ideas permeate the course, encouraging the student to be precise in dealing with both arithmetic and nonarithmetic operations. However, the presentation of computer-related ideas in the course will <u>not</u> depend on the availability of actual machines. Topics of everyday concern, such as how bills are prepared by a computer, calculation of interest in installment buying, quick estimation, analyses of statistics appearing in the press, and various job-related algebraic and geometric problems, are mainstays of the syllabus.

We have tried to make the proposed course coherent; that is, after a topic is introduced, it should be used in other parts of the course and not left dangling. The students must be actively involved throughout the course and should be encouraged to formulate problems on their own, based on their experience. Full advantage should be taken of the playful impulses of the human mind; interesting tricks and seemingly magic ways of solving problems are to be exploited. This point is more fully discussed on page 278.

It should be quite obvious from the foregoing that spirit is more important than content in the proposed course. In order to make as clear as possible the exact intent of the Panel's recommendations, we have included below a detailed discussion of the relationship of the proposed course to certain other courses widely offered at this level, a topical outline of a preferred version of the course, and an extensive commentary on how these topics can achieve our objectives. In addition, we give in Section V a description of the laboratory we propose as a means of dealing with remedial problems and meeting individual goals. In Appendix I there is a brief compilation of problems representative of those that might embody the philosophy of our program; Appendix II contains exercises illustrating the use of flowcharts.

Because the needs of students in the various institutions across the country vary, the Panel believes that a number of different adaptations of the proposed course may be developed. Thus, our topical outline should not be construed as a rigid description of a single course but rather as a flexible model.

II. THE BASIC MATHEMATICS CURRICULUM

There is a large number of well-populated courses in that part of the mathematics curriculum which precedes college algebra and trigonometry. One large group of such courses consists of different versions or repackagings of subjects for special groups of students, especially those in various occupational programs. In addition, there are what might be termed the low-level liberal arts courses, perhaps designed to satisfy a mathematics requirement.

The Panel has gathered anecdotal evidence concerning the reasons students are enrolled in such courses:

1. Some students, despite poor preparation, are ambitious to proceed to more advanced courses in mathematics. We observe that only a small minority of such students do in fact proceed as far as a course in calculus. As described below and in greater detail in Section V, our course would make this possible for the minority of students who will actually go on, even though it may not be as efficient as the traditional route.

2. Many students are advised or required to take certain existing courses in mathematics in the hope that such courses will give the students in some peripheral fashion the kind of mathematical literacy that is the central purpose of our proposed course in basic mathematics.

3. Many mathematics requirements are made in the hope that the mathematics courses prescribed will contain quite specific techniques of value in a student's proposed area of specialization or vocation. Through use of the laboratory, students can be taught such specific material almost on an individual basis.

4. Frequently, there is an institution-wide requirement of one or more courses in mathematics intended primarily for cultural or general education purposes. The basic mathematics course we propose is to be broad and relevant to the actual concerns of students and, therefore, could perhaps serve as a genuine liberal arts course for students of this level of mathematical maturity better than do most of the courses currently taught for that purpose.

Therefore, we believe that the course we propose will meet the objectives listed above better than current courses in the basic mathematics curriculum. Our aim is not to add a new course to the profusion of courses already existing. We wish, rather, to replace several of them by a flexible, more relevant course which will come closer to meeting the genuine requirements of this kind of student. In order to clarify this point, we examine in somewhat greater detail the relationship between the proposed course and certain widely offered courses.

1. Arithmetic. We feel that the courses customarily offered in colleges under this title are subsumed by the course we propose.

2. Elementary Algebra. A student finishing our course should be able to acquire (considering both his work in the classroom and in the laboratory) the computational and technical facility expected of a student finishing a course in elementary algebra. In addition, he should be able to make use of this knowledge in various concrete circumstances. Therefore, it would appear that the introduction of the course we propose could quite properly result in the elimination of courses in arithmetic and elementary algebra.

3. General Education Courses. It is necessary to distinguish between two quite different levels of general education or liberal arts courses in mathematics. The higher-level course may frequently contain material of considerable complexity and depth and might be thought of as an alternative to calculus (or at least to college algebra and trigonometry) for students majoring in the humanities or social sciences. Such courses, because of the relatively high level, are not properly part of the basic mathematics curriculum as defined above and, therefore, lie outside the scope of this report.

The second or lower-level courses, although motivated by much the same philosophy, may be thought of as forming a part of a program of general education for a broader category of students not necessarily restricted to those majoring in the liberal arts. Such courses are sometimes used as a substitute for remedial courses. A general education course is usually intended to develop an interest in and appreciation of mathematics, beginning with the concept of mathematics as an art or as a discipline and working gradually outward to broader issues.

Mathematics E (although it has its remedial aspects) is not primarily a remedial course. From the standpoint of a general education program the proposed course is a broad one; it can be termed the mathematics of human affairs, and as such should be a reasonable alternative to the usual general education mathematics course. Moreover, the prospective students for Mathematics E are likely to be of a pragmatic turn of mind. For them an appreciation of mathematics seems likely to stem from seeing how mathematical ideas illuminate areas in which they have an established interest.

4. Courses for Students in Occupational Curricula. It must not be presumed that all courses designed specifically for students in occupational programs are at a low level. For example, students in physical science related curricula such as Engineering Technology will generally begin their college mathematical training at a higher level and often continue through the calculus. In addition, there are many occupational curricula which by their nature must contain mathematics courses taught in extremely close relationship to the major program. For these categories of students the proposed course, Mathematics E, will not serve. However, there remain large numbers of students in occupational curricula whose mathematical needs are less specialized.

These include students in some programs in business and health professions as well as students in other occupational programs not containing a strong element of scientific training. These programs very likely encompass a large majority of students in occupational curricula. We believe that the principal mathematical need for such students is basic mathematical literacy together with some work in the laboratory directed toward their special needs.

5. Mathematics for Prospective Elementary School Teachers. These students have very special and pressing needs to which our course does not address itself. However, Mathematics E might be needed by some as preparation for the teacher-training courses recommended by CUPM. [See Recommendations on Course Content for the Training of Teachers of Mathematics, page 158.]

6. Mathematics A. This course is described in A Transfer Curriculum in Mathematics for Two-Year Colleges, page 205. Mathematics A is, briefly stated, an improved and extended version of college algebra, trigonometry, and analytic geometry interwoven with certain remedial topics. The Panel believes that Mathematics A should prove as viable in some four-year institutions as in the two-year colleges for which it was originally designed. Mathematics A is the most natural continuation of Mathematics E for the minority of students who continue with further courses in mathematics.

7. Intermediate Algebra. We take intermediate algebra to mean a semester course containing a rather systematic and extensive review of topics normally encountered in some form in elementary algebra, followed by new material on such topics as exponents and radicals, functions and graphs, quadratic equations, systems of equations, and inequalities, with selected topics from among complex numbers, logarithms, permutations and combinations, and progressions. The emphasis is on technical algebraic proficiency with occasional digressions into the applications of specific techniques.

Many of the students for whom the Panel would prescribe the year course in Mathematics E now take a year sequence composed of elementary and intermediate algebra. Mathematics E contains much nonalgebraic material and touches certain broader issues which the corresponding algebra sequence does not cover; Mathematics E, however, cannot be expected to provide as great a degree of technical algebraic proficiency as the conventional algebra sequence.

We suggest that what is appropriate here is a serious and realistic appraisal by the mathematics department at a given institution as to which sequence is more directly related to the real reasons why students take or are required to take courses at this level.

Intermediate algebra is not a more advanced course than Mathematics E, but rather one with different goals. The Panel feels that Mathematics A rather than intermediate algebra is the more natural continuation of Mathematics E. Thus, a school offering Mathematics E and Mathematics A might have no further need for intermediate algebra courses unless it has a significant number of students who (a) have an adequate competence in elementary algebra and (b) need a mastery of specialized algebraic techniques as opposed to a more generalized mathematical competence and (c) are not prepared for Mathematics A.

Finally, the introduction of Mathematics E, depending upon local circumstances, will make possible a considerable economy and simplification in the basic mathematics curriculum, an economy which should have special appeal to small colleges or to colleges having only a small number of poorly prepared students.

III. OUTLINE OF MATHEMATICS E

In this section we present an outline of one sequence of topics which the Panel feels is appropriate for use in implementing the purposes of the course. It should be re-emphasized that coverage of these topics is in itself neither necessary nor sufficient for the course to fulfill the spirit of the Panel's recommendations. One purpose in the presentation of the outline in detail is to demonstrate the existence of at least one sequence of topics which have obvious relevance to the interests and needs of present-day students and yet can be mastered by the students for whom the course is designed.

Although we feel we are recommending something more than coverage of a list of topics, the Panel has given extensive consideration to the question of which topics would best serve our purposes. We feel that others who give equally serious consideration to these questions will arrive at a sequence of topics in substantial agreement with ours.

Although the details of how much time should be allotted to each specific topic can only be determined by actually teaching the course, the Panel has in mind that approximately 50 per cent of the year's work would be devoted to Parts 1 through 4, approximately 25 per cent to Parts 5, 6, and 7, and the remaining 25 per cent to Parts 8 and 9.

The page references in the Outline are to the Commentary which follows.

Part 1 Flowcharts and Elementary Operations

There are two reasons for beginning this course with an introduction to computing. First, there is the rather obvious matter of initial motivation of the students. Second, we wish to introduce early the idea of a flowchart, which will permeate the entire course. (See pages 271-277.)

1-a Brief introduction to the nature and structure of digital computers. Specimens of computer programs and computer output, but no real programming until Part 5. Flowcharting as a preliminary device for communicating with the computer.

1-b Flowcharts. Further illustration of flowcharts by non-mathematical examples including loops and branches. Sequencing everyday processes.

1-c Addition and multiplication of whole numbers. Addition and multiplication as binary operations. The commutativity and associativity properties illustrated by everyday examples. Multiplication as repeated addition, illustrated by examples. Drill in these operations. Flowcharts for these operations, notion of variable, equality and order symbols. Introduction of the number line as an aid in illustrating the above and to provide for the introduction of the coordinate plane.

1-d The distributive property and base 10 enumeration. Distributive law done very intuitively and informally by examples on 2- or 3-digit numbers in expanded form. (See page 276.) Illustrate these two topics by means of simple multiplication.

1-e Orders of magnitude and very simple approximations. Relate order of magnitude to powers of 10. Motivate approximations to sums and products by means of simple examples. Lower and upper bound for approximations, no percentage errors. Introduction of the symbol $\approx$. (See pages 281-283.)

1-f Subtraction of whole numbers. Three equivalent statements:

$$a + b = c; \quad a = c - b; \quad b = c - a$$

Commutativity and associativity fail for subtraction, operation not always possible. Multiplication distributes over subtraction. Approximations as in 1-e. Drill.

1-g Exact division of whole numbers. Three equivalent statements for a and b not zero:

$$ab = c; \quad b = c/a; \quad a = c/b$$

Division is not always possible, division is noncommutative and nonassociative. Flowcharting. Computational practice.

1-h <u>Division with remainder</u>. Informal discussion of division with remainder. Flowchart process as handled by a computer. Approximations as in 1-e.

1-i <u>English to mathematics</u>. Translations of English sentences taken from real-life situations into algebraic symbolism. (See pages 277-281.)

Part 2 Rational Numbers

It is intended that this part shall have an informal and pragmatic flavor. Even though some references are made to certain of the field axioms, we do not wish to approach the number system from a structural point of view.

2-a <u>Extending the number line to the negatives</u>. Absolute value and distance.

2-b <u>Rational operations on the integers</u>. To be derived from plausibility arguments as novel as possible but not from the field axioms. Drill in these operations.

2-c <u>Fractions with the four rational operations</u>. Special case of the denominator 100 as percentage. Simple ratio and proportion. Drill in manipulations with fractions.

2-d <u>Decimals</u>. Use base 10 notation with negative exponents. Relation between fractions and decimals via division. Many practical applications and practice.

2-e <u>Roundoff and truncation errors</u>. (Most computers truncate rather than round off.) Significant digits and scientific notation.

2-f <u>More on English to mathematics</u>. Use the new ideas developed in this Part. More flowcharting with examples drawn from interest computations and financial problems, including the use of the computer. (See page 280.)

Part 3 Geometry I (See also Part 7)

The purpose of this material is to refresh the student's acquaintance with basic geometric vocabulary and then to present that minimal geometric background sufficient for the introduction of coordinate systems. (See pages 285-287.)

3-a <u>Introduction to geometric ideas</u>. Informal discussion of points, planes, segments, lines, angles, parallel and perpendicular lines.

3-b <u>Geometric figures</u>. Circles, triangles, special quadrilaterals, notion of congruence.

3-c <u>Use of basic instruments</u>. Ruler, protractor, compasses, T-square. Error in measurements.

3-d <u>Conversion of units</u>.

3-e <u>General introduction to linearity and proportion</u>. Many examples. Notion of similarity.

3-f <u>The coordinate plane</u>. Points and ordered pairs, road maps, etc.

3-g <u>The graph of $y = mx$</u>. Slope.

Part 4 Linear Polynomials and Equations

In this part there is to be a treatment of algebraic ideas at a level sufficient for the applications that follow, but which stops somewhat short of the technical algebraic competence usually sought in conventional courses in algebra. Students who wish a higher degree of technical competence may obtain it through appropriate work in the laboratory. (See Section V.)

4-a <u>English to mathematics</u>. A few word problems leading to one linear equation in one unknown as motivation for algebraic manipulation. Solve some equations by trial and error. Devise flowcharts for trial-and-error solutions.

4-b <u>Transformations of one equation in one variable</u>. Both identities such as $2x + 3x = 5x$ and $3(x + 2) = 3x + 6$ as well as transformations such as: if $4x + 5 = 11$, then $4x + 3 = 9$.

4-c <u>Flowchart for solving $ax + b = c$</u>. Include a variety of other forms.

4-d <u>Applications</u>. Word problems drawn from many different areas.

4-e <u>Situations leading to one equation in two variables</u>. (Motivation for next section.)

4-f <u>Transformations of one equation in two variables</u>. Leading, for example, to the form $y = mx + b$, being careful not to restrict the names of the variables to x and y.

4-g <u>Graphs of linear equations in two variables</u>. Slope of $y = mx + b$. Relation of $y = mx + b$ to $y = mx$.

4-h Solutions of two linear equations in two variables. Graphical and analytical methods, applications.

Part 5 The Computer

The amount of time devoted to this Part will turn out to be quite short or quite long depending on whether actual use is to be made of a computer. (See pages 16-17.)

5-a General discussion of the computer. Ability of a computer to respond to well-defined instructions. Illustrate with simple programs. Brief discussion of error due to truncation. Memory, operations, speed, with reference to the available equipment.

5-b Uses of the computer in modern society. Many different applications, with limitations of the computer stressed.

5-c Elementary instruction in programming. Language appropriate to the institution; writing programs from flowcharts.

5-d Varied applications. Drawing from material already presented, including more sophisticated financial problems. Run programs on computers when available.

Part 6 Nonlinear Relationships

We have included this material primarily to display the power of a mathematical model and to provide for development of the themes of flowcharting, approximation, and graphing which have been introduced earlier.

6-a Some examples of nonlinear relationships. Repeated doubling, and exponential growth of populations. Compound interest.

6-b The graph of $y = x^2$. Concept of square root and graphical evaluation of square root. Use of tables and approximation of square roots by averaging.

6-c Pythagorean theorem and distance formula. Very brief discussion of irrational numbers and the fact that lines and curves have no gaps.

6-d The graph of $y = ax^2$. Applications.

6-e The graph of $y = ax^2 + bx = x(ax + b)$. Roots and intercepts, maximum and minimum, applications.

6-f <u>Graphing of $y = ax^2 + bx + c$</u>. Use vertical translation from $y = ax^2 + bx$. Note that there may be 0, 1, or 2 roots of the corresponding quadratic equation.

6-g <u>Approximation of roots</u>. Use of the computer. (See page 276.)

6-h <u>Inverse, joint, and combined variation</u>. Applications.

6-i <u>Suitable bounds for accuracy and estimates</u>. Products and quotients, relative and percentage error, graphical illustrations. (See pages 281-284.)

Part 7 Geometry II

Our hope here is that the geometrical material presented will be made relevant and that there will be suitable links with the mathematical ideas introduced earlier. (See pages 285-287.)

7-a <u>Areas and perimeters of plane figures</u>. Rectangles, triangles, parallelograms, and circles. No extensive involvement with theorems and proofs. Perhaps compute area of irregular regions by use of rectangles and Monte Carlo methods.

7-b <u>Surface areas and volumes</u>. Use of formulas for areas and volumes of spheres, cylinders, parallelepipeds.

7-c <u>Applications</u>. Consumer problems, pollution problems, conversion of units.

7-d <u>Elementary constructions</u>. Use of straightedge and compasses. Include special triangles like isosceles right triangles, 30-60 right triangles, etc.

7-e <u>Further extension of work on similar figures</u>.

Part 8 Statistics

Besides the obvious interest and relevance of this material, it offers opportunities for use of virtually all of the ideas previously introduced in the course. We have in mind the use of statistical ideas in making practical decisions among realistic alternatives. (See pages 287-288.)

8-a <u>The role of statistics in society</u>. Problems of interpretation of charts, graphs, percentages.

8-b Descriptive statistics. Various kinds of graphs; mean, median, and mode; range and standard deviation; quartiles and percentiles.

8-c The normal distribution. Informal discussion.

8-d Statistics and the consumer. Informal discussion of bias; choosing samples. Flowcharts and computing should be used whenever appropriate.

Part 9 Probability

This Part represents a rather minimal introduction to the subject, avoiding any heavy involvement with combinatorics, but including one or more applications of complexity sufficient so that the methods of earlier chapters can be displayed to good advantage. (See pages 289-293.)

9-a Empirical probability. Mortality tables, long-run relative frequencies.

9-b A priori probability. Tossing coins, rolling dice, selecting discs from box. Experiments in which relative frequencies are compared with theoretical probabilities.

9-c Elementary counting principles. Emphasis on devising a procedure for listing of outcomes of an experiment, the procedure suggesting a principle or formula for obtaining the count. (See pages 291-292.)

9-d Further a priori probability. Independent trials of an experiment. Examples selected from everyday experiences such as athletics.

9-e Informal decision theory with examples. (See Exercise 15 in Appendix I.)

IV. COMMENTARY ON THE OUTLINE

An outline for a course is a good device for specifying the order of topics but a poor device for emphasizing the ideas which are to receive continuous attention. This section is devoted to making clear the intentions of the Panel concerning certain special topics and certain recurrent themes. Other topics of equal importance have been omitted from this discussion because the kind of treatment they should receive is relatively clear from the outline.

1. Computers and Computing

Mathematics E is to begin with a very brief description of how a digital computer receives its instructions and the operations it is capable of performing, preferably made concrete by exposure to actual computing machinery. The course returns to the subject in somewhat greater depth in Part 5 of the outline. At other places in the course opportunities to discuss and illuminate special topics from a computer point of view are to be exploited as these opportunities arise. The concept of a flowchart as a device for analysis is to be a central theme.

The purpose of introducing and utilizing computer-related ideas is basically the psychological one of getting the students involved with and thinking about mathematics. In a course of the kind we describe, this point is likely to be of crucial importance. The Panel has more than adequate anecdotal evidence that the injection of computing will provide very strong motivation for the student. A secondary reason is the ubiquity of the computer in our present society and the need for the students to understand its potentialities and limitations in order to function as citizens and employees.

The Panel feels very strongly that the basic validity of the course we propose would not be compromised by the absence of computing equipment at the institution where the course is being taught, nor would the lack of computer access lead us to prescribe a significant reduction in the emphasis on computer-related ideas such as flowcharting.

However, the Panel feels equally strongly that even a small amount of direct experience with computers will greatly enhance the motivational powers of these ideas and will result in a more effective course.

At the very least, the following should be done:

a. The students should have one or more guided field trips to a computer center to see how a problem is actually handled by a computer, e.g., they should understand what a computer program is and how it is related to a computer.

b. Students should be involved in team efforts to write quite simple programs.

c. The instructor should demonstrate how these programs are read by the computer and how the computer imparts its results.

It is clear that such a minimal degree of exposure would require only very limited access to computing equipment and would not imply actual possession of a computer.

In Part 5 we suggest further experience in programming. In various places that follow we mention some of the opportunities which exist for making some topics both more realistic and more challenging by actual computer usage.

The question of how extensively such opportunities can be utilized is clearly dependent on costs, even at colleges with extensive computer installations. The Panel feels that computer usage should certainly be carried beyond the minimum outlined above if at all possible. However, neither our experience nor the experience of others seems adequate at present to make any accurate estimates of the educational benefits to be expected as a function of costs.

2. Flowcharts

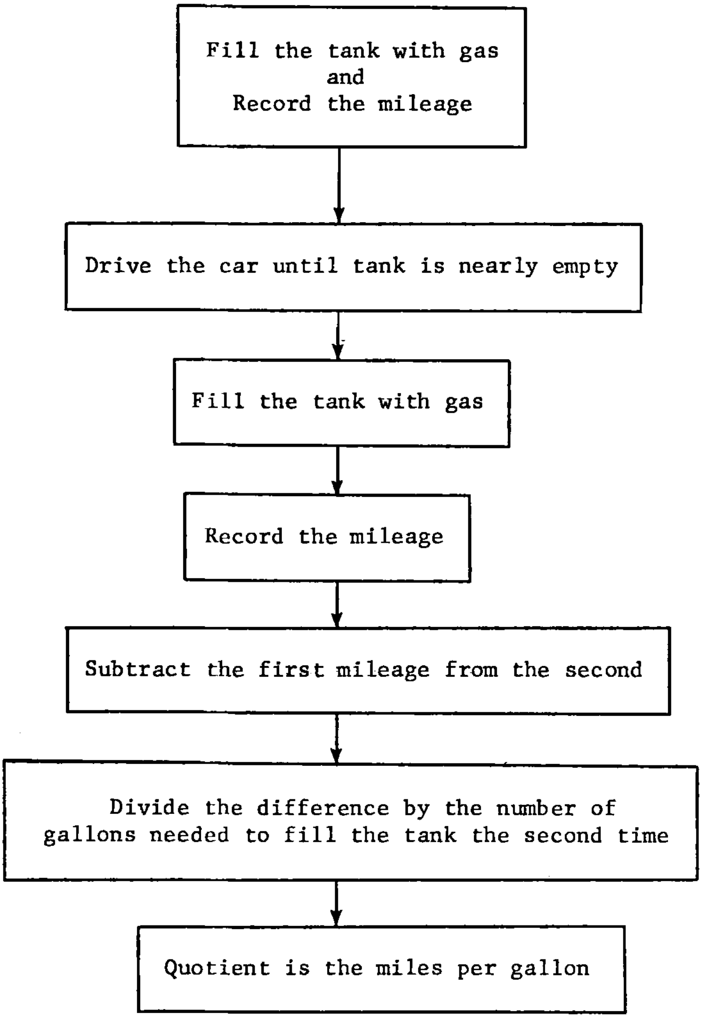
We have chosen to discuss flowcharting in the first part of the course both in order to arouse immediate student interest and because this technique is used extensively throughout the course. It must be made clear to the student that the computer operates only through very specific instructions presented to it in a special language. The idea of flowcharting can be first presented as an essential step in the process leading to the production of such detailed instructions.

The utilization of the construction of flowcharts as a technique in the analysis of problems recurs throughout the course being described. A flowchart is an extremely valuable way of describing a procedure to be followed either by another person or a computer. Its object is to break a problem up into easily managed steps whose interrelationships are clear.

It is not necessary to have a detailed familiarity with the actual capabilities of the computer to begin describing algorithmic processes with flowcharts. Perhaps the best examples to use in introducing flowcharting ideas are those of a nonmathematical nature such as the ones which follow. These first four examples illustrate the four kinds of elementary structure which flowcharts can have.

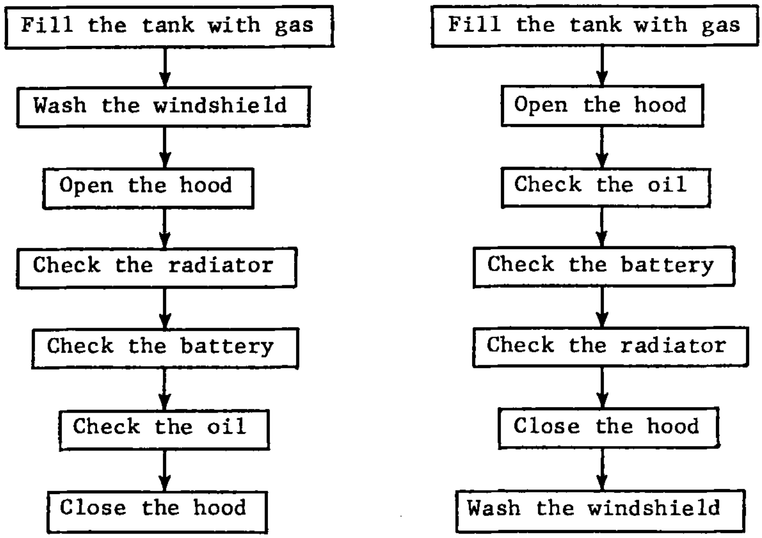
A. Sequential operations with no choice.

To determine the gas mileage for your car.



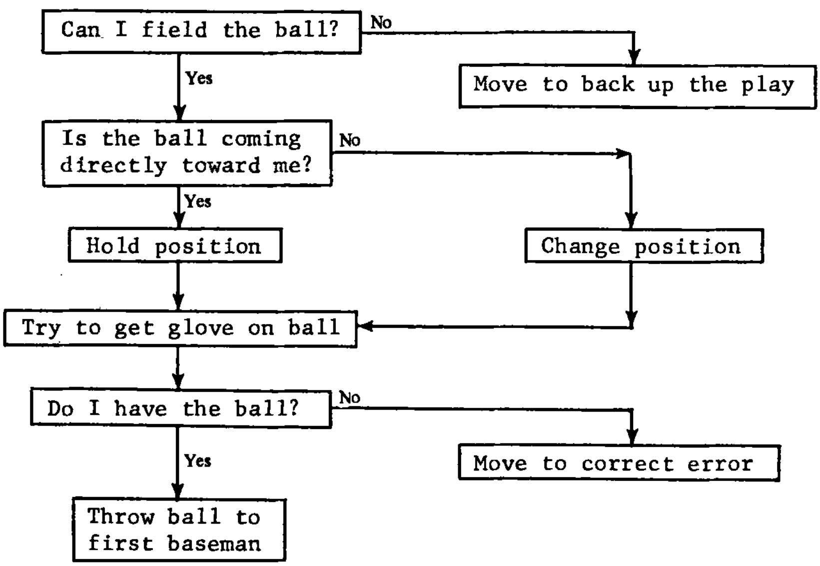
B. Sequential operations with a choice of order.

To service a car at a filling station.



C. Branches.

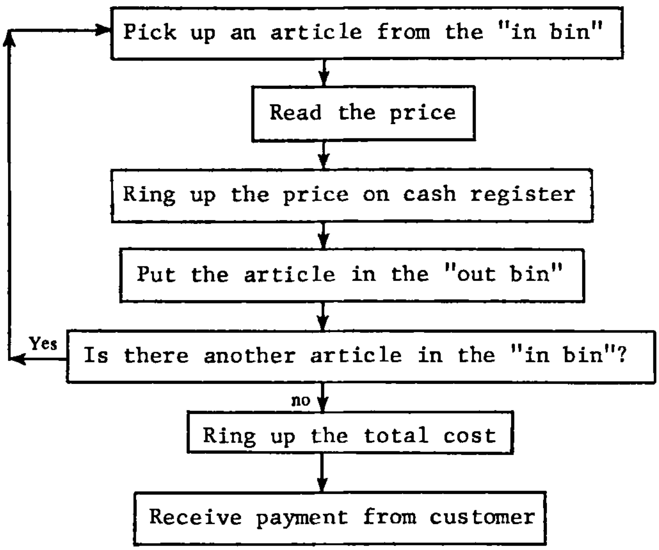
Reaction of a baseball shortstop to a groundball hit toward his position with no runner on base.



D. Loops

An algorithm may call for a process to be repeated any number of times to achieve a desired result. Thus, instead of repeating the same instructions for a set of objects, one may simply call for the same instructions to be applied to the set of objects until the set is exhausted. Representing such an algorithm by a flowchart gives rise to the so-called loop concept.

Action of cashier at a grocery store.



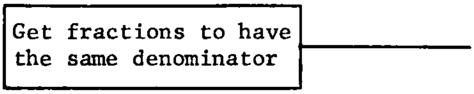
These flowcharts are all incomplete in the sense that the operations described within a given box are susceptible to further analysis into smaller steps. If a flowchart is to result in a computer program, the capabilities of the computer will determine completeness.

The student should be encouraged to construct flowcharts based on his own experience.

We propose the study of flowcharting not only as an end in itself, but also for its usefulness throughout the course to explicate concrete details in situations which may not be purely computational--for example, the standard situation faced by a student unable to solve a problem in which he does not know where to begin. Thus the source of a student's difficulty in adding two fractions may be his inability to break the problem down into simpler parts. Such a student would probably be able to add fractions with the same denominator. Let him then write

Add fractions having the same denominator

It is then apparent that if this could be preceded by

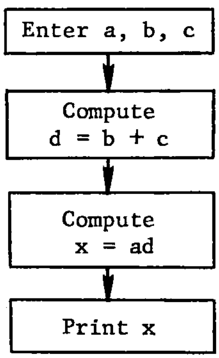


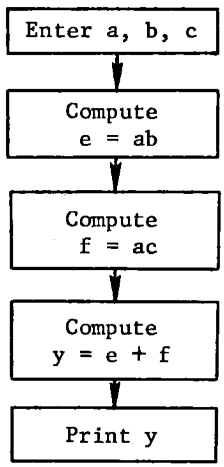
an important first step toward the solution of his problem would have been taken. A further analysis of the contents of this box via flowcharting should result in the solution.

Flowcharts can illuminate mathematical assertions that fail to have the desired impact when presented in the usual manner. For example, the two sequences of operations indicated in the equation for the distributive law can be dramatized by the exhibition of the two flowcharts that describe the sequences:

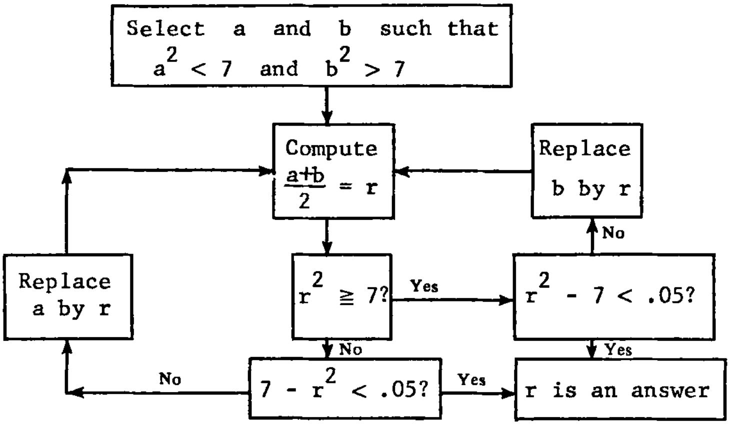
Compute

$x = a(b + c)$ $\qquad\qquad$ $y = ab + ac$





Once the student has been introduced to the graph of the quadratic equation, he should have no difficulty in understanding, for example, that if the graph of $y = 7 - x^2$ is above the x-axis at $x = a$ and below the x-axis at $x = b$, then the square root of 7 lies between a and b. This idea is exploited in the following flowchart of a simple algorithm for approximating the square root of 7. We have chosen this algorithm for its simplicity in spite of the fact that it is not the most efficient one for hand computations. The relative speed with which a computer would perform the work would be impressive to the students.



This flowchart illustrates graphically the idea of an iterative process and suggests the economy of presentation which loops in such charts provide. The example itself affords drill for the student who performs the successive approximations. Of course, access to a computer so that a program based on this chart could actually be run would provide added motivation for the student.

3. The Role of Applications and Model Construction

Since the development of mathematical literacy relevant to participation in human affairs is a major objective of the course, a strong relationship between the topics and their applicability must be established. The purpose of this section is to offer suggestions on the kind of applications that should be emphasized and the points at which they should be included.

Although it would be expected that separate sections devoted to "word problems" might be a part of the course, the mere inclusion of such sections is not sufficient to accomplish the desired goals. To aid in establishing the relevance of the topics, examples of practical problems should be used to motivate the topics. Additional examples need to be interspersed with sufficient frequency to insure that interest is maintained.

In order to help in overcoming one of the major difficulties associated with practical problems, separate sections entitled "English to Mathematics" are listed in the suggested outline. The purpose of those sections is to assure that attention is actually given to the process of translating English statements into mathematical equations and other mathematical models.

Care must be taken in the selection of examples to insure that the majority of the problems are actually practical. Too often in existing courses and texts, problems are "cooked up" to lead to a particular kind of equation. Consequently, some of the motivational value of the examples is lost because of their lack of relation to reality. Problems with sentences such as "John is three times as old as Mary was five years ago," or "Jack has twice as many quarters as dimes and seven more nickels than dimes," may lead to nice equations, but they are uninteresting simply because the situations are never encountered.

It is not being suggested that examples taking advantage of the playfulness of the human mind should be ignored. However, problems in which variables must represent quantities that are likely to be known in the situation being described should occur infrequently and only when a note of whimsy is desired.

Perhaps the most important problem that the instructor will face in teaching the typical Mathematics E student is to arouse his interest and to induce him to participate with some degree of enthusiasm in the course. It is hoped that the computer-related aspects of the course will help to bring this about. However, an appeal to the playful and puzzle-solving impulses of the student should also be used whenever possible. Most of the student audience under consideration is willing to think quite hard, and, indeed, in a mathematical fashion, when it comes to certain pursuits that it regards as pleasurable--for example, when it comes to playing card games. Moreover, a tricky puzzle will capture the attention of many of those students while a straight mathematical question will rarely do so. It is therefore very likely that if these elements are used whenever appropriate in Mathematics E they will jolt the students into awareness. For example, the old puzzle about the bellboy and the five dollars or the three warring species crossing the river two at a time[1] can be used to illustrate logical analysis and step-by-step consideration of problems. The elements of these puzzles should be brought out and used throughout the course in order to point out the intellectual similarity between dealing with them and with solving ordinary mathematical problems. Of course, in Part 9 many references should be given to the games that our students actually play, but even the seemingly more arid parts of the course can surely benefit from this kind of material. The Thirteen Colleges Curriculum Program[2] has made very successful use of this technique and has found it a very useful and efficient way to lead students into a study of mathematics that the students rejected originally. This material contains stimulating problems that have been used successfully by participants in the Thirteen Colleges Curriculum Program to engage

1. Many such problems can be found in <u>Mathematical Recreations</u> by Maurice Kraitchik (New York, Dover Publications, Inc., 1942).

2. Thirteen Colleges Curriculum Program Annual Report. <u>Analytical and Quantitative Thinking (Mathematics)</u>. Florida A and M University, Tallahassee, Florida, 1969.

the attention and reasoning power of students taking courses at approximately the level of Mathematics E.

The following points should be considered in the selection and presentation of examples and exercises: (1) The majority of problems should describe familiar situations in which variables represent quantities that could reasonably be expected to be unknown. (2) To develop the ability to translate English into mathematics, it may be necessary to begin with problems that students are able to solve without the aid of an equation or other model. However, at several points in the course, students will have the background necessary to construct models for problems that they will not be able to solve completely. If full advantage is taken of this situation, then applications can be used to motivate the need for being able to solve equations and otherwise manipulate models to produce solutions. (3) Students should frequently be asked to estimate an answer to an exercise prior to writing the equation or formulating a model, and to explain the process by which the estimation was made. Such explanations are often surprisingly close to being a correct verbal equation which can be more easily described by symbols than could the original problem. (4) Students should be encouraged to describe problems which they have actually encountered and which they would like to be able to solve. The experiences of students with charge accounts, savings accounts, tax problems, other college courses, vocational experiences, and situations that occur in playing cards and other games can be excellent sources of problems at various times within the course.

As an illustration, a sequence of problems that might be used near the beginning of the course to develop the ability to translate English into linear equations is given below. The problems begin with one that could be solved without a formal statement of the equation and progress to the point that the majority of students would need an equation to complete the solution.

1. A student has grades of 65 and 76 on two exams. In order to maintain the average that he desires, he must have accumulated 210 points after the third exam. Write an equation that can be used to find the grade which he must make on the next exam.

2. Suppose that the student in the previous problem wishes his average for the three exams to be 72. Write an equation that can be used to find the grade which he must make on the next exam.

3. Suppose that the third exam in problem 2 is a final exam and will count as two regular exams in computing his average. Write an equation that can be used to find the grade which he must make on the final exam in order that his average will be 72.

4. Suppose that in problem 3 the student may choose to use his textbook on the final exam, but if he so chooses, his final exam score is lowered by 10 points. Write an equation that can be used to find the grade which he must score on the final exam (before deduction) in order that his average will be 72.

Note that the equation for the first problem may be given by a sentence as simple as $P = 210 - 65 - 76$, which the student could certainly solve. In fact he could probably solve this problem without the aid of an equation. An equation for problem 4 is given by

$$\frac{65 + 76 + (X - 10) + (X - 10)}{4} = 72,$$

which the student probably could not solve at this point.

There is a wealth of possibilities for motivating and illustrating various topics throughout the course with consumer problems. Such problems seem to be especially adaptable to the notion of flowcharting. Two examples of such problems are given below. By adding such complicating factors as additional purchases, variable payments, and minimum service charges, one can expand the problems to the point where a flowchart is almost essential for describing the payment process. (See Appendix II.)

1. Suppose that a customer makes purchases on credit totaling \$560.00. Interest and all other charges on the debt make the total balance \$610.00. The customer must pay \$25.00 per month until he owes less than \$25.00. He will pay the remainder as a final payment. Construct a flowchart that contains boxes giving the amount owed and the number of payments remaining after n payments.

2. Suppose that a customer makes purchases on credit totaling \$560.00. On the last day of each month $1\frac{1}{2}\%$ of the unpaid balance is added to his account. The customer plans to pay \$25.00 each month until he owes an amount less than or equal to \$25.00. He will pay the remainder as the final payment. Payments are paid on the first of the month. Describe the payment process with a flowchart that contains boxes giving the amount paid and the amount owed after n payments.

The problems should include some that are more open-ended than is usually the case with textbook problems. Some problems should request a reasoned choice between alternative courses of action, and some should ask what additional information is needed in order for an ill-posed problem to become solvable.

It should be stressed that the Panel intends that "word" problems should be emphasized much more in this course than is usually the case. We do not have in mind teaching artificially neat procedures for solving quite special classes of problems (as, for example, in the rote methods frequently used to drill students in the solution of time-rate-distance problems). We believe that students in a course of this level can learn to create and analyze simple mathematical models. Many students are inhibited by their attitudes toward mathematics and beset by a fear of failure. If the problems are realistic enough to seem significant to the students and initially are simple enough to insure a ready solution by most students, we believe that the students can be helped to develop a regular use of mathematical ways of thinking in the analysis of practical situations.

Appendix I contains a list of problems and examples that illustrate the kind of applications that should be a part of the course. The list is not meant to convey any sort of desirable achievement level for the course. Neither is it meant to be comprehensive with respect to the areas from which applications should be selected. It does represent the spirit of the course relative to meaningful applications.

4. Estimation and Approximation

In a wide variety of practical problems an approximate solution will serve as well as the result of an exact calculation. In other situations (such as that of a shopper in a supermarket) estimation is the only practical course of action available. Even if an answer has been calculated exactly, it is useful as a check to obtain a quick approximate solution.

We hope that a graduate of this course will be an habitual estimator. The topic of estimation, like that of flowcharting, should appear throughout the course and should be presented in as many places as possible, taking advantage of opportunities as they arise. The course should make the student moderately proficient in estimating products, reciprocals and quotients, as well as powers and roots, of 2- and 3-digit numbers. Although the underlying theory is elementary, the approach must be through trial and error leading to the formulation of certain principles which in turn lead to the basic question of tolerance and control.

Because the audience we have in mind is not generally well prepared in arithmetic, one should start slowly, perhaps with relative errors discussed as percentages. These should be taken from everyday life: population figures, sports, betting, etc. Then one needs to present a crude technique of converting relative errors expressed as percentages. Thus, we discuss "48 parts in 99" and point out that it is roughly "50 parts in 100." Also, a relative error of 16/53 is approximately a 30% relative error, since 16/53 is approximately 15/50.

The product 13×27 can be estimated in several useful ways. For example,

(a) We could make each number smaller; replace 13 by 10 and 27 by 25 and obtain

$10 \times 25 = 250$, which we regard as roughly correct. We write

$13 \times 27 \approx 250$,

where the symbol $\approx$ means "is approximately equal to."

(b) We could make only one of the numbers smaller, using 10 and 27 for example, and write

$13 \times 27 \approx 10 \times 27 = 270$.

Clearly (a) and (b) are "underestimates," i.e.,

$13 \times 27 > 250$ and

$13 \times 27 > 270$.

(c) We could make each number larger; replace 13 by 15 and 27 by 30 and obtain

$15 \times 30 = 450$, writing

$13 \times 27 \approx 450$.

(d) We could make only one number larger, using 13 and 30, for example, and write

$13 \times 27 \approx 13 \times 30 = 390$.

Of course (c) and (d) are "overestimates." Thus, we can write

$270 < 13 \times 27 < 390$,

which the student must learn to read as two statements: 270 is less than 13×27, and 13×27 is less than 390. At this stage, we might suggest that an average be used. Thus, the average of 270 and 390 is 330; hence,

$13 \times 27 \approx 330$.

(e) In estimating the product we can make one factor larger and one smaller; this frequently leads to a better estimate. Thus, we could replace 13 by 10 and 27 by 30 and write

$13 \times 27 \approx 10 \times 30 = 300$.

The student will see that this estimate is perhaps better, but that we have lost control in the sense that we no longer know whether the estimate is larger or smaller than the correct answer.

Some students might wish to pursue these ideas further in the laboratory. They might be encouraged to consider questions such as: if a, b, and x are positive, x is small compared to a and b, and $a < b$, then is it the case that

$$\frac{a - x}{b - x} < \frac{a}{b} < \frac{a + x}{b + x} \ ?$$

Such a formulation might be conjectured empirically, and then an algebraic analysis attempted.

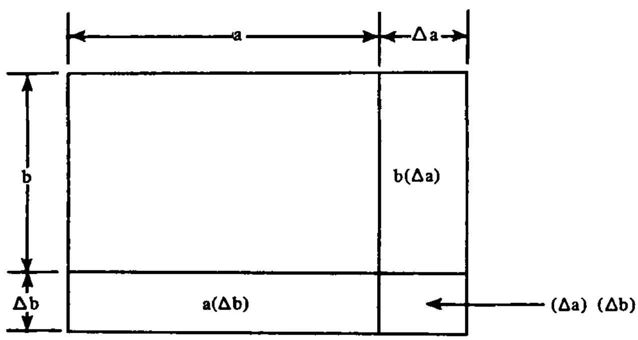
Somewhere in the course it might be well to introduce the student to the use of the slide rule, although the theory should be postponed until the student himself wants it and is ready for it. At the beginning it is just an instrument which serves to estimate products and quotients. Scientific notation would be introduced in discussing the position of the decimal point.

Before any formula about estimation is presented, the students should have been exposed to problems in which a quick rough estimate is useful.

<u>Example</u>: Steve wishes to mail 28 books to his new residence. The prices of the books range from \$6.95 to \$12.50. For how much should he insure the books?

The theory of approximation must, of course, be presented in stages, following the algebraic readiness of the student. Once the student has the necessary algebraic readiness, he might be introduced to the concept of relative error. Thus, if Δx denotes the error in x, then by the relative error $R(x)$ we mean $\Delta x/x$.

After doing a sufficient number of problems involving this concept, a student might be enticed to consider $R(ab)$. This can be motivated by considering the following diagram:



Thus, $\Delta(ab) = b \cdot \Delta a + a \cdot \Delta b + \Delta a \cdot \Delta b$.

Now, dividing by $a \cdot b$, we obtain

$R(ab) = R(a) + R(b) + R(a) \cdot R(b)$.

From this picture, a student can perceive that if $R(a)$ and $R(b)$ are "small enough" (depending on the situation), then an estimate for $R(ab)$ is

$$R(ab) \approx R(a) + R(b).$$

The student should, of course, have had the opportunity to conjecture this by considering several examples.

Having arrived at this formula, the student might be ready to understand, for example, that if $|R(a)| < 30\%$ and $|R(b)| < 20\%$, then the estimate for $R(ab)$, namely $R(a) + R(b)$, is correct to within 6%.

At the point where the student has learned to graph, we can use the graph of $y = x^2$ to get estimates of squares and of square roots (see also flowcharting). It should be interesting to the student that a smooth graph is a good "estimator." Of course, from the relation

$$R(ab) \approx R(a) + R(b),$$

we obtain

$$R(a^2) \approx 2R(a)$$

or

$$R(\sqrt{a}) \approx R(a)/2,$$

from which we conclude that the relative error in the square root is about half the relative error in the number.

Following this, the interested student could be shown (this should probably be done in the laboratory) that if $R(b)$ is "small enough," then

$$R\left(\frac{1}{b}\right) \approx -R(b),$$

and that if $R(a)$ is also "small enough," then

$$R\left(\frac{a}{b}\right) \approx R(a) - R(b).$$

At this stage, the student will have accepted the usefulness of these relations and will enjoy working examples and seeing how well he can quickly estimate products, quotients, powers, and roots.

5. Comments on Geometry

Geometric ideas should be presented in two stages. At the beginning (Part 3 in the outline) only basic geometric vocabulary and those ideas necessary for introducing the coordinate plane are presented. Once the student has gained some mathematical experience (Part 7 in the outline), he can be exposed to slightly more complicated geometric notions.

As early as in Part 3 it is fairly safe to assume that most of the students of whom we speak will have some notions of the basic geometric concepts of points, planes, segments, lines, angles, parallel lines, and perpendicular lines. It is hoped that the course can reinforce some of the correct notions the student already has, while at the same time pointing out a number of concrete uses of geometry in his daily life. Abstract definitions and an axiomatic approach <u>must be avoided</u>. It is appropriate to give the student experience with specific geometrical figures by performing constructions in the laboratory with the use of the ruler, compass, protractor, and draftsman's triangle. With even rudimentary ideas of similarity, one could discuss problems of indirect measurement.

After the student has been introduced to the idea of a plane, the coordinate plane can be regarded as analogous to the familiar devices employed in designating a specific section of a road map (by letter and number) or some particular locale in an ocean (by latitude and longitude).

The coordinate plane can then be introduced in the usual way, i.e., by reproducing on each axis the number line discussed in Parts 1 and 2, choosing the intersection of the two axes as the origin, and by making use of the concepts of perpendicular and parallel lines discussed earlier.

The graph of $y = mx$ can now be introduced, perhaps with an example such as the following:

<u>Example</u>: The owner of an ornate gift shop in a high-rent district has determined that he can make a reasonable profit and cover all overhead expenses (stock costs, rent, shipping costs, employees' salaries, taxes, utilities, office supplies, advertising, insurance, breakage, etc.) by making the selling price of each item equal to twice its cost. Let x be the cost of an item and y its selling price; then $y = 2x$. A table of corresponding values of x and y should be constructed and these ordered pairs plotted, with appropriate emphasis on the fact that the points are arrayed in a straight line.

The concept of steepness or slope might first be discussed in an already familiar context, such as the "pitch" of a roof or the "grade" of a road, with emphasis on the fact that the pitch, grade, or slope is determined by "rise over horizontal run." The slope of a line can be introduced by graphing functions such as $y = 4x$,

$y = 2x$, and $y = \frac{1}{2}x$ in the same plane and inviting students to compare the three lines. Such comparisons should make it difficult for them to avoid the notion that the constants 4, 2, and $\frac{1}{2}$ have something to do with the relative steepness of the lines.

Taking specific points on each of the lines $y = 4x$, $y = 2x$, and $y = \frac{1}{2}x$, a series of right triangles should be constructed by "walking away" from each point a varying horizontal distance and then "walking up" the required vertical distance to the line. Hopefully, the student will have done enough laboratory work on similarity (in Parts 3-5) to understand that all right triangles with hypotenuse on the same line and one leg horizontal are similar. Thus, successive computations of corresponding ratios will provide a simple illustration of the fact that the ratio of rise to horizontal run remains constant for the same line, and that the slope of each line is equal to the coefficient of x in $y = 4x$, $y = 2x$, and $y = \frac{1}{2}x$. Similar demonstrations with different examples can be used to show that the slope is positive if the line slants up to the right, is negative if the line slants down to the right, is nonexistent if the line is vertical, and is equal to the constant m in the equation $y = mx$.

In Part 7 the student should be introduced to the basic formulas for areas and perimeters of rectangles, triangles, parallelograms, and circles. He should also learn the formulas for surface areas and volumes of parallelepipeds, cylinders, and spheres. This, of course, provides an opportunity to do more work on approximations. The student will find this work more interesting if it is presented through useful examples such as the following:

Example 1: A man wants to cover his front yard with top soil before he plants his lawn. The yard is 100 ft. by 30 ft. and he wants to have the top soil 6 inches deep. How many cubic yards of top soil should he order?

Example 2: An oil discharge of fixed volume may spread over a very large area as the thickness of the slick decreases. Find an algebraic expression for the area covered by a given volume of oil in terms of the thickness of the slick.

Example 3: A certain noxious substance is discharged into a river as a by-product of a factory. Suppose the rate of flow of the stream in winter is at least 1,500,000 gallons per day and the rate of discharge of the pollutant is variable but may be as great as 2,000 gallons per day. If a concentration of 150 parts per million of the pollutant is the maximum that is considered "safe," what conclusion can be drawn as to the safety of the stream near the site of the factory? If the stream is safe at the factory site, how much additional pollutant can be discharged without exceeding a safe concentration? If the stream is unsafe at the factory, consider the situation downstream where additional inflow from tributaries may be thought of as diluting the concentration. How much inflow from tributaries will be needed before the augmented stream will be "safe"?

How does the situation change when the spring rains double the flow of the stream?

Some work on conversion of units, making use of mensuration formulas, should also be given--for example, the following:

Exercise: The base of a rectangular container of water is 2 ft. by 5 ft. Suppose that a metal ball 13 inches in diameter is dropped into the water and is completely submerged. How much will the level of the water rise?

The usefulness of similarity can be exemplified by pointing out that when a house is built, the contractors use plans to help them. These plans may be drawings or blueprints. In these the length of a line segment is much smaller than the actual length it represents. The ratio of the length of a line segment to the actual length it represents is called a scale. Maps are other scale drawings that students are familiar with. Many examples can be given of proportion using the idea of such scale drawings.

6. Remarks on the Material on Probability and Statistics

These two Parts can be considered as an opportunity to synthesize and apply many of the ideas of the preceding Parts. In addition, a case can be made for these topics as being among the most important mathematical concepts a citizen can acquire. He needs some knowledge of statistics in order to understand the news he hears on radio or television. Many of the problems facing the various governmental agencies in regard to taxes, welfare, education, and other public matters are comprehensible only to one who has some knowledge of statistics. Without an understanding of these problems, it is difficult, if not impossible, for a citizen to vote intelligently. The advertising of consumer goods is frequently couched in statistical or pseudo-statistical terms, and it has a direct bearing upon how one allocates one's income. In short, some knowledge of statistics seems important for everyone. Since the real payoff from a knowledge of statistics lies in using statistics to make decisions, and because most decisions of the sort considered daily are made in the presence of uncertainty, a basic grasp of elementary probability seems equally important.

What is proposed in Part 8 is a treatment, at a level accessible to the students to which the course is addressed, of the most fundamental notions of statistics. The main thrust of the Part should be the preparation of students to become intelligent consumers of statistical information rather than to be statisticians. Accordingly, they should be alerted to the pitfalls of interpretations based on some rather commonly encountered misuses of statistical information. Distortion of the scales of a statistical chart, faulty use of percentages, and poor sampling techniques are examples of causes of misinterpretations. An excellent and very readable book (for students, too) on this topic is How to Lie with Statistics by Darrell Huff and Irving Geis (New York, W. W. Norton and Company, Inc., 1954).

For example, one might pose the following question:

> Statistics show that in 1954 among fatal accidents due to automobiles, 25,930 occurred in clear weather, 370 in fog, 3,640 in rain, and 860 in snow. Do these statistics show that it is safest to drive in fog?

Students should become accustomed to analyzing and interpreting sets of numerical data. As a first step in this process, they need to be familiar with the construction and use of histograms, bar charts, line graphs, and pie diagrams. The World Almanac is a good source of material, e.g., census figures on U. S. population over a period of years, current distribution of population by age groups, personal income per capita, etc. Of course, daily newspapers and magazines contain other examples. The computation and use of measures of central tendency (mean, median, mode) and the use of percentile or other similar ranking indices should be presented with examples for actual hand-computation chosen so as to avoid an excessive amount of tedious labor and restricted to a list of numbers rather than grouped data. If a computer or a desk calculator is available, it would be possible in the laboratory to have interested students compute some of these measures using more meaningful data which they have collected or might be expected to encounter in vocational areas of particular interest to them.

Examples relevant to these matters might include discussions raised by questions such as:

> Is it safe for an adult who doesn't swim to step into a pool whose average depth is 4 feet?
>
> A newspaper reports that the average American family consists of 3.6 persons. What, if anything, does this mean? Is your family average?
>
> City A has average daily temperature of 75 degrees, as does city B. In city A, temperatures range from 10 to 99 degrees during the year. In city B, temperatures range from 60 to 80 degrees. For someone preferring moderate climate and hence considering moving to city A or city B, is there any reason to prefer one over the other and why?

An informal discussion of the normal distribution can be based upon the study of approximately bell-shaped histograms such as might be obtained, for example, from recording heights of a large number of college men. It may be observed that for such data the mean, median, and mode are approximately equal, so that phrases like "the average height is 69 inches" are not ambiguous. That this is not generally the case should also be noted. For instance, a distribution of family incomes would probably not have this property. The notion of standard deviation as a measure of dispersion and its relationship to the shape of distributions can be briefly touched upon--e.g., approximately 2/3 of the cases lie within one standard deviation of the mean, 95% within two standard deviations of the mean, and practically all are included within three standard deviations of the mean. In Part 9, after the notion of probability has been studied, it is possible to return to these properties in discussing problems of statistical inference.

An informal discussion of bias in sampling and the idea of a random sample might be enlivened by an experiment involving use of random number tables to select a random sample. A flowchart could be constructed to describe the steps in such a procedure.

The probability concepts proposed for Part 9 are simply those that bear upon the students' ability to make judgments in the presence of uncertainty. Little is proposed beyond introducing the general notions of the meaning of <u>a priori</u> and empirical probability, and developing an ability to understand the application of these notions in common-sense ways. These concepts are detailed in the outline, but their development should focus on the establishment of confidence in general impressions and interpretations rather than on the manipulation of formulas.

Empirical probability might be introduced by an example such as the American Experience Mortality Table. This table is based on birth and death records of 100,000 people alive at age 10. Entries give the number of these people living at various ages; for age 40 the entry 78,106 means that of the original group this many were still alive at age 40. The ratio 78,106/100,000 or .78 is called the empirical probability that a child of 10 will live to be 40. Students should be given some practice on the use of this table and caution on its misuse (e.g., probability .8 of living from age 10 to age 40 does not mean that in every group of ten individuals two will die before age 40).

The notion of <u>a priori</u> probability should be introduced from the point of view of an experiment performed with a set of objects in which certain outcomes are of interest. Some simple examples are: (1) A penny is tossed to see which of the outcomes, heads or tails, results. (2) One of several discs, each painted red or white, is drawn from a box and its color noted. (3) An individual is selected from a group of people and his opinion asked concerning a specific proposal of current interest to students.

It should, of course, be pointed out that experiments (1) and (2) are idealizations of certain processes having an intrinsic interest, such as the interaction of genes or the selection of a tax return for audit by the Internal Revenue Service.

It would be natural for these experiments to be performed in the laboratory with the results used to motivate the fundamental definitions. For example, in experiment (2), if it is known that the box has 50 white and 50 red discs, students will probably agree to assigning probability 1/2 for outcome "white disc." What if, instead, it is known that the box has 10 white and 90 red discs? What if the number of red and the number of white discs is unknown to the student? Experiments in the laboratory using boxes of discs with different proportions of the two colors might be undertaken with each student performing the experiment of drawing out a disc, then replacing it, say, 20 times, and recording the color of disc on each draw. A histogram of the relative frequencies collected by the class for outcome "white disc" illustrates the idea of variability and can lead to a discussion of how much confidence can be placed in information from a single, small sample. Combining the relative frequencies of groups of two students and then of five students, one can study the effect of sample size on variability. From an analysis of several histograms the class might be led to agree on a sample size when the composition of the box is unknown.

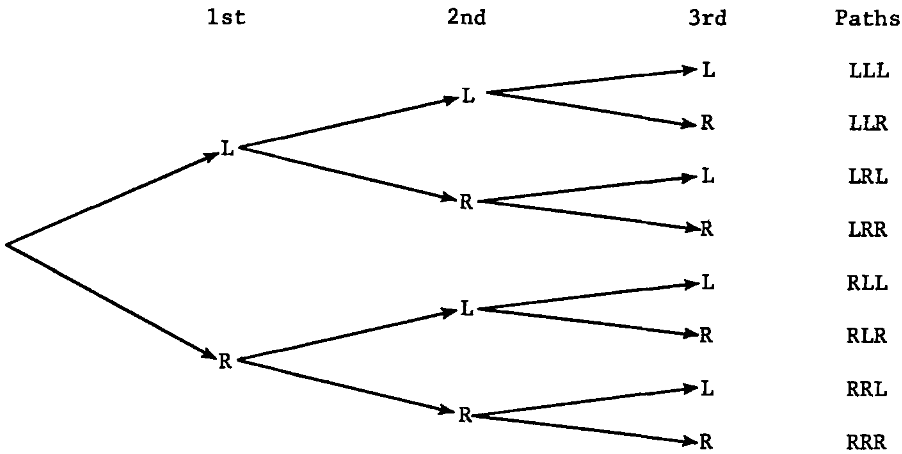
In experiment (3), if the percentage of the population in favor of the proposal is unknown and the problem is to estimate the unknown percentage, one should be careful to obtain a random selection from the population. For example, the percentage of the population in favor of building a new sports arena should not be estimated from those people who are leaving the old arena after a game. Estimation of the percentage of a population who favor a proposal is an example of statistical inference in which probability models are used to evaluate the likelihood with which assertions based on simple data are valid.

After some preliminary excursions into the computation of probabilities, it soon becomes apparent that a few guidelines to counting are needed. These should be based on the notion of first attempting a listing of (equally likely) outcomes. For problems involving a sequence of actions in succession--first do this, second that, etc.--the tree diagram provides a natural vehicle for presenting one of the so-called counting principles, the multiplicative one.

For example, in a psychology experiment a rat is placed at the entrance of a T-maze from which he runs either to the left arm, L, or the right arm, R.

(a) Suppose that the experiment is performed 3 times. List the possible paths of the rat on the three trials.

(b) If the food is always placed at L, how many of these paths could the rat take to receive the food at least two of the three times?



(a) See last column above; (b) 4: LLL, LLR, LRL, RLL.

With this basis for counting, the notion of the set of permutations of r objects chosen from n distinguishable objects, $n \geqq r$, can be pictured as an r-stage tree. Starting with many numerical examples, one arrives at the formula for the number $P(n,r)$. Simple discussion of factorials arising from the case of $P(n,r)$ is needed to facilitate arriving at the formula for the number of combinations (subsets) of r objects selected from a set of n objects. Students should compute a few factorials to see how fast they grow. A flow diagram for computing $n!$ might be developed. The formula for the number of combinations or subsets of r objects from a set of n should be derived after considering lists of permutations and of combinations in cases such as $n = 4$, $r = 2$; $n = 4$, $r = 3$. It should be stressed that this formula and the one for permutations are treacherous and should not be used until one has first ascertained (by an attempt at listing) that one of them is appropriate. Special combinatorial problems (e.g., circular permutations, permutations with repetition of symbols, etc.) should be avoided.

The formal treatment of probability theory should culminate in a discussion of the binomial distribution patterned after and illustrated by coin-tossing, die-throwing, etc. It is here that the convenience of the counting principles and formulas is noticeable. Some simple binomial tables should be available for student use in sampling (e.g., for $n = 10$, with various cases for p).

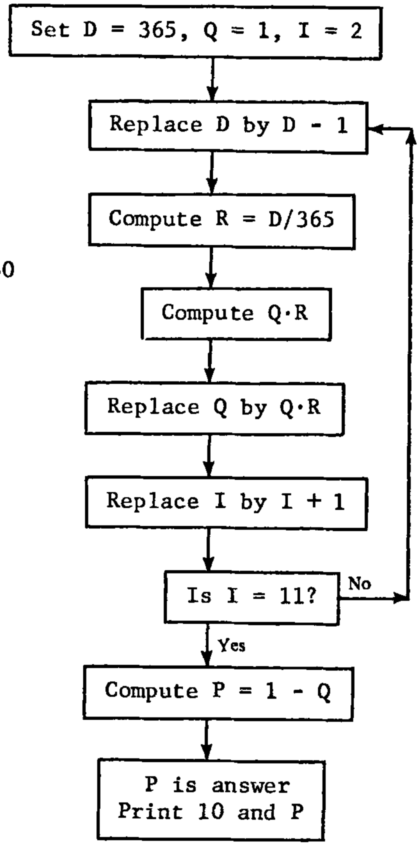
At this point let us consider an example to show the uses of the material of earlier chapters. (See Exercise 15 in Appendix I.)

Example: A famous probability example which interests beginning students is the so-called "birthday problem." The question is, "In a room containing N people, how large a number should N be so that the odds are 50-50 that at least two persons have the same birthday (month and day, not year)?" The answer, $N = 23$, defies the intuition of most people. The problem might be introduced as stated, various guesses given, and obvious extreme answers discarded. For example, for birthdays occurring in an ordinary year, if there are 366 people in the room, then it is certain that at least two have the same birthday. No doubt some will suggest that one half of 366 or, say, 180 is a good guess for the answer to the question, but the probability of coincidence of two or more birthdays for 180 people is still indistinguishable from 1! At this point the class can be encouraged to find the probability of coincidence in the case of small numbers N. For example, if $N = 2$, the number of possible (ordered) birthday pairs is $365 \cdot 365$, while the number of differing ordered pairs among these is $365 \cdot 364$, so that the required probability is $1 - \frac{365 \cdot 364}{365 \cdot 365}$ or $1 - 364/365 = 1/365$, indicating a small probability of two birthdays on the same day. Similarly, for $N = 3$ we find the probability of at least two birthdays falling on the same day to be $1 - \frac{365 \cdot 364 \cdot 363}{365 \cdot 365 \cdot 365}$, etc. It will be clear that the computation becomes prohibitive in a short time as one increases the size of N. Also, some remarks on approximating the fractions arising in such computations should tie in with previous discussions. Actual multiplication of numerator factors followed by division by the product of denominator factors would certainly lead to overflow on most computers. Writing, for example, the fractional part in the case $N = 3$ as $1 \cdot \frac{364}{365} \cdot \frac{363}{365}$ suggests a more hopeful procedure. A computer program (in BASIC language), with a corresponding flowchart, to compute and print out the values of this probability in the cases N = 10, 22, 23, 24, 25, 30, 40, 50, 60 is given below. Even with 60 people in the room, it is practically certain (probability .99) that at least two will have matching birthdays!

Birthday Problem BASIC Program

```
10  Read N
15  Let D = 365
20  Let Q = 1
25  For I = 2 to N
30  Let D = D - 1
35  Let R = D/365
40  Let Q = Q*R
45  Next I
50  Let P = 1 - Q
55  Print "N = "N; "P = "P
60  Go to 10
65  Data 10,22,23,24,25,30,40,50,60
70  End
```

Flowchart for case N = 10



The printout is given below:

```
N = 10      P = .116948
N = 22      P = .475695
N = 23      P = .507297
N = 24      P = .538344
N = 25      P = .5687
N = 30      P = .706316
N = 40      P = .891232
N = 50      P = .970374
N = 60      P = .994123
```

Now the student could be asked to consider just how significant the above 6-digit numbers are. What is the underlying assumption in the mathematical model? Is it true that each day of the year is equally likely for a birthday?

V. THE MATHEMATICS LABORATORY

The term "laboratory" will be used to indicate arrangements for teaching other than classroom instruction or undirected individual study. The word "laboratory" is also to mean a place (or places) in which would be located programmed materials, various audio-visual devices, books, and perhaps computer terminals. It might also include facilities for individual or group conferences. Laboratory facilities and organization may differ widely from school to school, and no prescriptive suggestions seem warranted.

Although we have not studied the role of mathematics laboratories in general, some form of a laboratory is an integral part of Mathematics E. Without a laboratory it is difficult to see how the flexibility necessary to deal with individual differences in preparation, ability, and goals can be achieved. Whatever form the laboratory may take, or whatever kinds of materials or devices are used, the total laboratory program must have three distinct goals:

(1) to correct deficiencies in preparation
(2) to make provision for individual goals
(3) to reinforce and extend the classroom instruction

We will discuss each of these three goals in turn.

1. <u>The Remedial Goal</u>

It is false to assume that any student knows all about mathematics up to a certain point in the standard curriculum, after which he knows nothing. Both the knowledge and the deficiencies of the students in the course will be scattered throughout their previous mathematical work, probably in small pieces. Small gaps in prerequisite knowledge and techniques can be diagnosed and treated in the laboratory.

The approach we have in mind is to determine as specifically as possible the individual deficiencies of a student and to prescribe as specifically as possible materials which the student can use individually to correct these deficiencies <u>in advance</u> of the time when these ideas and techniques are needed in the course.

From the outline one can see that Part 1 is essentially without prerequisites and that the prerequisites for Part 2 are minimal. Thus, while Part 1 is being taught in class the student can use the laboratory to master, by as many cycles of diagnosis and treatment as are necessary, the skills and concepts which are needed for Part 2. This leads in general to the idea that while one Part is being taught in class the student will be using the laboratory program to identify and correct any deficiencies he may have in the prerequisites for the next Part.

The problem of determining student needs for the remedial work of the laboratory is not trivial. Clearly, one requirement is the use of existing or teacher-constructed diagnostic tests. These tests must be highly discriminating and yet easily administered. They must pinpoint the student's deficiencies and indicate appropriate remedial activity.

Persons concerned with the design of the tests should also plan to minimize the degree of professional skill necessary to evaluate the results. Hopefully, the tests will be such that persons other than professional teachers can administer them and use the outcomes to assign appropriate remedial work.

Diagnostic tests should not constitute the only avenue for referring students to the laboratory for remedial work. If, on the basis of routine homework performance or class test results, it seems necessary for some students to repeat instruction in certain areas, instructors should have a means of referring these students to the laboratory for such reinforcement.

2. Provision for Individual Goals

Students will take (or be assigned to) this course for quite varied reasons. Beyond the mathematical literacy which is our fundamental purpose, some students who intend to continue in further mathematics courses will require more thorough grounding in or a broader coverage of fundamental algebraic skills. Others enrolled in technical or occupational curricula may need a further development of certain particular aspects of mathematics relevant to their curricula as well as applied problems directed specifically to their proposed major fields.

We imagine that corresponding to a given Part in the classroom presentation there might be several special-interest blocks composed of applications or illustrations of the same basic principles in different fields. For example, in connection with Part 8 on statistics, the prospective nurse might be studying statistics relative to public health, the prospective businessman the statistics of income and employment, and the prospective police administrator the statistics of crimes. Similarly, even such a straightforward subject as linear equations has applications to many subjects.

In addition to variant versions of an application, there may be need for optional blocks which endeavor to teach mathematical techniques not embodied in the main sequence of the course. For example, it is likely that a student interested in drafting might profit from some work in numerical trigonometry. We can also imagine that a student intending to take a course in chemistry might well profit from specific instruction in certain special techniques commonly used in chemistry courses. Finally, a student who does not possess any specific vocational objectives might even elect special-interest blocks in such subjects as number theory or more algebra.

We believe it is important that the student be allowed to choose freely from among the special-interest blocks that may appeal to him. Not only would this enhance the appeal of the laboratory material, but it might even be important to the student that he has been allowed to make some choice in order to adapt the course to his own goals as he sees them.

The availability of adaptation of the course to individual needs might well go far toward eliminating the demand for separate courses for special groups of students. In many local situations this flexibility might be good for the diplomatic relations between the mathematics faculty and the rest of the institution. In this connection, faculty members in other disciplines could well be invited to give advice or even aid in the construction of special-interest blocks for students interested in specializing in their fields.

An important class of students will be those who initially desire to use Mathematics E as preparation for Mathematics A or science or vocational courses having algebra as a prerequisite. We have previously remarked that the majority of students will not, in fact, go on. However, it would be desirable for this to be possible for those students with the requisite ability. Therefore, we suggest that at the beginning of the second semester a student who has such a goal and who is not heavily involved in the remedial aspect of this course could elect to devote the major portion of his laboratory time to studying programmed material on the techniques of algebra. In this way a course which will be terminal for the majority of students can become a nonterminal course for those willing and able to continue.

3. Reinforcement of Classroom Instruction

The laboratory may also be used to provide common extensions of the classroom work. As examples of such use of the laboratory we mention (a) the performance of certain geometric constructions by means of more effective devices than straightedge and compass, (b) computer programming and the use of computer terminals, (c) experiments in probability and statistics.

4. Management of a Mathematics Laboratory

The complexity of a mathematics laboratory may vary greatly, depending on the number of students at the level of Mathematics E which a given institution must serve. If only one or two sections of students are involved, procedures of diagnosis and prescription of suitable materials may reduce to conversation between the professor and the student. On the other hand, if several thousands of students and a large number of professors are involved in the basic mathematics program, the management of a laboratory may be a quite complex matter.

It may even be the case that some assistance from a computer is needed in order to keep records, to schedule conferences, and to give some feedback to the professor from the laboratory. If a laboratory reaches this degree of complexity, it would be natural for its services to be offered also to students whose mathematical deficiencies reveal themselves in other courses.

In some institutions the laboratory may be combined with similar endeavors in other disciplines and housed in a central learning resources center, thus relieving the mathematics faculty of the details of such an operation, much as a library relieves the mathematics faculty of the details of caring for collections of books. The difficulty with the latter arrangement is that the personnel of the laboratory would normally be expected to serve a tutorial function as well as performing the functions of clerks and librarians. Some colleges with well-developed laboratories have found students to be effective tutors and librarians provided they are properly directed and supervised. It is very important that professors have control over what the students are doing in the laboratory and that there is a method for professors to receive feedback from the laboratory.

There is naturally a question regarding the relative proportion of laboratory and classwork within the course. The answer would depend on local circumstances, and we have no desire to be prescriptive. The more strongly one believes in the importance of providing for individual differences and in the importance of insuring student activity, the greater the proportion of time one would tend to assign to the laboratory.

VI. QUALIFICATIONS FOR TEACHERS OF MATHEMATICS E

CUPM has considered in other reports the mathematical qualifications which it feels are necessary for the teaching of other mathematics courses in two- and four-year colleges. The present Panel feels that it would be unfortunate if teachers of Mathematics E were to constitute a subfaculty separate from those teaching other freshman and sophomore mathematics courses. Compared with other courses such as Mathematics A or Calculus, the teaching of the course would place somewhat smaller demands on depth in graduate mathematics training and somewhat greater demands on breadth in mathematics and related subjects. Thus, we do not see the need for a training program distinct from the usual one as preparation for the teaching of this course. However, teachers having little or no acquaintance with computing or statistics might require some small amount of additional training such as might be obtained at a summer institute or through some form of departmental or inservice training.

The teacher of Mathematics E must have certain attributes. Above all, he must be convinced of the value of the objectives of the course. In addition, he must have the ability to relate to a heterogeneous group of students. He must be willing to acquire an understanding of the applications of mathematics to the varied occupations to which these students may aspire; he must have sufficient understanding of computers, flowcharting, and estimation to weave these threads through the fabric of the course; he must have the versatility and experience in teaching needed to experiment with the laboratory to insure its success.

APPENDIX I

Sample Exercises

The following list of exercises was selected to demonstrate the existence of realistic problems that are likely to seem significant to the majority of students taking this course. The information given in the exercises is of the kind that would normally be encountered in the situation being described, and the questions asked in the problems are pertinent to the situation.

As indicated earlier in this recommendation, the exercises are not meant to convey any sort of desirable achievement level. In fact, the exercises given are generally more difficult than what might be termed an average exercise for the course. Most of the exercises can be reduced to a sequence of shorter and simpler problems. Some are appropriate for classroom discussion and perhaps would culminate in only a partial solution.

1. A builder quotes a prospective customer a price of $18 per square foot to build a certain style of house. The lot on which the house is to be built will cost an additional $4000. The customer knows that he can plan to spend at least $30,000, but no more than $40,000, on land and construction costs. Write an inequality whose solution will yield the range in which the size of the house must fall.

2. A grocery store advertises a peanut butter sale at a price of 2 jars for 85 cents. You notice that the net weight of peanut butter in each jar is 9 ounces. The same brand can be bought at 55 cents a jar containing 12 ounces of peanut butter. Should one buy the jars which are on sale?

3. Suppose you were required to take a 20 per cent cut in wages. If you are then given a 5 per cent increase in wages, does this mean that 1/4 of your cut has been restored?

4. What is wrong with the following procedure? To find the probability that an American citizen, chosen at random from a list of citizens, was born in a specified state, divide the number of favorable cases, 1, by the total number of states, 50, to obtain 1/50.

5. Two teams are playing a basketball game. A supporter of team A is willing to give you 3 to 1 odds and a supporter of team B will give you 2 to 1 odds, each betting on his favorite team. It is possible for you to bet x dollars with the first man and y dollars with the second and be \$10 ahead no matter which team wins. Write two equations involving x and y which express that fact.

6. A car salesman receives \$75 commission for each sale of one model of car and \$100 for each sale of another model. In a certain period of time he would like to receive a commission of \$3300. If he sold only cars of the first model, how many would he have to sell in order to earn the desired commission? Answer the same question for sales of only the second model of car. Set up an expression for the commission when cars of both models are sold. List additional items of information we would need in order to determine exactly how many cars of each model he must sell. Can he reach his goal exactly if he sells an odd number of the cars for which the commission is \$75? Explain. When three or four pairs of solutions have been found, what do you notice about the number of sales of the \$75 commission model? How many possible answers does the problem have?

7. A machinist is using a boring mill to rough-cut a collar for a steel shaft. In this process the speed of the lathe must be set carefully. Cutting too fast could burn the steel, while cutting too slowly is inefficient and produces a ragged edge. The speed in number of revolutions per minute (R) is given by $C = (\pi RD)/12$, where C is the best cutting speed for the materials being used in feet per minute and D is the diameter of the drill in inches. The value of π is approximately 3.14. If the drill is tool steel and the shaft is machine steel, D should be between 50 and 70 feet per minute. If the drill is 1 inch in diameter, what are the slowest and fastest rates of rotation at which the lathe should be set?

8. The current yearly gross salary of a state employee is \$7500. Each year he is given a raise equal to the rise in the cost of living. Each month 10% of his salary is withheld for federal income tax, 5% for his state income tax, 4.8% for social security tax, and \$9.50 for insurance. The rise in the cost of living during the current year is 6%. Write an equation whose solution will give his monthly take-home pay for the next year.

9. A housewife has a recipe for making brownies which calls for an 8" x 11" pan. By experience she knows that this will make two dozen servings. She is giving a big party and feels she needs 60 servings. She has two 8" x 11" pans and two 4" x 5" pans in the house. How many pans of which size should she use and how should she adjust her recipe? Note: This kind of problem gives an opportunity for some use of estimation. If the pans available have an area, say, 3.1 times that of the pan for which the recipe was written, the students should be made aware of the fact that tripling the recipe is good enough and that they shouldn't worry about the .1.

10. A risky operation used for patients with no other hope of survival has a survival rate of 80 per cent. What is the probability that at least four of the next five patients operated on will survive.

11. Assume that you are considering the purchase of a piece of land which costs $40,000. If a highway being planned passes through the land, the land will be worth $100,000. If, instead, the highway goes through nearby, the land will be worth $20,000. The probability that the highway will pass through the land is estimated to be .30. Evaluate the investment in the light of the probabilities given.

12. The following (with only trivial changes) is taken from the 1958 Boston and Maine Time Table:

Miles			A.M.
0	Dole Junction	Lv.	8:15
3	Hinsdale	Lv.	8:30
6	Ashuelot	Lv.	8:45
8	Winchester	Lv.	9:10
14	Wesport	Lv.	9:25
16	West Swanzey	Lv.	9:35
18	Swanzey	Lv.	9:45
21	Keene	Ar.	10:00

Observe that this trip of 21 miles requires an hour and three quarters and then check that this means that the average speed is 12 miles per hour. However, prove that an engine whose maximum speed is 20 miles an hour could not have made this trip on schedule.

13. Use data from the latest World Almanac giving U. S. population (official census) at 10-year intervals from 1880 to 1970 to make a line graph showing the population growth in the U. S. over this period. On the vertical scale start at 0 even though the population figures start at 50 million. In plotting points round off each population figure to the nearest million and choose scales of years on the horizontal axis in such a way as to make your picture approximately square.

Now make another line graph of the same data starting your vertical scale at 50 million and making the extent of your vertical scale about 4 inches while your horizontal scale is such that it extends across the width of your paper.

Compare the effect of the two graphs in portraying population growth.

14. A typical credit agreement reads:

Within 30 days after the billing date shown on each such monthly statement, Holder agrees to pay (1) the outstanding indebtedness ("New Balance") for "Purchases"; or (2) an installment of not less than 1/20th of such New Balance or $10, whichever is greater, and in addition a Finance Charge on the previous month's New Balance less "Payments and Credits" at the following rates: 1½% per month on so much of such amount as does not exceed $500; 1% per month on the excess of such amount over $500, or if the Finance Charge so computed is less than 50¢, a minimum Finance Charge of 50¢.

(Construct a flowchart for computing monthly payments. Do this for a $750 and a $100 purchase.)

15. Suppose that a person observing a carnival man in a game involving two tosses of a coin suspects the manner of tossing these coins favors both coins landing the same way, i.e., both heads or both tails. Such an outcome is unfavorable to the player. His decision on whether or not to play the game is based on the following rule: on the next throw of the two coins, if both show the same face he will not play; otherwise he will. Express the probability P that this person will play in terms of p, the probability with which the carnival man tosses a head. Determine P for $p = 0, .1, .25, .5, .75, 1$ and interpret the results.

16. One section of the Tax Reform Act of 1969 reads as follows:

"Low Income Allowance--

(1) The low income allowance is an amount equal to the sum of--

(A) the basic allowance, and
(B) the additional allowance.

(2) Basic Allowance--the basic allowance is an amount equal to the sum of--

(A) $200, plus
(B) $100, multiplied by the number of exemptions.

The basic allowance shall not exceed $1,000.

(3) Additional Allowance--

(A) the additional allowance is an amount equal to the excess (if any) of $900 over the sum of--

(i) $100, multiplied by the number of exemptions, plus
(ii) the income phase-out.

(B) Income Phase-out--The income phase-out is an amount equal to one-half of the amount by which the adjusted gross income for the taxable year exceeds the sum of--

(i) $1,100 plus
(ii) $625, multiplied by the number of exemptions."

The preceding statement offers a wealth of possibilities for problem construction including several possibilities for flowcharting problems.

(a) Write an equation that gives the basic allowance for a taxpayer with four exemptions.

(b) Write an equation that gives the basic allowance for a taxpayer with n exemptions.

(c) The statement limits the basic allowance to $1000. Write an inequality whose solution will yield the maximum number of exemptions allowable for computing the basic allowance.

(d) Write an equation that gives the additional allowance when the income phase-out is $400.

(e) Suppose the adjusted gross income for the taxable year is $4400. Write an equation that gives the income phase-out. (See Appendix II.)

17. The following is the procedure for computing social security benefits:

(1) Determine the "number of years" figure. If you were born before 1930, start with 1956. If born after 1929, start with the year you reached 27. Using your appropriate starting year, count that year and each one thereafter up to but not including the year in which you will be 65 if a man, or 62 if a woman.

(2) List the amount of taxed earnings for all years beginning with 1951. List no more per year than the amount subject to social security tax. These amounts have been: $3600 for 1951 through 1954; $4200 for 1955 through 1958; $4800 for 1959 through 1965; $6600 for 1966 and 1967; and $7800 for 1968 and succeeding years.

(3) Cross off this list your lowest earnings until the number remaining is equal to your "number of years" figure.

(4) Using the reduced list, add up all the earnings that are left, and divide by your number of years. This gives "average earnings," a figure which can be used with the aid of a table to determine the social security monthly benefit.

Construct a flowchart that describes the above process. (See Appendix II.)

18. Suppose you wish to buy a new car and you have determined that you do not want to pay more than \$130 per month for 30 months. If the interest is 5% per year on the original amount borrowed, then what price car should you consider? Approximately how many monthly payments are used to pay the interest? If you now decide that you might pay up to \$150 per month for 30 months, can you quickly estimate the price of a car you can consider?

19. The federal tax on capital gains in the stock market is roughly determined as follows: If you hold the stock for more than six months before selling it (a long-term gain), only half your profit is taxed; if you hold your stock six months or less before you sell it (a short-term gain), then your entire profit is fully taxed. For example, suppose you are in the 40% tax bracket and you have a stock which cost you \$1000. If you sell this stock for \$2000 within six months after you bought it, then your tax is 40% x \$1000, or \$400, so that your profit is \$600. If you sell this stock for \$2000 after holding it for more than six months, then your tax is 20% x \$1000, or \$200, so that your profit is \$800. Thus, long-term capital gain is clearly better than an equal amount of short-term gain.

Now suppose that your stock has shown a gain of \$1000 (paper profit), but six months has not lapsed since you purchased it and you fear that the stock will decline in price. Can you afford to risk losing part of your gain while you wait for it to become a long-term gain? Can you determine how much of your profit you can afford to lose and still be as well off after tax because of the lower tax on long-term gain?

If you think about it, you can show that the answer is

$$\frac{40\% - 20\%}{100\% - 20\%} = \frac{1}{4};$$

i.e., you can afford to lose 1/4 of your paper profit to wait for six months to pass and still have the same profit after tax. So a short-term gain of \$1000 is equivalent to a long-term gain of \$750 after tax. If you are in the 30% bracket, then what fraction of the paper profit can you afford to lose while waiting for six months to pass and still have the same profit after tax.

Examples of Artificial Word Problems

The exercises given as examples in this section are listed to demonstrate the kind of problem that is not in keeping with the spirit of the course. These problems are not necessarily considered to be innately "bad." In fact, some of the problems could justifiably be included in the course. However, the problems are obviously artificial and, hence, may fail to stimulate the pragmatically oriented students.

1. John is three times as old as Mary was five years ago. In twelve years he will be exactly twice as old as Mary. How old is Mary?

2. George and Frank can paint a house in 5 days by working together. Frank could paint the house in 8 days by working alone. How long would it take George to paint the house by working alone?

3. The tens digit of a 2-digit number is twice the units digit. The number is 20 more than the units digit. Find the number.

4. John has twice as many quarters as dimes and seven more nickels than dimes. If he has $2.30 in all, how many nickels does he have?

5. A certain room is 6 feet longer than it is wide. If the perimeter of the room is 82 feet, what is its width?

6. Compute three consecutive integers whose sum is 54.

7. Bob leaves city A at noon and drives 60 mph toward city B. Bill leaves city A at 1:30 p.m. and drives at 65 mph toward city B. At what time will Bill overtake Bob?

8. Frank bought thirteen pounds of meat for $10.55. He paid $1.10 per pound for beef and $.55 per pound for pork. How many pounds of each kind of meat did he buy?

APPENDIX II

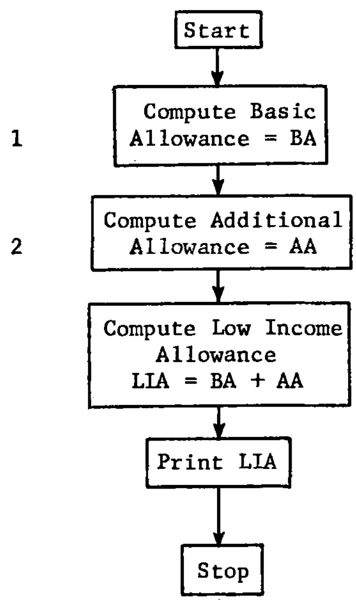
This Appendix contains analyses for exercises 15, 16, and 17.

15. This example, or one like it, might be used to introduce the notion of the uses of probability in decision making. Suppose the man can toss each coin so that for each, independently, p is the probability of heads. (If he tosses each fairly, $p = \frac{1}{2}$.) Then the probability of a head-tail combination is $P = 2p(1-p)$; this is

the probability that the observer will decide to play the game. Of course, p is unknown and can take on any value between 0 and 1. A graph of this quadratic, which students have made before (in the form $y = 2x(1 - x)$, probably), reveals that the maximum P is $\frac{1}{2}$, occurring when $p = \frac{1}{2}$ and the carnival man is tossing the pennies fairly. In the extreme cases of $p = 0$ or 1 (meaning what on the part of the carnival man?), we have $P = 0$, i.e., the observer will never play. Some inbetween values of p are of interest; should the carnival man have the finesse to insure always a $\frac{1}{4}$ chance of heads for each coin, $P = 3/8$--i.e., about $37\frac{1}{2}$ per cent of the time the observer engages in the game. The value $p = .1$ would result in $P = .18$, or less than 1 chance in 5 of the observer's succumbing. The main point is that the computation of P in terms of p, or the graph of P as a function of p, has provided the observer with some (probability) basis of evaluating the consequences of his rule of action. In this case, the decision rule, although based on a minimum amount of data, is a reasonable one.

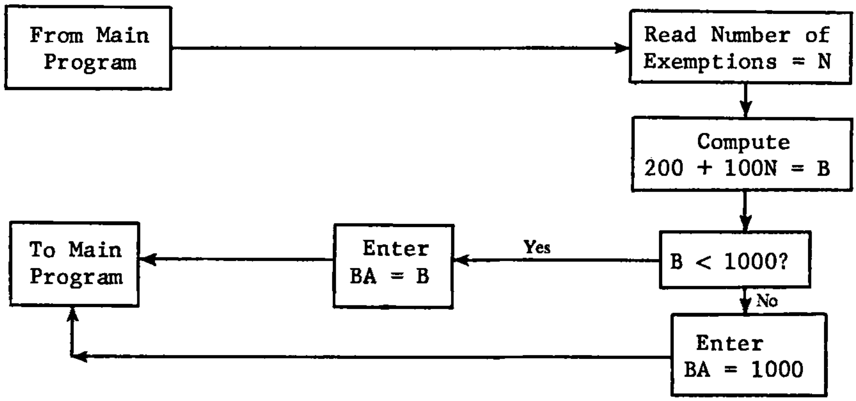
The flowcharts for Exercises 16 and 17 are especially illustrative of how flowcharting may be used to break large problems into sequences of smaller ones. Flowcharts for those two problems are given below:

(A) Flowchart for Exercise 16:

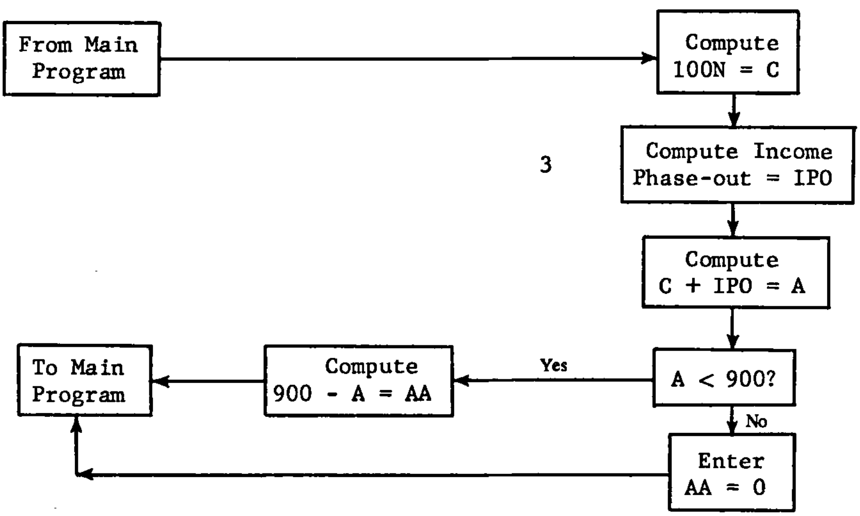


See next pages for expanded flowcharts for boxes 1 and 2.

Expansion of Box 1: Computing Basic Allowance

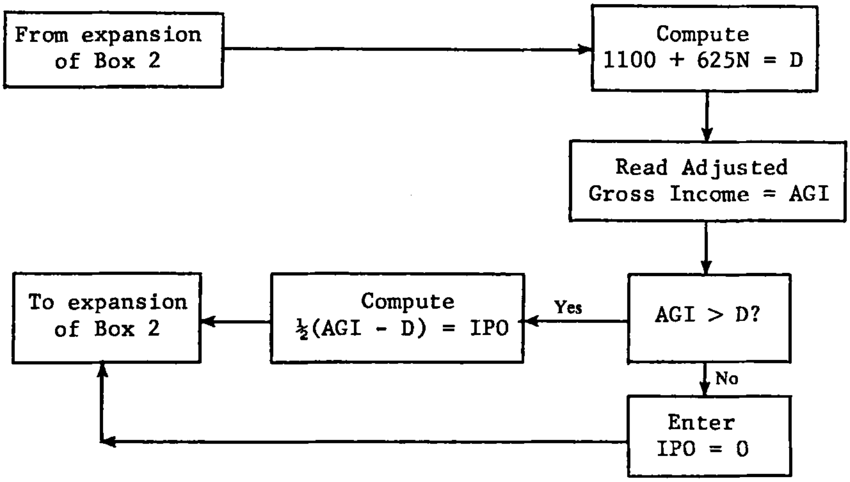


Expansion of Box 2: Computing Additional Allowance



See next page for expanded flowchart for Box 3.

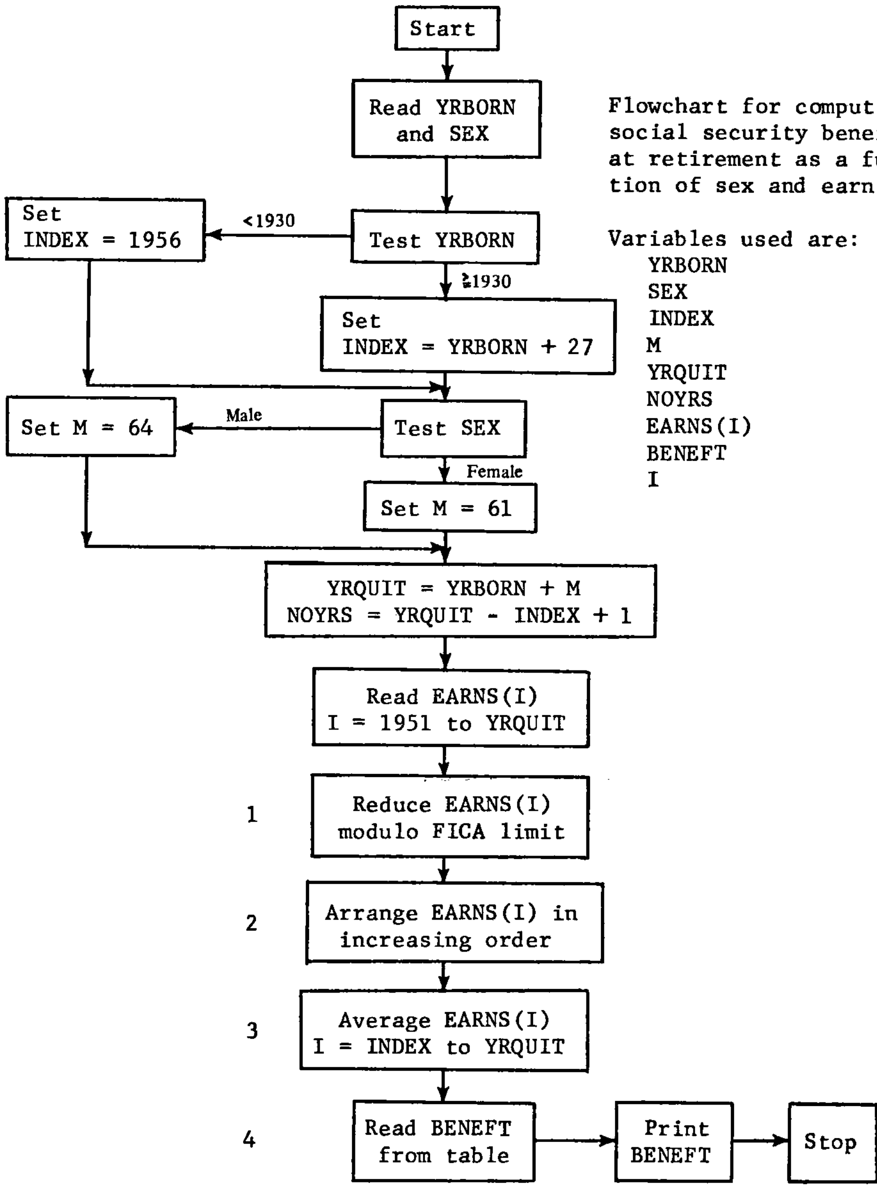
Expansion of Box 3: Computing Income Phase-out



(B) Flowchart for Exercise 17:

The table referred to in Part 4 of Exercise 17 is quite long. A portion of the table for a retired worker who has reached 65 is given below:

Average Earnings	Benefit (Per Month)
\$899 or less	\$55.00
\$900	\$70.00
\$1800	\$88.40
\$3000	\$115.00
\$4200	\$140.40
\$5400	\$165.00
\$6600	\$189.90
\$7800 or more	\$218.00

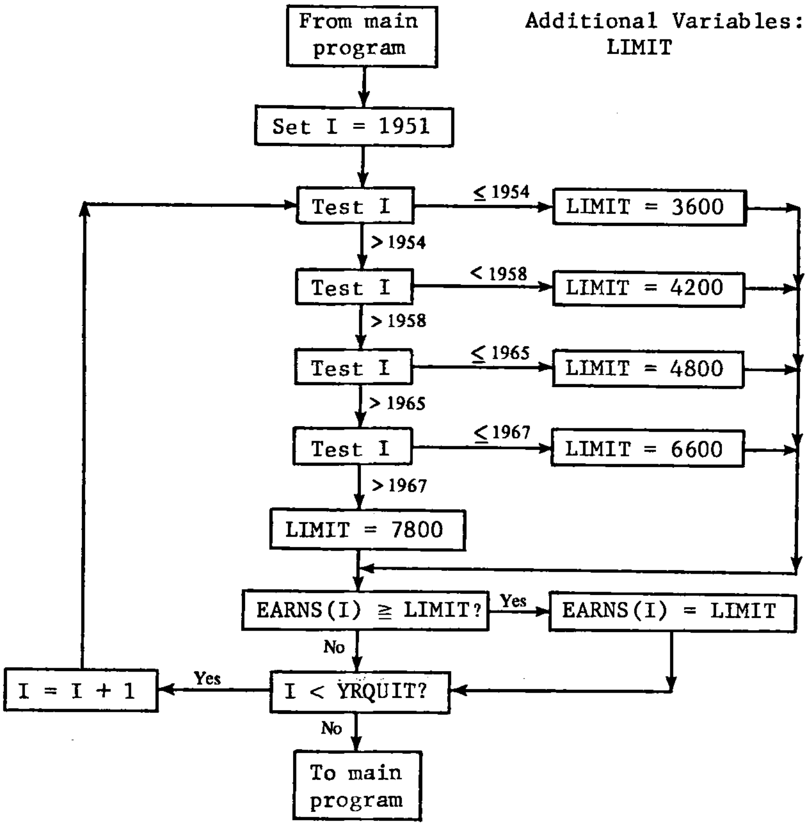


Flowchart for computing social security benefits at retirement as a function of sex and earnings

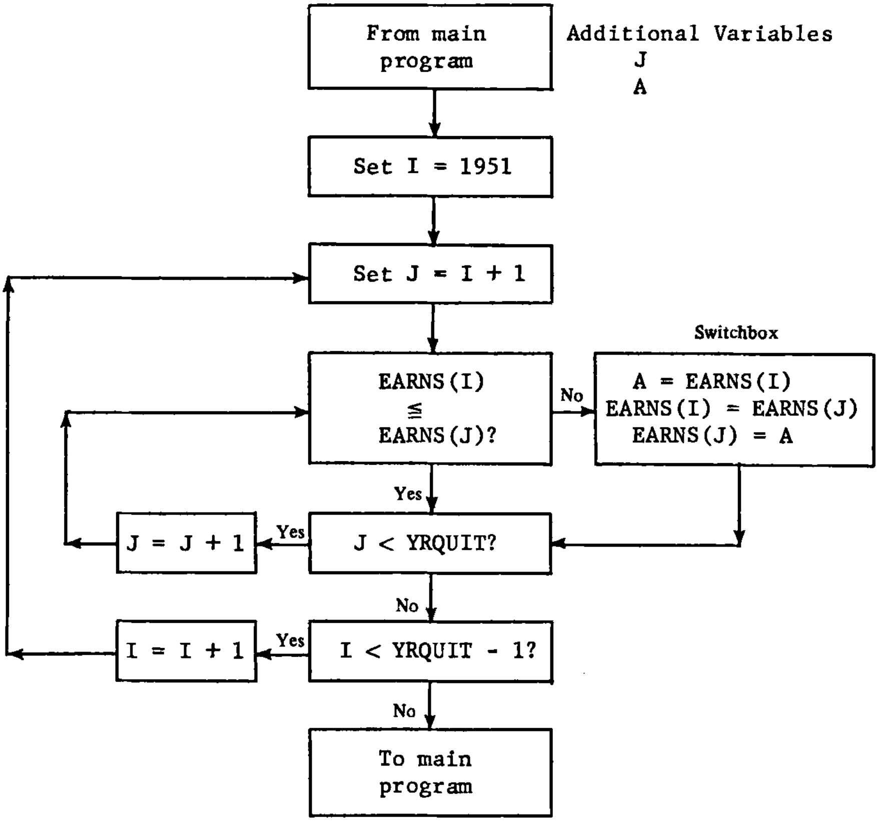
Variables used are:
YRBORN
SEX
INDEX
M
YRQUIT
NOYRS
EARNS(I)
BENEFT
I

See other pages for expanded flowcharts for Boxes 1, 2, 3 and a discussion of Box 4.

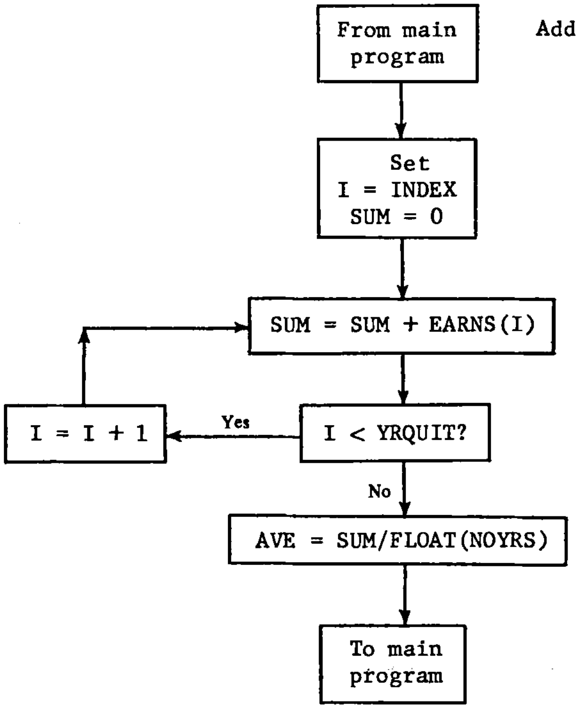
Expanded flowchart for Box 1--reducing actual earnings modulo the FICA limit for each year from 1951 to retirement



Expanded flowchart for Box 2--arranging FICA earnings in increasing order of magnitude from 1951 to retirement



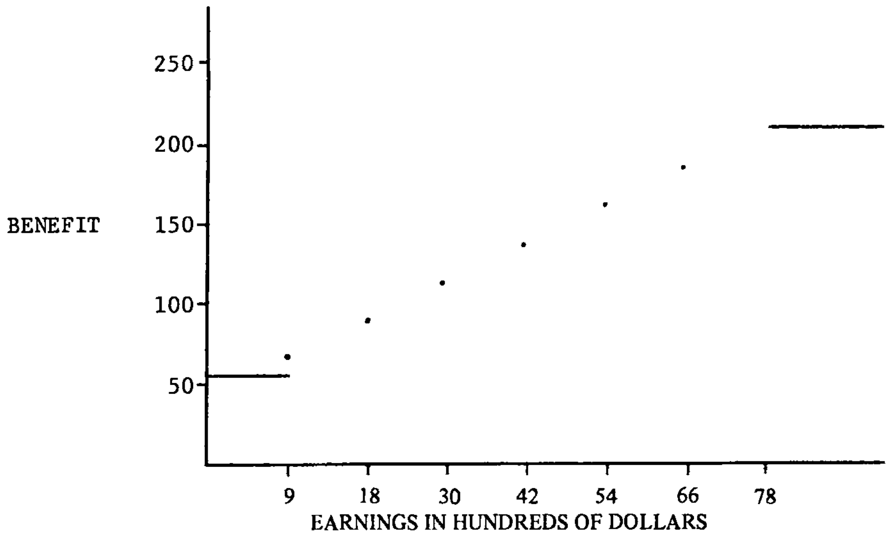
Expanded flowchart for Box 3--computing average FICA earnings from index year to retirement



Additional Variables
SUM
AVE

FLOAT is a FORTRAN word

Box 4 of the flowchart offers several pedagogical opportunities. First, the student should plot a graph of benefit vs. earnings from the table on page 307. The graph might look like this:



Note that there is a jump of $15 in benefit between earnings of $899 and $900.

Now, several approaches may be used for approximating the benefit associated with earnings between $900 and $7800. We list three possibilities that are all related to the fact that the isolated points on the graph seem to lie on a line.

1. The table could be stored in the computer memory and a program provided for interpolating linearly between adjacent points.

2. The student could select a line which actually passes through any two of the isolated points of the graph and determine its equation. There are several ways of doing this. For instance, the line on the points (3000, 115) and (5400, 165) has equation

$$\text{BENEFIT} = 115 + \frac{1}{48}(\text{EARNINGS} - 3000).$$

   This is a "good" line in the sense that it nearly contains the other points of the table, except for the first, the approximation of benefit being within $2 of the table value in every case.

3. One could use a least-squares fit on some points of the graph. For instance, the equation

$$\text{BENEFIT} = \frac{253}{12000} \times \text{EARNINGS} + \frac{46071}{900}$$

fits the five points with abscissas 1800, 3000, 4200, 5400, and 6600 in this sense. This is a "good" line too. It yields benefits corresponding to tabular values of earnings that are within $1 of tabular benefits in all cases.

The table given in this example was for a man retiring at age 65. A class project could be to find similar information for a woman retiring at age 62 and to complete the program.