APPLIED MATHEMATICS

IN THE

UNDERGRADUATE CURRICULUM

A Report of

The Panel on Applied Mathematics

January 1972

TABLE OF CONTENTS

I. INTRODUCTION AND STATEMENT OF RECOMMENDATIONS

Traditionally, attempts to solve problems in the physical sciences have stimulated and, in turn, extensively utilized basic developments in mathematics. This essential interaction between mathematics and the sciences is experiencing new vigor and growth. Recently, mathematical methods have been introduced into the social and life sciences, and even into some areas of the humanities. This has led to the development of new mathematical ideas and to new ways of using mathematics. The Committee on the Undergraduate Program in Mathematics (CUPM) appointed a Panel on Applied Mathematics to consider the implications for the undergraduate curriculum of this new growth of the uses of mathematics.

Instead of training students to handle all of the steps involved in solving a realistic problem, typical courses in applied mathematics generally confine themselves to a treatment of various mathematical techniques; in particular, mathematical model building is neglected. While courses in mathematical techniques are necessary, they do not provide a sufficiently broad training for students interested in applied mathematics.

The Panel therefore makes the following recommendations:

1. Every mathematics department should offer one or two courses in applied mathematics which seriously and comprehensively treat realistic problems and which emphasize model building.

2. Mathematics courses in the first two years of college should contain many realistic applications.

3. Every student taking a substantial number of courses in mathematics should include at least one course in applied mathematics.

4. A concentration in applied mathematics should be made available if the resources of the college permit.

The Panel is aware that the fourth recommendation is the most difficult to implement, especially in smaller departments. However, we feel strongly that most college departments can begin to implement the first three recommendations without undue difficulty or delay. For instance, having one instructor offer a course emphasizing model building could be an initial step toward implementing the first recommendation. Although the course may not have all of the desired characteristics the first time it is taught, the instructor's experience, along with ideas from this report, should enable him to come closer to meeting the objectives described here when he teaches the course again. Instructors in calculus, for example, can help to implement the second recommendation by introducing in their courses some applications different from the usual ones. In any case, the first of these recommendations can be effected by instituting one or two new courses at the upper-division level, and the second by incorporating applications in the lower-division courses.

II. DISCUSSION

Pure mathematics has undergone tremendous development during the past 25 years. Consequently, the recent generation of mathematicians is concerned primarily with pure mathematics, not only in research but also in educational activities. This is evidenced by the abstractness of some high school mathematics courses and the early introduction of axiomatic courses in colleges.

While the Panel applauds the advances in pure mathematics, it feels that it is unfortunate that education in applied mathematics has not received the same attention as that in pure mathematics. As a result, many other departments offer courses having substantial mathematical content, and mathematics faculties have tended to be unaware of the mathematization of many areas. It is encouraging, however, that there seems to be a recognition of this tendency and that a sympathetic interest in applications of mathematics is spreading. There is much more emphasis now than there was ten years ago on areas which directly attack problems of contemporary society such as ecological studies, city planning, water and atmosphere restoration, etc. This interest manifests a return to an attitude held in earlier times when mathematics was viewed as closely related to other areas such as the physical sciences and engineering. The unique way in which mathematics can contribute to an understanding of important problems in modern society is acknowledged, and many mathematicians have been attracted to the new ideas involved in recent applications because they are eager to have their teaching and research contribute to solutions of problems which are practical and contemporary.

These recent applications have contributed to changes in applied mathematics, both in its nature and in its methods. Applied mathematics may once have been identified exclusively with areas of analysis which had particular bearing on physics and engineering. But because mathematics is used in the social, life, and managerial sciences, and even in the humanities, applied mathematics must now include topics such as linear programming, graph theory, optimization theory, combinatorics, game theory, and linear algebra, in addition to those which have been traditionally associated with it. Similarly, methods of applied mathematics may have been thought of as involving complicated calculations with numbers or analytic expressions. While techniques for calculation are important, they are only part of the professional resources of an applied mathematician. Theory construction and model building are now essential for him. In studying the role of applied mathematics in the undergraduate curriculum, the Panel has taken into account these new topics and methods.

Having considered all of these points, we conclude that undergraduate instruction in applied mathematics must have a strong component specifically devoted to model building, and that undergraduates generally should be more aware of the many uses of mathematics in other areas.

III. NEW COURSES IN APPLIED MATHEMATICS

In our considerations we have been guided by the steps a working applied mathematician follows in studying a given situation. This process has been described in many ways by various authors. We use a description which is reminiscent of the one given by Murray Klamkin in the American Mathematical Monthly, 78 (1971) pp. 55-56 (ascribed to Henry O. Pollak):

1. Recognition of the nonmathematical problem.
2. Formulation of the mathematical model.
3. Solution of the mathematical problem.
4. Relevant computations.
5. Explanation of results in the context of the original problem.

Courses in mathematical topics give training in the solutions of mathematical problems (step 3), and courses in computer science and numerical analysis explain computational and approximative techniques (step 4), but very few courses adequately treat the processes involved in recognition, formulation, and explanation (steps 1, 2, and 5). While the student must, of course, have sufficient mathematical and computational techniques at his command to solve the mathematical problems he confronts and to obtain the numerical results which are needed, we are convinced that the training of a student of applied mathematics must be more comprehensive. He must be thoroughly grounded in the techniques of mathematical model building, and he must have ample practice in interpreting the results of his mathematical solution in the original setting.

The first recommendation of the Panel is that each department should offer a course or two in applied mathematics which treat some realistic situations completely, beginning with a careful analysis of the nonmathematical origin of the problem; giving extremely careful consideration to formulation of a mathematical model, solution of the mathematical problem, and relevant computations; and presenting thoughtful interpretations of the theoretical results to the original problem. In other words, there should be a few courses which give the students the experience of grappling with an entire problem from beginning to end.

To aid colleges in implementing the first recommendation, the Panel has constructed outlines of courses which emphasize model building. These courses are not intended to replace courses stressing mathematical techniques which are offered for students majoring in other areas, nor should they replace those standard mathematical offerings in which applications play a useful motivational role. Service courses are valuable and should continue to be offered by the mathematics department; indeed, they should be designed in active collaboration with members of concerned departments. Courses in mathematical topics which have their origins in applications are also important. However, the courses we recommend here provide a complementary training by giving students active experience in mathematical model building.

The Panel has given at the end of this report three course outlines which illustrate how a course stressing model building can be designed. These outlines are centered around the topics of optimization, graph theory and combinatorics, and fluid mechanics. The optimization course is intended as an example of a sophomore-junior course, the course on graph theory and combinatorics is appropriate at the junior level, and a course along the lines of the fluid mechanics option can be taught at the senior level. The optimization course and the course in graph theory and combinatorics can be offered at various levels by changing the level of rigor, varying the pace, concentrating longer on problems from a specific area, etc. These particular topics were chosen as unifying themes because of the experiences, interests, and competencies of individual Panel members, and because the courses on optimization and on graph theory and combinatorics illustrate the use of topics not traditionally viewed as being part of applied mathematics. In choosing these topics, the Panel does not mean to exclude other topics which might be used as the unifying element of an applied mathematics course. On the contrary, we hope that these outlines will stimulate instructors to construct similar courses around other topics. In fact, within reasonable limits, the particular topics chosen are not nearly so crucial as the emphasis on the model building process.

IV. GUIDELINES FOR TEACHING THE NEW COURSES

In planning or in teaching courses which emphasize model building, the instructor should keep in mind certain points which are essential for proper implementation of our recommendations.

First, the role of model construction must be made clear and amply illustrated throughout the course. The student must have as much experience as possible in constructing models. Real-life situations are often so complex that it is impossible to formulate a satisfactory model immediately; quite often it is necessary to construct a succession of models in an effort to find a satisfactory one. The student should have experience with this process. Furthermore, he should be aware that there may be several approaches which lead to essentially different mathematical models for the same problem. Therefore, a critical evaluation of the steps in constructing a model is essential in order that the student know what kind of information he can expect or cannot expect from a model and that he be able to choose the model which is most effective for his purpose.

Constructing a model for a given situation requires originality and a thorough understanding of the original nonmathematical situation. To appreciate what is involved, students must be active in formulating models. This aspect of the training is so important that the instructor should be willing to sacrifice some topics to insure the student's thorough grounding in model building. If the instructor conducts his class in the traditional lecture fashion, then he should prepare homework projects which require his students to formulate and to refine models for various situations. However, the Panel explicitly calls attention to the possibility of conducting these courses as seminars in which students and faculty members work cooperatively. Such a seminar could be organized around various problems, or it could develop a model for a complicated system which can be subdivided into smaller units. A benefit of the latter format is the experience of teamwork. Another possibility is for students to choose projects which they pursue independently. These projects could range from original investigations to reports based on the literature. In this case, students should periodically report their progress to the other participants in the seminar.

It is important to realize that model building has many forms. The activity which is most usually associated with the term modeling and which is actually always present in some form consists of formulating in explicit terms the dependence of the phenomenon under investigation upon the relevant factors. A classic example is the construction of a model for the motion of a vibrating string leading to a linear partial differential equation. In this case the factors which are to be neglected as well as those which have considerable effect on the motion can be identified, and the sort of physical assumptions which simplify the model are relatively clear. With appropriate assumptions, an analysis of the physical laws governing the motion of a particle lead to a mathematical model for the motion of the string consisting of a partial differential equation and suitable boundary conditions. The solution of this mathematical problem aids in the description of the motion of the string. The degree to which the solution of the mathematical problem contributes to an understanding of the physical one depends upon the degree to which the assumptions fit the real situation.

The model for motion of a vibrating string is a deterministic one; that is, it is based on the assumption that the physical laws and the initial conditions determine the response of the system exactly. Such models are not always appropriate, and there are instances in which uncertainty in the real situation should be reflected in the model, as, for example, in stochastic models. As an illustration, consider the construction of a model for the spread of a disease. The number of people who become ill during an epidemic depends on a number of factors associated with the disease--its virulence and period of contagion for example--and also on the random contacts between infected and susceptible individuals. In some instances the results obtained by ignoring the probabilistic features of the situation may be adequate, while in others inclusion of the probabilistic features may be required in order to obtain a satisfactory fit between the predictions of the model and the results of observations.

Alternatively, it may be that any model which accounts for what appear to be the essential features and which is formulated for mathematical analysis will lead to mathematical problems which are either totally intractable or beyond the scope of investigation. In such cases a computer simulation may be useful. Simulations may be performed on both deterministic and stochastic models, and they may provide much of the same type of information that is obtained from a mathematical analysis when such analysis is feasible.

The point is that there are many kinds of models, and the student of applied mathematics should be aware of them. Consequently, the topics for investigation must be chosen carefully so that different types of models will be illustrated.

Second, the problems chosen for investigation must be realistic. In this report, when we use the term "realistic" referring to problems or situations, we have in mind those which arise directly from nature or from social behavior and which have some current significance. We label as "artificial" those problems which seem to be designed purely to illustrate some mathematical point. While some artificial problems have undeniable pedagogical value, relying almost exclusively on such problems will not instill the attitude of mind which should characterize the modern applied mathematician. In a contrived situation it is difficult to create and maintain interest in the multitude of concerns which arise in problems occurring in the real world. Since it is the Panel's intention that the student recognize the complexities of the real world and that he come to terms with these complexities in his model building process, the student must face real problems. In the course outlines we have given references to assist those who wish to acquaint themselves with significant problems in other fields.

Third, the original nonmathematical situation should not be forgotten once a mathematical formulation has been achieved. The results of the mathematical study need to be interpreted in the original setting. Stopping short of this gives the impression that manipulation of symbols or that techniques of computation or approximation are the important points, whereas they are only intermediate steps, although absolutely essential, in studying a realistic nonmathematical situation. For this reason we urge that in these courses the situations should not only be realistic but that they should be treated as completely as possible.

Fourth, the mathematical topics treated should be worthwhile and have applicability beyond the specific problem being discussed. They should be appropriate to the level at which the course is offered; problems and examples should be chosen to illustrate more than just elegant or ingeneous applications of mathematically trivial ideas. It is impossible for a single course to contain all of the mathematical techniques which all students may need; nevertheless, it is possible to select as illustrative techniques those which will be valuable to a large portion of the students.

Finally, an instructor of applied mathematics should not view his work as being confined to one academic department or, for that matter, restricted to his college or university. Applied mathematics affords unique opportunities for cooperative projects with other members of the college community and with people outside the college whose professional work is related to mathematics. We encourage instructors to invite active participation by students and faculty members from other departments in planning and conducting courses or seminars. In some instances it may be valuable to include nonacademic professional people having interests and competencies related to the area being studied; their experience and point of view may add a new dimension to the investigations. It is our view that instructors in applied mathematics are in a particularly good position to initiate cooperative ventures of this type.

V. USE OF COMPUTERS IN APPLIED MATHEMATICS

Mathematics education has been influenced in several ways by the recent trend toward the widespread use of computers. This is particularly true of instruction involving applications. The role of computing in the mathematics curriculum is being studied in detail by the CUPM Panel on the Impact of Computing on Mathematics Courses [see Recommendations on Undergraduate Mathematics Courses Involving Computing, page 571], and comments on computing as a part of a concentration in applied mathematics can be found in Section VIII of this report. The purpose of this section is to draw attention to the ways in which machine experience can reinforce ideas and techniques which the student is learning and thereby contribute to the teaching of applications.

The use of computers makes it possible to consider situations having a much greater complexity than would be possible if the associated numerical work were to be carried out by hand or with the assistance of a desk calculator. This is true not only in courses specifically oriented toward applications but also in the standard undergraduate courses. As an illustration of the sort of activity which illustrates the process of applied mathematics and which becomes feasible through the use of computers, consider the example of determining as a function of time the position and velocity of a rocket traveling to the moon. The depth of the study obviously depends heavily on the audience, but certain versions are appropriate for students in courses in elementary calculus or ordinary differential equations. A sample discussion in the spirit of this report would include the following features.

1. Newton's laws of motion and gravitation and a mathematical model for the system. A careful discussion of the idealizations and approximations made in constructing the model.

2. Derivation of the differential equations governing the motion of a rocket in one dimension between the earth and moon.

3. Discussion of the qualitative features of the solution.

4. Selection of a numerical method.

5. Preparation and testing of a computer program for the integration of a system of first-order ordinary differential equations.

6. Use of the program to obtain quantitative information on the motion of the rocket. Determine the escape velocity of the earth-moon system and compare it with that of the earth alone.

7. Comparison of results predicted by the 1-dimensional model with observed phenomena and a discussion of the inadequacies of such a model.

8. Derivation of the differential equations describing the motion of a rocket in two dimensions.

9. Numerical solution of these equations in two dimensions [repeat steps 3, 4, and 5 in this case]. Use a plotter to graph the trajectories as functions of initial velocity and firing angle.

10. Comparison of these results with observations. Discussion of discrepancies.

In addition to its use in the activities described above, the computer can also be used to obtain the best values of parameters occurring in the model and to test the validity of the model. The latter usually involves comparing predictions based on the model with experimental data by using statistical techniques. Finally, both analog and digital computers are useful tools for simulation when the situation cannot be modeled in a form susceptible to mathematical analysis.

VI. RECOMMENDATIONS CONCERNING MATHEMATICS COURSES IN THE FIRST TWO YEARS

The Panel believes that many students lose their enthusiasm for mathematics even as a tool because their mathematics courses seem unrelated to their own discipline. A large segment of students in lower-division mathematics courses is primarily interested in fields outside mathematics. These students want to use the ideas and techniques of mathematics in their fields of interest; they are not interested in majoring or minoring in mathematics. We feel that the best way to demonstrate the power and utility of mathematical ideas to these students and thereby to sustain their interest is to introduce applications to other fields in the early mathematics courses. Therefore, <u>the second recommendation of the Panel is that a greater number of realistic applications from a greater variety of fields be introduced into the mathematics courses of the first two years</u>.

The suggestions made earlier about the choice of problems and examples apply here too. Instructors should strive to avoid artificial or contrived examples and applications. It is especially important to formulate the problem clearly and to mention explicitly the assumptions, approximations, and idealizations used to obtain a reasonable mathematical model. If simplifications are needed to make the mathematical problem workable, then they should be clearly stated and discussed. In other words, the applications should be significant and their treatment should be as complete and intellectually honest as the level of the course will allow.

The applications should be chosen from various fields in order to illustrate the use of a mathematical model or idea in different settings. If the course and the background of the students permit, some problems should be treated which require one to construct a succession of mathematical models in an effort to conform better to experimental data. Numerical methods might be included.

As we have already mentioned, some students who are not mathematics majors lose enthusiasm for mathematics because their courses do not contain applications. However, the Panel is also concerned that the mathematics major have an appreciation for the importance of mathematics in other areas. Even if he becomes a research mathematician, he is very likely to teach some undergraduate mathematics courses. His effectiveness in these courses can be greatly increased by a grasp of the relations among different branches of mathematics and the relations between mathematics and other disciplines. Therefore, we feel that he should see many significant applications in his elementary mathematics courses.

Unfortunately, very little literature on applications of elementary mathematics exists at the present time. One source is the Proceedings of the Summer Conference for College Teachers on Applied Mathematics held at the University of Missouri--Rolla with the support of the National Science Foundation, published by CUPM. These proceedings contain applications of elementary calculus, linear algebra, elementary differential equations, and probability and statistics.

Textbooks for most undergraduate mathematics courses vary considerably in their emphasis on applications, and instructors should consult various books so that they can provide their classes with a variety of interesting applications. For example, in differential equations there are many modern texts which contain discussions of genuine applications. Two books which contain a variety of applications not duplicated in many other places are:

Bellman, R. and Cooke, K. L. Modern Elementary Differential Equations, 2nd ed. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1971.

Carrier, G. F. and Pearson, C. E. Ordinary Differential Equations. Waltham, Massachusetts, Blaisdell Publishing Company, 1968.

Also, modern texts in general physics and mechanics usually have examples suitable for discussion in a course on differential equations.

Another standard undergraduate course--linear algebra--has many applications to both physical problems and linear programming. In addition to the references listed in connection with the optimization outline given later in this report, the following text deserves mention:

Noble, B. Applied Linear Algebra. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1969.

The following book is a collection of realistic problems suitable for undergraduate mathematics courses. The problems are catelogued according to the mathematical tools used in their solution. Every teacher of freshman and sophomore mathematics should be aware of this source of applications.

Noble, B. Applications of Undergraduate Mathematics in Engineering. New York, The Macmillan Company, 1967.

As an example of the way in which a specific subject matter area may be used to provide applications for elementary courses, consider the biological sciences. Population growth, for example, can serve as a motivation for the introduction of elementary differential equations. Also, population growth problems can be considered from a probabilistic point of view; indeed, many problems in the biological and social sciences admit both deterministic and stochastic models, so it may be wise to introduce probability along with calculus in order to be able to study both kinds of models. Books are now available which take this approach; for example,

Chover, J. The Green Book of Calculus. Menlo Park, California, W. A. Benjamin, Inc., 1971.

Stein, Sherman K. Calculus in the First Three Dimensions. New York, McGraw-Hill Book Company, 1967.

The instructor who wishes to include applications to the biological sciences will find the following references useful. Although some of this material can be treated with little modification in lower-division classes, these sources are more suitable for the instructor than for the student.

Gerstenhaber, M. et al. AMS Lectures on Mathematics in the Life Sciences: Some Mathematical Problems in Biology, vol. I. Providence, Rhode Island, American Mathematical Society, 1968. (See particularly the first paper.)

Pielou, E. C. An Introduction to Mathematical Ecology. New York, John Wiley and Sons, Inc., 1969.

Smith, J. M. Mathematical Ideas in Biology. New York, Cambridge University Press, 1968.

VII. COMMENTS ON SECONDARY SCHOOL TEACHER TRAINING

Our third recommendation is that every student whose degree program includes a substantial number of courses in mathematics should take at least one course in applied mathematics. This recommendation clearly should apply to mathematics majors, but the Panel wishes to emphasize that every prospective secondary school teacher of mathematics should also have at least one course in applied mathematics. The role of applied mathematics in the training of teachers of secondary school mathematics has been underscored by the American Association for the Advancement of Science* and by other CUPM panels. [See Recommendations on Course Content for the Training of Teachers

---

* Guidelines and Standards for the Education of Secondary School Teachers of Science and Mathematics. Washington, D. C., American Association for the Advancement of Science, 1971.

of Mathematics, page 158.] The AAAS recommendations state that "an undergraduate program for secondary school mathematics teachers should ... provide substantial experiences with mathematical model building so that future teachers will be able to recognize and construct models illustrating applications of mathematics." The CUPM Panel on Teacher Training recommends that prospective teachers should complete a major in mathematics and that the courses in the program should include not only a mixture of motivation, theory, and application but also an introduction to model building. Indeed, that Panel recommends that a course in applied mathematics is particularly desirable as an upper-division option for the mathematics major.

The Panel on Applied Mathematics strongly supports these recommendations and emphasizes the following reasons for a secondary school teacher of mathematics to have a knowledge of applications:

1. Appropriate applications provide excellent motivational material.

2. The teacher should be aware that most of the mathematics encountered in the secondary school has its origins in problems in the real world, and he should know what these origins are.

3. The teacher should be aware of the applications of mathematics in the social and life sciences as well as in the physical sciences. Since mathematical notions are occurring with increasing frequency in elementary texts in the social and life sciences, and since it is unlikely that most teachers of these subjects have adequate mathematical training to appreciate this material, the mathematics teacher may well be called upon to serve as a resource person for other teachers.

4. Carefully selected applications may aid significantly in developing the student's ability to recognize familiar processes which occur in complex situations.

Further discussion of these and other ideas can be found in references [E] and [P] at the end of this section.

We make the following recommendations:

1. In those courses of the basic curriculum which are taken by substantial numbers of prospective secondary school teachers (viz., Mathematics 1, 2, 3, 4 and 2P of Commentary on A General Curriculum in Mathematics for Colleges (CGCMC)), applications of the subject to problems arising outside mathematics should receive more attention than is generally given now.

2. Each prospective teacher should be strongly encouraged to take one of the courses proposed in Section III of this report or a course in applied mathematics designed especially for secondary school teachers. Sample materials appropriate for an applications-oriented course for teachers include [B], [Po], and [S].

References

[B] Bell, M. S., ed. Some Uses of Mathematics: A Source for Teachers and Students of School Mathematics. (Studies in Mathematics, vol. XVI, SMSG) Pasadena, California, A. C. Vroman, Inc., 1967.

[E] Engle, A. "The relevance of modern fields of applied mathematics for mathematics education." Educational Studies in Mathematics, 2 (1969), pp. 257-269.

[P] Pollak, H. O. "How can we teach applications of mathematics?" Educational Studies in Mathematics, 2 (1969), pp. 393-404.

[Po] Pólya, G. Mathematical Methods in Science. (Studies in Mathematics, vol. XI, SMSG) Pasadena, California, A. C. Vroman, Inc., 1963.

[S] Schiffer, M. M. Applied Mathematics in the High School. (Studies in Mathematics, vol. X, SMSG) Pasadena, California, A. C. Vroman, Inc., 1968.

VIII. RECOMMENDATIONS CONCERNING A CONCENTRATION IN APPLIED MATHEMATICS

The fourth recommendation of the Panel is that an undergraduate concentration in applied mathematics should be offered if the resources of the college permit. In many institutions there are students who desire such a program. These students should take some courses in model building such as those described in Section III, and they should be trained in mathematical topics useful in applications. We are concerned both that the training of the students properly reflect the changes taking place in applied mathematics and that a department of mathematics be able to begin implementation of our recommendations immediately with a relatively small change in course offerings. For these reasons our recommendations center around courses of the type we have already described and courses in various mathematical techniques which are common in many colleges.

A student interested in a concentration in applied mathematics should take three courses in calculus (Mathematics 1, 2, 4 of CGCMC) and a course in linear algebra (Mathematics 3 of CGCMC). (For those who notice the omission of differential equations, we point out that Mathematics 2 of CGCMC contains an introduction to differential equations.) To insure training and practice in modeling, he should take at least one and preferably two of the new courses described in this report. A student who has a particular area to which he wishes to apply his mathematics should select courses in mathematical topics which are useful in that area as well as courses in the field of application which utilize significant mathematics. The topics suggested below can be organized into courses in various ways. However, we do recommend that applications be introduced in these courses, and we feel that the comments made in Section VI on applications in the freshman and sophomore courses are particularly appropriate here.

A student who is interested in applications to the physical sciences or in some areas of life sciences (e.g., ecology) should take a physical science version of an applied mathematics course such as the one in fluid mechanics outlined in this report. His further mathematics courses should include as many of the following topics as possible: probability theory; elementary partial differential equations (some of this is already contained in the fluid mechanics course); topics in ordinary differential equations such as asymptotic solutions, stability, and periodic solutions; boundary value problems (including Fourier series); computer-oriented topics from numerical analysis such as those which emphasize numerical solutions of ordinary differential equations, numerical linear algebra, solution of nonlinear equations, or numerical quadrature.

A student interested in applications to business and social sciences should take courses such as the optimization course and the graph theory course outlined in this report. His further mathematics course work should include as many topics as possible from the following: probability theory and applications as described in the report of the CUPM Panel on Statistics, Preparation for Graduate Work in Statistics; statistics as described in the same document; computational linear algebra.

Furthermore, because much work in applied mathematics involves computations, approximations, and estimates, it is clear that students concentrating in applied mathematics should have training in the use of computers. Beyond increasing computational power, a knowledge of the uses of computers can provide a new perspective for formulating and analyzing problems of applied mathematics. Consequently, the Panel strongly recommends that the following phases of computer experience be included in the program of every student of applied mathematics:

1. Computer programming. The student should have sufficient familiarity with a programming language to be able to use computer facilities in ways that are appropriate for his mathematical course work.

2. Computational mathematics. The approximations, estimations, algorithms, and programming necessary to derive numerical solutions of mathematical questions should be presented.

3. Training and experience in the use of a computer at the various stages of solving a problem in applied mathematics. The

student should have experience in using the computer to organize large quantities of numerical data and to simulate complicated processes.

IX. COURSE OUTLINES

To exemplify the kinds of courses recommended in Section III, the Panel has constructed three course outlines. These courses do not deal merely with mathematical topics; they are courses in which the momentum comes from real situations. In particular, stress should be placed on model building and on interpretation of mathematical results in the original nonmathematical situation.

These outlines are not offered as perfect models of the kinds of courses we recommend. Rather, they represent our present best efforts to construct courses with these new emphases. We hope that they will produce a thoughtful response in the form of even better outlines for applied mathematics courses.

It is essential that these outlines be read with the recommendations of Section IV in mind. Also, the reader should have in hand one or two of the primary references in order to find examples of the kind of treatment we are suggesting.

In reading these outlines, in teaching these courses, or in constructing other courses along the lines of our recommendations, instructors should strive to stay well between the extremes of: (a) a course about mathematical methods whose reference to science consists mainly of assigning appropriate names to problems already completely formulated mathematically, and (b) a kind of survey of mathematical models in which only trivial mathematical development of the models is carried out.

The course in optimization was planned as a one-quarter course, with additional material in the sections marked * bringing the total to a one-semester course. The courses in graph theory and combinatorics and in fluid mechanics were designed as one-semester courses.

The number of lectures specified indicates the relative emphasis we have in mind for the various topics and serves as an actual time estimate for a well-prepared class. The Panel appreciates the fact that some instructors will find these time estimates somewhat unsuitable (for instance, they do not take into account the pursuit of finer points or the review of prerequisite material) and will find it necessary to make modifications in the courses for their classes. The Panel was tempted to construct less ambitious outlines but decided against this, because it felt that a prospective instructor would be helped by having more examples of the treatment we recommend rather than fewer. Nonetheless, a valuable course can be constructed by choosing a few of the topics listed and treating them carefully and thoroughly. Furthermore, if the students become actively engaged in the model building activity, then the time estimates given are not appropriate. In any case, we encourage instructors to engage in open-ended discussions with class participation in the modeling aspect of the course and, if necessary, to restrict the subject matter content of the course in order to accommodate this.

IX.1. OPTIMIZATION OPTION

This course was designed to provide an introduction to the applications of mathematics in the social and management sciences. The goals of this course, as stated in Sections III and IV, are a study of the role of mathematics as a modeling tool and a study of some mathematical notions of proven usefulness in problems arising in the social and management sciences. The mathematical content consists of programming and game theory. This selection is a considered choice, although it is recognized that several other alternatives could serve as well.

The proposed course can be taught at several levels to fit the competencies and interests of the class. In particular, one version might be appropriate for freshmen whereas another might be appropriate for upper-class students in the management and social sciences. The course outlined here is intended for an average junior-level class. The students should have completed the equivalent of two semesters of calculus and should have some familiarity with elementary probability theory. Linear algebra is not included as a prerequisite, as the necessary background is developed in the course. No specific knowledge of any other discipline is assumed.

A bibliography and an appendix, important adjuncts to the course coutline, are found after the outline. References to the bibliography are enclosed in square brackets [ ], and references to the appendix are enclosed in braces { }. The bibliography contains a selection of books and other references which have proved useful in courses of this sort. Certain references have been designated as primary references, and comments have been provided which indicate those features of particular interest for an instructor. Most of the citations in the course outline are to the primary references. The instructor should have at least one of the primary references at hand while reading the outline. The appendix contains examples of the types of problems which can be studied using the ideas and methods of this course.

COURSE OUTLINE

1. Mathematical foundations of model building (4 lectures)

The real world and abstractions to mathematical systems; axiom systems as used in model building.

The ideas of a mathematical model and model building are introduced by using several examples which can be developed quickly and which illustrate applications in several different fields. Typical examples might be drawn from business (programming models for resource allocation), ecology (linear programming models of pollution control), psychology (2- or 3-state Markov chain models for learning), and sociology (game theory models for conflict). Assumptions made in the construction of these models should be carefully identified. The status of empirical "laws" should be discussed: law of gravity, law of reflection, law of supply and demand. It should be pointed out that all model building requires some essentially arbitrary decisions on the part of the person who is constructing the model. For example, whether to select a deterministic or a stochastic model is ultimately a decision of the investigator. In most instances there is no single best model. A model which was constructed to account for observed phenomena of one type may not give predictions which agree with other observations. The role of approximation and idealization in model building is fundamental. Approximations which are made and justified for real-world reasons should be distinguished from those whose basis is mathematical. Students need practice in making connections between assumptions about the real world and axioms in a mathematical system. Some of the examples should bring out the fact that an important (and frequently difficult) part of model building is asking the right question and viewing the real world problem from the right perspective. Some attention should be given to the practical problems of critically evaluating models and estimating parameters.

Most of the references contain some comments on model building. The initial chapters of primary references [D], [KS], and [Sa] have more comprehensive discussions. The books [ABC] and [LR] discuss modeling from the point of view of the social scientists.

2. <u>Linear programming models</u> (18 lectures)

a) Construction of linear programming models. (1 lecture) A detailed discussion of a real-world situation which can be reasonably modeled in terms of a linear program.

Examples similar to {1} or {2} might be used. Assumptions which lead to the axioms of linearity should be explicitly noted and adequately justified. It may be that the linear model is meant to serve only as a first approximation to a more complicated situation. Also, a linear model is frequently realistic only for restricted values of some variables. Such questions need to be considered. It is desirable to introduce both deterministic and stochastic models and later to compare two models of the same situation. The history of the development of linear programming during and after World War II is interesting. The book [D] is a useful reference for this material.

b) The basic problem. (6 lectures) The algebra and geometry of systems of linear inequalities in R^n. Matrix and vector notation and elementary linear algebra. Systems of linear equations and their application to systems of linear inequalities (e.g., if A is an $m \times n$ matrix and $\underline{b} \in R^m$ ($\underline{b} \neq 0$), then there exists $\underline{x} \in R^n$ satisfying $A\underline{x} = \underline{b}$ or there exists $\underline{y} \in R^m$ satisfying $A^t\underline{y} = 0$, $\underline{b}\cdot\underline{y} \neq 0$).

The notion of duality and the fundamental theorem should be introduced and illustrated. Consider complementary slackness and its economic interpretation. Selections from the primary references [D], [Ga], [SpT], and [W] provide appropriate sources.

* Proof of the fundamental duality theorem.

c) Algorithms: the simplex method. (6 lectures)

Much of the usefulness of linear programming models rests on the fact that the resulting mathematical problems can be efficiently solved. Accordingly, it is important to give some attention to computation, although only a bare introduction is proposed here. The method can be introduced as a sequence of replacement operations similar to a method for solving systems of linear equations. Algebraic and economic considerations can be used to describe and motivate the method. The concept of degeneracy arises naturally, but a complete discussion of this idea is beyond the scope of the course. Larger and more realistic problems should be solved, and students with computer competence should be encouraged to use it. The references are the same as those cited in b).

* Further remarks on degeneracy.

* A proof of the convergence of the simplex algorithm.

d) Refined models: linear programming and uncertainty. (5 lectures)

These models should be introduced by discussing the inadequacy of deterministic models for certain problems. One example is the allocation of aircraft to routes (this is discussed in Chapter 28 of [D]). There is no single formulation for stochastic models, as for deterministic ones, and there is little general theory. However, this is an important modeling technique which serves to demonstrate how models can be refined to take account of additional information. Examples can be given which show that one is not usually justified in simply substituting expected values for coefficients which are actually random variables. The basic problem is to formulate the stochastic model in such a way that relevant information can be obtained by studying an ordinary deterministic model. Chance constrained programming provides an interesting special case. Primary references [D] and [W] contain this material.

* Multistage models and dynamic programming.

* Geometry of the simplex method.

* Linear models of exchange and production.

3. Game-theoretic models (10 lectures)

a) Games and decision-making with uncertainty models for systems involving opposing interests. (3 lectures) The role of games as a modeling technique in the social sciences. The basic assumption of rational behavior and its validity.

Introduce utility theory, in both its qualitative and quantitative aspects. Consider individual decision-making under uncertainty and compare this to games. Discuss examples and, in particular, the relevance of a mathematical theory of games for the real world. The basic reference [LR] is useful here. Both [LR] and [BN] discuss game theory from the social scientist's point of view.

b) Games with two sets of opposing interests. (3 lectures) Two-person zero-sum matrix games and the connection between such games and linear programming.

Although such games are of limited use in applications, they provide a convenient vehicle for introducing basic notions of strategy and payoff. The fundamental (minimax) theorem of two-person zero-sum games. In primary references [D] and [Ga] this material is closely connected with linear programming. The discussion in [LR] is more comprehensive, and the notions of extensive and normal forms for games are introduced.

c) Nonzero-sum games. (3 lectures)

Games of the "prisoner's dilemma" type are of particular interest to the social scientist and can be used to illustrate the difficulties which arise in more complex models. The theory for such games is not nearly so well developed as for the games of b), but the study should bring out many of the questions that arise in mathematical work in the social sciences. Primary reference [LR] contains some of this material; more detailed expositions can be found in [BN] and, among the additional references, in [R].

d) n-person games. (1 lecture)

There is a qualitative difference between two-person situations and those involving three or more independent interests. Thus, there are new difficulties which arise in modeling three-interest conflict situations. The notion of a "solution" to such games requires careful analysis. The role of bargaining and coalitions is important in such models. See primary reference [LR].

* Games of timing. Reference [Dr] is especially complete on this topic.

* Two-person cooperative games.

References

A bibliography consisting of several hundred items on the topics listed in the course outline could easily be compiled. Thus, with some exceptions, the list of references is restricted to those sources specifically cited in the course outline. Several of the books listed here contain extensive bibliographies. The books given extended annotation are, with one exception, examples of writing which reflect the spirit of the course. The exception [O] is a mathematics textbook which presents some of these notions from a purely mathematical point of view. Critical reviews are indicated according to the following scheme: AMM, American Mathematical Monthly; MR, Mathematical Reviews; OR, Operations Research; and SR, SIAM Review. Also, each reference has been broadly classified according to whether it is primarily concerned with the mathematical content (M) or applications (A), and whether it is most useful for the student (S) or instructor (I). Several of the other references have been given a one-line annotation where useful.

Primary References

[D] Dantzig, George B. Linear Programming and Extensions. Princeton, New Jersey, Princton University Press, 1963, 625 p. AMM 72, p. 332; MR 34 #1073; OR 14, p. 734. (M and A, S) A textbook on mathematical programming written by one of the founders of the field. It includes chapters on the history of the subject and on model formulation. Chapter 3 contains five detailed examples. Standard topics in linear programming, extensions to integer, stochastic, and nonlinear programming, and many applications. Connection between programming and matrix games is included. Basic linear algebra is covered rapidly, and some probability is needed for the chapters on stochastic programming and games. No other prerequisites. Last two chapters contain detailed examples of formulation and study of models for nutrition and transportation. Extensive bibliography, many examples, and exercises.

[Dr] Dresher, Melvin. Games of Strategy: Theory and Applications. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1961, 184 p. AMM 69, p. 243; MR 22 #13310; OR 10, p. 272. (M and A, S) The basic theory of two-person zero-sum games is developed in the first three chapters independent of its connection with linear programming. The remaining chapters include methods of solution, extensions to games with an infinite number of strategies, and games of timing. Basic calculus and probability are required. Several thoroughly discussed examples and applications, particularly to military problems. No exercises.

[Ga] Gale, David. The Theory of Linear Economic Models. New York, McGraw-Hill Book Company, 1960, 330 p. MR 22 #6599. (M and A, S) Theoretically oriented presentation of linear programming and game theory with emphasis on the use of these concepts in developing linear economic models. An advanced undergraduate textbook with no specific mathematics prerequisites, but almost everything is proved and several proofs require considerable sophistication. Chapter 2 provides a self-contained study of linear algebra including

linear inequalities. The approach is both algebraic and geometrical throughout. Validity of the models is not discussed. Exercises are a definite asset; they vary from routine to nontrivial extensions of the theory.

Kaufmann, A. and Faure, R. Introduction to Operations Research. New York, Academic Press, Inc., 1968. (A, S) Collection of 18 chapters, each a completely worked-out independent example, generally written in anecdotal form. Few specific mathematics prerequisites; calculus and finite mathematics certainly sufficient. Basic ideas thoroughly explained but involved mathematical arguments are avoided. Applications are mostly to business situations and problems have an aura of reality.

[KS] Kemeny, J. C. and Snell, J. L. Mathematical Models in the Social Sciences. Waltham, Massachusetts, Blaisdell Publishing Company, 1962, 145 p. AMM 71, p. 576; MR 25 #3797. (A, S) Collection of eight independent examples of the construction and study of mathematical models drawn from several scientific disciplines. Stated mathematics prerequisites are one year of calculus and a good course in finite mathematics, but most students will require more background. No specific social science knowledge is assumed. There is an introductory chapter on the methodology of mathematical model building. Exercises and projects at the end of each chapter.

[LR] Luce, R. D. and Raiffa, H. Games and Decisions. New York, John Wiley and Sons, Inc., 1957, 509 p. MR 19, p. 373. (A, S) This is more a book about the concepts and results of game theory than a mathematics textbook; there are almost no proofs. Modest prerequisites: some knowledge of finite mathematics plus a bent for mathematical thinking. Thoroughly motivated discussions of the heuristic considerations which precede the mathematical formulation of the problems. These discussions are colored by a social science point of view. The introductory chapters consider the role of game theory in the social sciences and give a relatively complete discussion of utility theory including an axiomatic treatment. Extensive bibliography. No exercises.

[O] Owen, Guillermo. Game Theory. Philadelphia, Pennsylvania, W. B. Saunders Company, 1968, 228 p. MR 36 # 7420. (M, I) Mathematics textbook on two-person (chapters 1-5) and n-person (chapters 6-10) game theory, including those aspects of linear programming which are important for the study. Assumes basic calculus and probability. Convexity used but developed in an appendix. Chapters on infinite games and utility theory. Written in Definition-Theorem-Proof style. Many exercises of varying difficulty. Few applications.

[Sa] Saaty, Thomas L. *Mathematical Methods of Operations Research*. New York, McGraw-Hill Book Company, 1959, 421 p. AMM 68, p. 188; MR 21 #1223. (M and A, I) A textbook on operations research consisting of three major units. Part I contains chapters on the scientific method, mathematical existence and proofs, and some methods of model formation. The first chapter is particularly relevant for this course. Part II includes classical optimization techniques as well as linear programming and game theory. Part III is devoted to probability theory and its applications, particularly to queueing. There are many examples with convenient references to the literature, and a large bibliography accompanies each chapter. Assumes basic calculus and matrix theory. Some sections require multidimensional calculus. No exercises.

Simonnard, Michele. *Linear Programming* (translated by William S. Jewell). Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1966, 430 p. MR 34 #1079, original French edition MR 25 #1952; SR 9, p. 608. (M, I) A textbook on linear programming covering the general theory (chapters 1-7), integer programming (chapters 8-9), and the transportation problem (chapters 11-15). The connection of linear programming with game theory and extensions to stochastic and dynamic programming are omitted. The book is oriented in a practical direction, and emphasis is on effective methods. It contains references to the literature, and there is an extensive bibliography but no exercises. Appendices on linear algebra, convex polyhedra, and graphs. No specific prerequisites.

[SpT] Spivey, W. A. and Thrall, R. M. *Linear Optimization*. New York, Holt, Rinehart and Winston, Inc., 1970, 530 p. (M and A, S) A mathematics textbook on linear programming with emphasis on the development of the simplex algorithm. The approach is a spiral one, and most topics are developed at several levels of difficulty. Chapter 2 discusses modeling and presents several examples. There is a chapter on game theory. The necessary background material on foundations, sets, functions, and linear algebra is given in appendices. Many exercises. Suitable as a text for students with limited backgrounds.

[W] Wagner, H. M. *Principles of Management Science with Applications to Executive Decisions*. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1970, 562 p. (A, S) A textbook of mathematical model building and optimization in a business setting. Prerequisites are a year of calculus and finite mathematics. No knowledge of business administration or economics is assumed. Emphasis on linear, dynamic, and stochastic programming with chapters on waiting-line models and computer simulation. Broad selection of exercises ranging from computational to "Form a mathematical

model for ... ." Some proofs, but many results are provided only heuristic justification.

Additional References

Ackoff, Russell L., with the collaboration of S. K. Gupta and J. S. Minas. Scientific Method: Optimizing Applied Research Decisions. New York, John Wiley and Sons, Inc., 1962. AMM 72, p. 216; OR 11, p. 157. The philosophy and formulation of mathematical models, including utility theory.

[ABC] Atkinson, R. C.; Bower, G. H.; Crothers, E. J. An Introduction to Mathematical Learning Theory. New York, John Wiley and Sons, Inc., 1965.

Bellman, Richard, ed. Mathematical Optimization Techniques. Berkeley, California, University of California Press, 1963.

Blalock, H. M. Theory Construction, from Verbal to Mathematical Formulations. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1969. AMM 77, p. 216.

[BN] Buchler, I. R. and Nutini, H. G., eds. Game Theory in the Behavioral Sciences. Pittsburgh, Pennsylvania, University of Pittsburgh Press, 1969.

Charnes, Abraham and Cooper, W. W. Management Models and Industrial Applications of Linear Programming, vol. I, II. New York, John Wiley and Sons, Inc., 1961. MR 28 #1003a,b.

Dantzig, George B. and Veinott, A. F., eds. Mathematics in the Decision Sciences, Parts 1, 2. (Lectures in Applied Mathematics, vol. XI, XII) Providence, Rhode Island, American Mathematical Society, 1968.

Duffin, R. J.; Peterson, E. L.; Zener, C. Geometric Programming. New York, John Wiley and Sons, Inc., 1967. AMM 77, p. 1024; SR 10, p. 235.

Freudenthal, H., ed. The Concept and the Role of the Model in Mathematics and Natural and Social Sciences. New York, Gordon and Breach, Science Publishers, Inc., 1961.

Gale, D.; Kuhn, H. W.; Tucker, A. W. "Linear programming and the theory of games." Cowles Commission for Research in Economics, Monograph #13. New York, John Wiley and Sons, Inc., 1951.

Goldman, A. J. and Tucker, A. W. "Theory of linear programming." Annals of Mathematics Studies No. 38. Princeton, New Jersey, Princeton University Press, 1956, pp. 53-98.

Hadley, G. Linear Programming. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1962. AMM 71, p. 815; SR 6, p. 191.

Rapoport, Anatol. Fights, Games, and Debates. Ann Arbor, Michigan, University of Michigan Press, 1960.

Rapoport, Anatol. Two-Person Game Theory: The Essential Ideas. Ann Arbor, Michigan, University of Michigan Press, 1966. MR 35 #1356. Nonmathematical exposition of basic ideas of two-person game theory.

[R] Rapoport, Anatol and Chammah, A. M. Prisoner's Dilemma; a Study in Conflict and Cooperation. Ann Arbor, Michigan, University of Michigan Press, 1965.

Shubik, Martin, ed. Game Theory and Related Approaches to Social Behavior. New York, John Wiley and Sons, Inc., 1964. OR 12, p. 637. Collection of articles on the applications of game theory in the social sciences.

Spivey, W. A. Linear Programming: An Introduction. New York, The Macmillan Company, 1963.

Vajda, S. Mathematical Programming. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1961.

Ventzel, E. S. Lectures on Game Theory. New York, Gordon and Breach, Science Publishers, Inc., 1961.

Wilder, R. L. Introduction to the Foundations of Mathematics, 2nd ed. New York, John Wiley and Sons, Inc., 1965.

Williams, J. D. The Compleat Strategyst, rev. ed. New York, McGraw-Hill Book Company, 1966. Intended for readers with no background in mathematics, this book develops the basic ideas of game theory through a variety of clever examples.

Appendix

The problems given here are indicative of the sorts of questions that can be studied using the techniques and ideas of this course. Problems similar to these should be approached in the spirit of Section 1 of the outline, where the question is phrased in real-world terms and a mathematical model is constructed. In such a discussion, close attention should be paid to assumptions, both explicit and tacit. The student should be made aware of the strengths and shortcomings of the resulting models.

1. Linear programming

Here are two linear optimization problems, one concerning diet and another concerning transportation, posed in a business context. The first is given in considerable detail, while the second is merely sketched. Possible extensions are indicated.

1.1 This problem, the determination of an adequate diet of minimum cost, was one of the first studied using a linear programming model. Detailed comments on the formulation of a mathematical model may be found in [D] and in the original paper of G. J. Stigler ("The cost of subsistence," J. of Farm Econ., 27 (1945), pp. 303-314). The following is a linear programming model.

Consider n different types of foods (apples, cheese, onions, peanut butter, etc.) and m nutrients (proteins, iron, vitamin A, ascorbic acid, etc.). In the original problem of Stigler, $n = 77$ and $m = 9$. Suppose that one can determine the daily allowance of each nutrient required by an individual and the nutrient values of the foods per dollar of expenditure. (These assumptions are at best approximations and should be presented as such.) Let

a_{ij} = amount of nutrient i obtained from an expenditure of one dollar on food j,

b_i = daily requirement of nutrient i,

x_j = number of dollars spent on food j.

With these definitions the condition that the diet provide at least the daily requirement of each nutrient becomes

$$\sum_{j=1}^{n} a_{ij}x_j \geqq b_i, \quad i = 1, 2, \ldots, m.$$

The problem of finding an adequate diet of least cost is then the problem of minimizing $\sum_{j=1}^{n} x_j$ subject to the above inequalities.

1.2 Suppose an oil company has m producing wells, n refineries, and pipelines connecting certain pairs of wells and refineries. Given the output at each well, the demand at each refinery, and the cost of transporting one barrel of oil through each pipeline, determine how the production of the wells should be allocated among refineries in order to minimize transportation costs.

1.3 In 1.2 consider the case that only allocations in whole barrels are permitted. Also consider the case where supply, demand, or other parameters are not known exactly, but instead some random behavior for each is assumed.

2. Game theory

Here are two examples involving decision making under uncertainty. The first example can be completely analyzed in terms of the elementary theory; the second cannot, but it illustrates a game that occurs frequently in the social sciences.

2.1 Two political parties compete for public favor by stating their views in n different media, labeled $1, \ldots, n$. Each party has finite resources and must distribute its expenditures among the various media without knowing the intentions of the opposing party. The payoff (a numerical measurement of the gain of one party or, equivalently, the loss of the other) resulting from use of medium i is given by a function $p(i,x,y)$ depending only on the medium and the resources x and y committed to that medium by the opposing parties. The payoff for the entire game is the sum of the payoffs in individual media. Given a knowledge of the resources and payoffs, how should each political party allocate its expenditures?

The following is a very simple model of a social situation involving conflicting interests. Models of this sort and their refinements are currently being studied by mathematically oriented social scientists. Although these models are only rough approximations to very complex situations, the results obtained from them are far from completely understood from a psychological and sociological point of view.

2.2 In an isolated and self-contained environment two retail stores compete for the local soft drink market. Each retailer handles only one brand of soda pop, different from the brand handled by the other retailer, and the two brands are identical in quality. In ordinary circumstances each retailer pays 70¢ for a carton of pop which he sells for \$1. However, the soft drink distributors realize that from time to time price competition will develop, and they agree to sell their products to the retailer at 60¢ per carton provided that it is offered at retail for 80¢ per carton. Every Saturday each retailer must decide independently what his price for soda pop will be for the following week. Each has available the following information concerning demands: At the usual price they will each sell 1,000 cartons per week. If one retailer discounts while the other does not, then the discount store will sell 2,000 cartons while the store maintaining the usual price will sell only 200 cartons. If both stores sell at the discount price, then the total demand will be for 2,300 cartons and each will sell half that amount. Supposing that this decision must be made each week, how should the store managers proceed?

IX.2 GRAPH THEORY AND COMBINATORICS OPTION

This is an outline for a one-semester course designed to acquaint students with some fundamental concepts, results, and applications of graph theory and combinatorial mathematics. Only high school mathematics is required, but the student needs to be thoroughly familiar with this material. It should be kept in mind that this course represents just one of a number of (essentially equivalent) possible courses and is intended to offer the student not only specific facts and applications but also a feeling for the underlying philosophy of combinatorial mathematics.

A bibliography and an appendix follow the course outline. References to the bibliography are in brackets [ ], and references to the appendix are in braces { }. The bibliography contains references to books and other sources, together with comments about the primary references. The appendix contains examples of problems which can be treated using the ideas and methods of this course.

COURSE OUTLINE

1. <u>Mathematical foundations of model building</u> (4 lectures)

Real models, mathematical models, axiom systems as used in model building. (For discussion, see Section 1 of the course outline for the Optimization Option.)

2. <u>Graph theory</u> (18-20 lectures)

a) Basic concepts: relations, isomorphism, adjacency matrix, connectedness, trees, directed graphs, Euler and Hamiltonian circuits. (3 lectures)

In this section the student is introduced to a number of elementary (but fundamental) ideas of graph theory. He should be given as soon as possible the opportunity to formulate and discuss various models of real situations in these terms. [BS] is an especially good source of appropriate, relatively simple examples.

This material is available from numerous sources. The presentation in [L] is suitable here; more technical treatments are given in [Har1] and [O1], while that of [O2] is probably too elementary. Other sources are [Bel] and {1, 2}.

b) Circuits, cutsets, spanning trees, incidence matrices, vector spaces associated with a graph, independent circuits and cutsets, orthogonality of circuit and cutset subspaces. (5-7 lectures)

The linear algebra required for this section is minimal and, if necessary, could be developed in several hours. The concepts covered here lead directly to one of the more important applications of graph theory, namely, electrical network analysis. This material is covered rather briefly in [L] with no applications, very compactly in [Bec], more completely in [BS], and comprehensively in [SR] (on which an entire course could easily be based). [SR] is also an excellent source of applications of these topics.

c) Flows in networks, max-flow min-cut theorem, Ford-Fulkerson algorithm, integrity theorem, applications (e.g., linear programming, König-Egerváry theorem, multicommodity flows, marriage theorem). (4 lectures)

An appropriate discussion of this material occurs in [L] and in selected passages of [BS]. An exhaustive treatment occurs in [FF], which is also a good source of examples and applications ({3} is typical).

This section allows for a wide selection of applications for which these techniques are appropriate. Examples of multicommodity flow problems might be given here in order to illustrate the difficulties often encountered in more complex models.

d) Planarity, Kuratowski's theorem, duality, chromatic graphs, matching theory. (6 lectures)

The concepts presented in this section allow the student to become familiar with some slightly more advanced material in graph theory. These can be used to model more complex situations, e.g., {4} and {5} (cf. [Si], [Ben]).

This material may be found in nearly all standard graph theory texts (e.g., [01], [Bel], [Harl], or briefly in [L]). Typical applications occur in [BS]. Example {4} gives a nice application of some of these subjects (cf. [Si]). These topics are perhaps not so fundamental as the preceding and may be omitted if time pressure is a problem.

3. <u>Combinatorial mathematics</u> (19-22 lectures)

a) Basic tools: permutations, combinations, generating functions, partitions, binomial coefficients, recurrence relations, difference equations, inclusion-exclusion. (10-12 lectures)

The concepts introduced in this section are fundamental and should be part of every applied mathematician's stock in trade. Typical applications of this material are literally too numerous to be singled out. See, e.g., [F], [Rio], [L], [Bec], [Sa], [Kn], [Pe].

Two standard sources are the initial chapters of [F] and [Rio], but these might tax some students a bit. [L] is easier to read but says less. Crisp discussions of most of the material are given in [Ry].

b) Somewhat more advanced material. Systems of distinct representatives, Möbius inversion, theorems of Ramsey type, block designs, Hadamard matrices. (3-4 lectures)

It is important for the student to see models which use somewhat more sophisticated concepts from combinatorial mathematics. Good examples of this are the studies of the dimer problem and the Ising model presented in [Pe] and the analysis of telephone switching networks in [Ben]. The topics listed in this section serve to introduce the student to more advanced ideas. (Of course, other similar topics listed in the available references may be substituted at the discretion of the instructor.) These subjects are covered adequately (although perhaps somewhat disjointly) in [Hal]. The treatment in [Ry] would be suitable for the better students. The relevant sections of [Hal] are suitable if more emphasis on block designs is desired. Historically, block designs arose primarily in the design of statistical experiments. Recently, these concepts have been useful in a variety of fields, e.g., coding theory [Berl], spectroscopy [SFP], and data compression. (Also see {5}.)

c) Pólya counting theory: equivalence classes, (permutation) groups, cycle structure, Burnsides' theorem, Pólya's theorem, generalizations. (6 lectures)

Historically, this subject arose from Pólya's work on enumerating chemical isomers ([Po]; see also {6}). Typical applications include enumeration of Boolean functions [Sle] and enumeration of random walks on lattices [Pe]. Other examples are also available in [L], [Be2], [Rio].

[L] is appropriate here if only minimal depth is required. [Bec, Ch.5] gives a more detailed picture. The presentation of [Rio] has a reputation of being somewhat hard to read. Pólya counting theory offers students an opportunity to apply some elementary concepts from group theory to their models. Of course, several additional lectures may be needed to prepare students who have had no exposure to the concept of an equivalence relation or a group. Numerous examples and applications of this material are available, e.g., [Sle], [L], [Be2], [Rio].

It should be kept in mind that the particular choice of models and results presented is not critical. The underlying object here is to develop in the student a feeling for the formulation and analysis of various models using the ideas of combinatorial mathematics.

Many of the topics covered involve techniques for which efficient algorithms are known (e.g., network flows, matching, connectivity, and planarity). It would be quite appropriate for students to implement these algorithms on computers if facilities are available. This very effectively illustrates the savings in time and money achieved by using an efficient algorithm rather than, for example, an enumerative search.

References

In the list of references below, there is no attempt to be exhaustive. Each primary reference is accompanied by a short description and a suggestion whether it is of interest mainly to the instructor (I) or to a student in the k^{th} year of college (S-k). References to Mathematical Reviews (MR) are given.

Primary References

[Bel] Berge, Claude. The Theory of Graphs and its Applications. New York, John Wiley and Sons, Inc., 1962, 247 p. MR21 #1608. (I; S-3,4) This is one of the original standard texts on graph theory. In addition to the standard topics, e.g., chromatic numbers, connectivity, planarity, Hamilton paths, and transport networks, this volume contains several nice chapters dealing with games on graphs and Sprague-Grundy functions. There are no exercises and a moderate selection of examples.

[BS] Busacker, Robert G. and Saaty, Thomas L. Finite Graphs and Networks: An Introduction with Applications. New York, McGraw-Hill Book Company, 1965, 294 p. MR 35 #79. (S-2) This is a nice introduction to the basic topics of graph theory, slanted somewhat toward applications to network theory. A major feature of this book is the 140-page section on applications. They are varied and interesting and include applications to economics and operations research (linear programming and PERT), combinatorial problems, games, communication networks, statistical mechanics, chemistry, genetics, human sciences, group theory, and a number of other subjects. Exercises are included.

[FF] Ford, L. R., Jr. and Fulkerson, D. R. Flows in Networks. Princeton, New Jersey, Princeton University Press, 1962, 194 p. MR 28 #2916. (I) This book, written by two of the principal developers of the field, contains the most complete treatment of network flows. A sampling of the contents includes the max-flow min-cut theorems, the König-Egerváry theorem, sets of distinct representatives, linear programming and duality, Dilworth's theorem, minimal cost flow problems, and 0-1 matrices. There are some examples but no exercises. It is probably more useful as a reference than as a text.

[Hal] Hall, Marshall, Jr. Combinatorial Theory. Waltham, Massachusetts, Blaisdell Publishing Company, Inc., 1967, 310 p. MR 37 #80. (S-3,4) In addition to most of the standard topics, some less common subjects such as Möbius inversion and finite geometries are touched upon. By far the chief emphasis of the book is on block designs, a topic on which the author is well qualified to write. Some exercises and a few examples are contained in the book.

[L] Liu, C. L. Introduction to Combinatorial Mathematics. New York, McGraw-Hill Book Company, 1968, 393 p. MR 38 #3154. (S-1,2) This is a very well-rounded presentation of most of the basic concepts of both graph theory and combinatorial mathematics mentioned in the course outline. Numerous exercises and examples are contained in the book, although it is not particularly strong in mentioning applications to other fields. However, this defect is offset by an extensive bibliography. The book is on the whole quite readable and could easily serve as a textbook for a freshman-sophomore course.

[Ol] Ore, Oystein. *Theory of Graphs*. Providence, Rhode Island, American Mathematical Society, 1962, 270 p. MR 27 #740. (I; S-4+) This is currently the most mathematical treatment of graph theory available. The subject material ranges widely and includes, e.g., product graphs, Euler paths in infinite graphs, homomorphic images of graphs, the axiom of choice, partial orders, and groups and their graphs. Unfortunately, the density of definitions is rather high (especially at the beginning), and this may discourage many readers. The patient reader will be well-rewarded for his perseverance, however. While numerous examples are included, the primary orientation of the book is toward basic concepts rather than applications of graph theory.

[Rio] Riordan, John. *An Introduction to Combinatorial Analysis*. New York, John Wiley and Sons, Inc., 1958, 244 p. MR 20 #3077. (I; S-3,4) This is *the* standard modern text on combinatorial analysis. It contains complete discussions of permutations and combinations, generating functions, inclusion and exclusion, occupancy problems, permutations with restricted positions, and, above all, enumeration (including Pólya theory). There are many exercises and examples at a variety of levels with which the reader may test his skill. An extensive list of references is also included. This is suitable as a text for upper-division and good lower-division students or as a reference for an instructor.

[Ry] Ryser, H. J. *Combinatorial Mathematics*, MAA Carus Monograph 14. New York, John Wiley and Sons, Inc., 1963, 154 p. MR 27 #51. (I; S) This short book, already considered by many to be a classic in its field, provides the reader with an accurate (although necessarily abbreviated) introduction to some fundamental ideas in combinatorics. The subjects include permutations and combinations, inclusion and exclusion, recurrences, Ramsey's theorem and applications, systems of distinct representatives, 0-1 matrices, orthogonal Latin squares and the Bruck-Ryser theorem, block designs, and perfect difference sets. Each chapter concludes with a number of references. There are few examples and no exercises. The initial sections of the book could be read by freshmen; the latter material would be more suitable for a good junior or senior.

Additional References

Avondo-Bodino, G. Economic Applications of the Theory of Graphs. New York, Gordon and Breach, Science Publishers, Inc., 1962.

[Bec] Beckenbach, E. F., ed. Applied Combinatorial Mathematics. New York, John Wiley and Sons, Inc., 1964. (I; S-3,4) A formidable collection of chapters written by a number of experts in combinatorial fields. Especially good is the section by deBruijn on counting theory.

[Ben] Beneš, V. E. Mathematical Theory of Connecting Networks and Telephone Traffic. New York, Academic Press, Inc., 1965.

[Be2] Berge, Claude. Principles of Combinatorics. New York, Academic Press, Inc., 1971.

Berge, Claude and Chouila-Houri, A. Programming, Games, and Transportation Networks. New York, John Wiley and Sons, Inc., 1965.

[Ber1] Berlekamp, E. R. Algebraic Coding Theory. New York, McGraw-Hill Book Company, 1968.

Combinatorial Structures and Their Applications. Proceedings of Calgary International Conference on Combinatorial Structures and Their Applications. New York, Gordon and Breach, Science Publishers, Inc., 1970.

[ER] Erdös, P. and Rényi, A. "On random graphs I." Publicationes Mathematicae, 6 (1959), pp. 290-297.

[F] Feller, W. An Introduction to Probability Theory and Its Applications, vol. 1, 3rd ed. New York, John Wiley and Sons, Inc., 1968.

[GP] Gilbert, E. N. and Pollak, H. O. "Steiner minimal trees." SIAM Journal, 16 (1968), pp. 1-29.

Guillemin, E. A. Introductory Circuit Theory. New York, John Wiley and Sons, Inc., 1953.

[Har1] Harary, F. Graph Theory. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1969. A coherent presentation of the basic results in what could be called "exact" graph theory (as opposed to the asymptotic graph-theoretic results of Erdös and others, cf. [ER]).

Harary, F. ed. Graph Theory and Theoretical Physics. New York, Academic Press, Inc., 1967.

Harary, F. Proof Techniques in Graph Theory. New York, Academic Press, Inc., 1969.

Harary, F.; Norman, R. Z.; Cartwright, D. Structural Models: An Introduction to the Theory of Directed Graphs. New York, John Wiley and Sons, Inc., 1965.

Harris, B. ed. Graph Theory and Its Applications. New York, Academic Press, Inc., 1970.

Harrison, M. A. Introduction to Switching and Automata Theory. New York, McGraw-Hill Book Company, 1965.

Kattsoff, L. O. and Simone, A. J. Finite Mathematics with Applications in the Social and Management Sciences. New York, McGraw-Hill Book Company, 1965.

Kemeny, J. G.; Snell, J. L.; Thompson, G. L. Introduction to Finite Mathematics, 2nd ed. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1966.

[KL] Kernighan, B. and Lin, S. "A heuristic algorithm for the traveling salesman problem." Operations Research, to appear.

[Kn] Knuth, D. E. The Art of Computer Programming. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., Vol. I (1968), Vol. II (1969), Vol. III (1971). The first two volumes of this (intended) seven-volume series are literally overflowing with a wealth of results, applications, references, and problems having to do with computational (and combinatorial) algorithms (among other things).

König, D. Theorie der endlichen und unendlichen Graphen. New York, Chelsea Publishing Company, 1950.

[Kr] Kruskal, J. B., Jr. "On the shortest spanning subtree of a graph and the traveling salesman problem." Proceedings of the American Mathematical Society, 7 (1956), pp. 48-50.

[Li] Lin, S. "Computer solutions of the traveling salesman problem." Bell System Technical Journal, 44 (1965), pp. 2245-2269.

Proceedings of the Louisiana Conference on Combinatorics, Graph Theory, and Computing. Baton Rouge, Louisiana, Louisiana State University, March, 1970.

MacMahon, P. A. Combinatorial Analysis, vols. I and II reprinted in one volume (originally published in two volumes in 1915 and 1916). New York, Chelsea Publishing Company, 1960. This is the old classical treatise on combinatorial analysis.

Marcus M. A Survey of Finite Mathematics. Boston, Massachusetts, Houghton Mifflin Company, 1969.

Milne-Thomson, L. M. The Calculus of Finite Differences. New York, St. Martin's Press, Inc., 1951.

Minty, G. J. "On the axiomatic foundations of the theories of directed linear graphs, electrical networks, and network-programming." J. Math. and Mech., 15 (1966), pp. 485-520.

Moon, J. Topics on Tournaments. New York, Holt, Rinehart and Winston, Inc., 1968.

Noble, B., ed. Applications of Undergraduate Mathematics in Engineering. New York, The Macmillan Company, 1967 (especially Ch. 2 and Sect. 4.6).

International Conference on Combinatorial Mathematics, proceedings. New York Academy of Sciences, July 31, 1970.

Recent Trends in Graph Theory. Proceedings of the First New York City Graph Theory Conference. Jamaica, New York, St. John's University, June, 1970.

[O2] Ore, Oystein. Graphs and Their Uses. New York, Random House, Inc., 1963. A good reference for graph theory as applied to electrical network theory.

[Pe] Percus, J. K. Combinatorial Methods. New York University, Courant Institute of Mathematical Sciences, 1969. An excellent reference for the application of combinatorial techniques to some of the classical lattice statistics problems of physics, e.g., the dimer problem and the Ising model.

[Po] Pólya, G. "Sur le nombre des isomètres de certains composes chimique." C. R. Acad. Sciences, 202 (1936), p. 1554.

Richardson, W. H. Finite Mathematics. New York, Harper and Row, Publishers, 1968.

Rota, G.-C. "On the foundations of combinatorial theory, I." Theory of Möbius Functions. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 2 (1964), pp. 340-368.

[Sa] Saaty, T. L. Optimization in Integers and Related Extremal Problems. New York, McGraw-Hill Book Company, 1970. A recent book which contains a wide variety of interesting results and examples from optimization theory, many of which are well in the spirit of this course.

[SR] Seshu, S. and Reed, M. B. Linear Graphs and Electrical Networks. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1961.

[Si] Sinden, F. W. "Topology of thin film RC circuits." Bell Systems Technical Journal, 45 (1966), pp. 1639-1662.

[Sle] Slepian, D. "On the number of symmetry types of Boolean functions of n variables." Can. J. Math., 5 (1953), pp. 185-193.

[SFP] Sloane, N. J. A.; Fine, Terrence; Phillips, P. G. "New methods for grating spectrometers." Optical Spectra, 4 (1970), pp. 50-53.

Whitworth, W. A. Choice and Chance, reprint of the 5th ed. (1901). New York, Hafner Publishing Company, Inc., 1965.

Appendix

1. Organization X has offices located in a number of cities. It wishes to establish a communication network among all its locations so that any two offices may communicate with one another, possibly by going through some of the other locations. Furthermore, it is desired to minimize the total length (cost) of the network. How should the cities be connected? If one is allowed to locate switching junctions arbitrarily rather than just at the office locations, then how can a minimal network be obtained? (See [Kr] and [GP].)

2. A (traveling) salesman has a fixed set of locations (farm houses) that he is required to visit. He leaves from his home office, travels to each location once in some order, and then returns. In what order should he visit the locations in order to minimize his total distance, cost, time (energy)? (See [Li] and [KL].)

3. An oil company has a number of oil wells (sources) and a number of refineries, customers, etc. (sinks), all connected by some intricate network of pipelines. The portions of pipeline between various points of the network have different (known) capacities. How can one route the oil through the system in order to maximize the flow of oil to the sinks? What if the direction of flow in certain pipelines is restricted? What if there are several grades of oil available in varying amounts from the sources and it is desired to maximize the value of the mixture received at the sinks? (See [FF].)

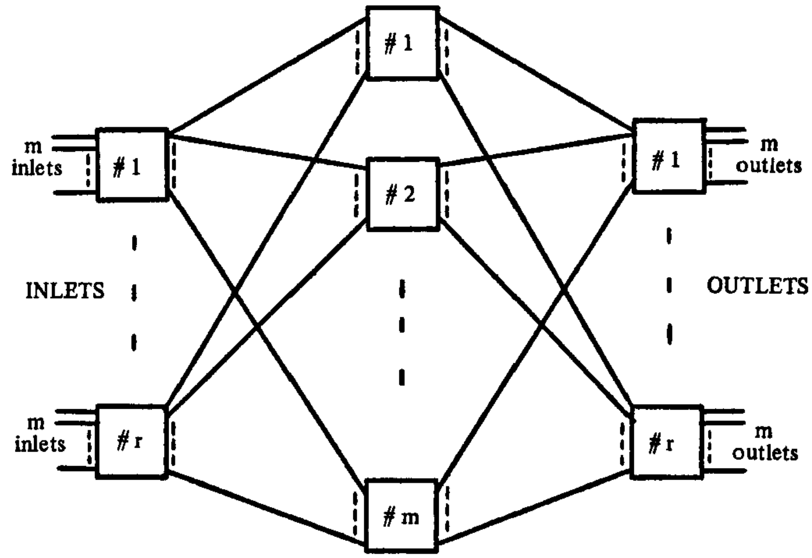
4. Certain integrated circuits can be made by depositing very thin metallic and dielectric films in suitable patterns on an insulating substrate. Ordinarily printed circuits are strictly planar; crossovers are made only by leading one of the conductors entirely out of the plane of the circuit. In the thin film technique, however, conductors can be separated by thin insulating layers within the plane of the circuit, causing a nonzero capacitance between the crossing conductors. Thus, crossovers can be permitted, provided this nonzero capacitance between the crossing conductors is permitted. The general problem is to determine which circuits can be realized by some suitable thin film circuit. This leads to a number of interesting questions in graph theory, one of which is the following: Given a set $S = \{s_1, \ldots, s_n\}$ of arcs or "strings," what are necessary and sufficient conditions on a set P of pairs $\{s_i, s_j\}$ so that there is some configuration of the s_k in the plane for which s_i and s_j intersect if and only if $\{s_i, s_j\}$ belongs to P? (See [Si].)

5. The Hall theorem on systems of distinct representatives occurs in a variety of applications. Several of these are:

a) In a certain company, n employees are available to fill n positions, each employee being qualified to fill one or more of these jobs. When can each employee be assigned to a job for which he is qualified? (See [Bel].)

b. An $m \times n$ chessboard has a certain subset of its squares cut out. When is it possible to place a collection of 2×1 "dominoes" on this board so that each of its squares is covered exactly once? (See [Pe].)

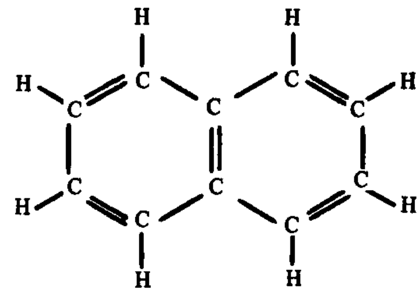
c) A telephone switching network connecting mr inlets with mr outlets is made up of three stages as indicated in the figure. (See [Ben].)



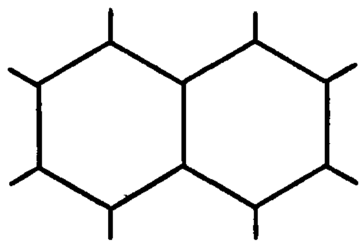
Each square box represents a switching unit for which any of the possible permutations of connecting its local inlets to its local outlets is possible. The problem is to show that this network is rearrangeable, i.e., given any set of calls in progress and any pair of idle terminals, the existing calls can be reassigned new routes (if necessary) so as to make it possible to connect the idle pair. How is the reassignment made so as to change the minimum number of existing calls? (cf. [Ben].)

d) If there are as many r-element subsets of an n-element set as there are k-element subsets, then it is possible to associate with each k-element subset a distinct r-element subset which contains it. How?

6. A naphthalene molecule $C_{10}H_8$ (See figure on next page.)



contains 8 hydrogen atoms which are available for substitution. The symmetry group G of the underlying figure



has order 4 and consists of the identity and three rotations about axes which are horizontal, vertical, or coming out of the page. Think of this group as just permuting the eight hydrogen atoms. The identity fixes them all and has cycle index S_1^8; each of the other three permutations moves them in four pairs of two each and contributes to the cycle index S_2^4. The cycle index of the group (considered as acting only on the hydrogen atoms) is thus

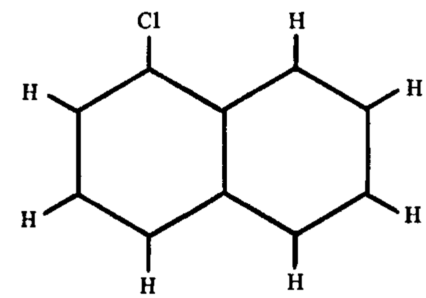
$$P_G(S_1,S_2) = \frac{1}{4}(S_1^8 + 3S_2^4).$$

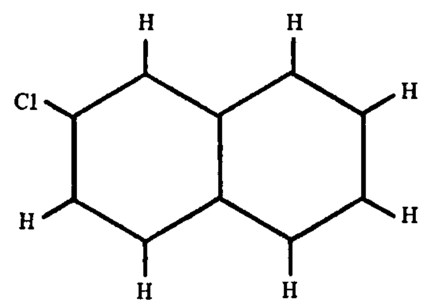
Now suppose we replace k of the hydrogen atoms by chlorine atoms and r of the hydrogen atoms by bromine atoms. How many different molecules can be formed? This is exactly the kind of question that Pólya's theorem answers.

Answer: In $P_G(S_1,S_2)$ replace S_1 by $1 + x + y$ and S_2 by $1 + x^2 + y^2$. Then the coefficient of $x^k y^r$ is the desired number. In fact, after making the substitution, we have

$$\frac{1}{4}\left[(1 + x + y)^8 + 3(1 + x^2 + y^2)^4\right] = 1 + 2x + 2y + 10x^2 + 14xy + 10y^2 + \ldots .$$

Each term in this series can be interpreted: 1 corresponds to the original molecule (no substitution); $2x$ corresponds to





(substituting one Cl for an H), etc. (For the notation used here, see [L] and [Bec]; problems of this type occur in [Pe].)

IX.3 FLUID MECHANICS OPTION

The course described below is based on a view of applied mathematics as a natural science distinguished from other natural sciences by a mathematical content that is significant in its own right. Fluid mechanics was chosen because it exemplifies applied mathematics in this sense: it is important historically, it encompasses many interesting physical problems, and it can be taught in the spirit of this report. However, to teach such a course at the undergraduate level requires special care in order to avoid the two possible extremes of, on the one hand, pursuing mathematical topics for their own sake and, on the other hand, studying physical models which involve only trivial mathematical ideas.

The approach to the subject proposed here has been selected with the audience and our objectives in view. Although this material can be taught from a more modern perspective, it would then require more sophisticated mathematical techniques and would be feasible only with very well prepared undergraduates. Our approach was selected because we feel that it is accessible to a wide audience and because it effectively attains our goals.

The course is intended for seniors. Prerequisites are elementary courses in calculus, differential equations, linear algebra, and physics. A course in advanced calculus or analysis is desirable. The student should be familiar, for instance, with the mathematical issues involved in the termwise integration of infinite series. This course should be valuable in solidifying and extending the student's grasp of areas of analysis and differential equations. The course outlined here does <u>not</u> assume prior knowledge of complex analysis, partial differential equations, or fluid mechanics.

A potential instructor of this course is faced with issues not present in the preceding outlines. It requires more specialized knowledge and would most easily be offered by someone with a background in applied mathematics. Nevertheless, we feel that the present outline is sufficiently detailed so that it can serve as a guide to instructors and so that it can encourage teachers to experiment with courses in this area. The main point which the instructor must keep in mind is that this is to be a course about applied mathematics using fluid mechanics as its representative element; it is not a course on fluid mechanics alone.

The main needs of the instructor, in addition to mathematics, are a basic knowledge of classical physics, a willingness to read, and perhaps above all an interest in nature. Those who are not specialists in fluid mechanics will find it particularly important to read this outline with one of the references at hand. While there are many books on fluid mechanics, there are very few which emphasize the point of view which the Panel has taken here. A list of books which may be helpful to the teacher is given in section 5 below, with brief comments. References to specific sections of some of these books are given for each of the topics in the detailed outline in section 4. Unfortunately, there seems to be no book which would be completely satisfactory as a text for this kind of a course; the book by Prandtl, which includes most of the topics in the outline discussed clearly from the physical point of view, is perhaps the most appropriate. But the mathematical side of many of these discussions will require appreciable expansion for the purposes of this course; for digressions of a more purely mathematical nature which will from time to time be appropriate, one can perhaps rely on the general mathematical background of the instructor.

COURSE OUTLINE

This course has two main parts, the first of a fairly general nature concerned with the mathematical formulation of continuum models for fluids and the second dealing with more specific problems illustrative of the more important simplified models. In the outline each part is broken down into several areas, for each of which some remarks in the style of a "catalog description," with some suggestions on the general approach, are given. These remarks are followed by a list of topics for each of several lectures on this area, with attention drawn to specific sections of the references in which a treatment of these topics is given. For definiteness, these specific references have been restricted mainly to the books of Prandtl and Yih. The format of the references is indicated by this example: [P] II:1.1 means section 1.1 in the second chapter of Prandtl's book.

1. Continuum models for fluids

This part of the course concerns primarily the formulation and basic properties of the principal mathematical models used in fluid mechanics. Here one can well emphasize the central role of model building in applied mathematics and the importance of models which are both mathematically self-consistent and capable of being critically compared with the experimental or observational facts which they are supposed to describe. Fluid mechanics is a particularly good example to illustrate that a mathematical model can be very helpful even though it is in a sense definitely incorrect (e.g., the molecular structure of matter is completely missing from continuum models) and that in reality all theoretical science is done in terms of models, none of which should be assigned any absolute validity.

a) The concept of continuous matter as a useful macroscopic model of real matter. (2 lectures) Mass and density. Kinematics: velocity field and the idea of a "fluid particle" as a theoretical concept in the continuum model, not the same thing at all as a molecule. "Eulerian" and "Lagrangian" variables and the mathematical form of the continuum model. The continuity equation.

Mathematical ideas: flow as a continuous mapping, Jacobians in the transformation of multiple integrals, the divergence theorem. One might well emphasize here the <u>reverse</u> of the familiar physical "proof" of the divergence theorem--the mathematical theorem shows that the continuum model is in accord with our intuitive ideas about the continuity of matter.

<u>References</u>. Continuum models for fluids; mass, density, "fluid particle," velocity, acceleration: [Y] I:1,2,3,4 and [P] II:1.1. Continuity equation, divergence theorem, and Jacobians: [Y] I:5,6 and [P] II:1.2. More kinematics: [Y] I:7,8,9.

b) Dynamics. (4 lectures) Introduction to the basic ideas from particle mechanics (momentum, force, kinetic energy) into the continuum model. Pressure and stress. Stress tensor and the momentum equation. Mechanical energy equation. Angular momentum and symmetry of the stress tensor in the absence of body torques and "torque-stresses."

Mathematical ideas: divergence theorem again, with more vector calculus. Tensors as geometric objects. Components of a symmetric second-order tensor form a symmetric matrix, hence have real eigenvalues and an orthonormal basis of eigenvectors (principal stresses).

<u>References</u>. Pressure and stress: [P] I:1,2,3. Concept of a tensor as a geometric object and its representation by components; stress tensor of a continuum: [Y] I:10. Yih's rather classical and formalistic view of tensors might well be given a more geometrical and contemporary flavor. Symmetrical second-order tensors, in particular stress and deformation tensors, relationship between a second-order tensor and the matrix of its components; application of linear algebra; principal stresses and eigenvalues, principal directions, etc.: [Y] I:11,12. Navier-Stokes form of the stress tensor and the corresponding fluid equations of motion: [Y] I:13 and [Y] II:1,2. Conservative body forces and the mechanical energy equation, vorticity equation: [Y] II:3,4.

c) Thermodynamics. (3 lectures) The equation of state. Internal energy, heat, and entropy. Heat conduction and the total energy equation.

In the absence of sufficient background in physics, this part may have to be limited mainly to equations of state in the simplest cases: incompressible fluids and the isothermal and adiabatic ideal gas. However, thermodynamics, where accessible, provides a good source of exercises in changing variables, Jacobians, etc., and also often illustrates rather well the advantages of a careful mathematical formulation over a loose intuitive description.

At this point various examples of hydrostatics problems can conveniently be introduced. Two important points to be emphasized here (and throughout the course in other contexts) are: i) Hydrostatics is a "simplified model," relevant not only when there is strictly no motion but also a good approximation in appropriate circumstances (vertical accelerations small compared with that of gravity). One can introduce here the idea of simplifying the model on the basis of the smallness of certain dimensionless parameters characteristic of the particular case in hand. ii) By discussing some problems related to familiar situations, one can help the student to form the habit of using mathematics to enhance his perception of nature. For example, the hydrostatics of the isothermal and adiabatic atmospheres can answer questions like: Is it plausible that oxygen should be needed when climbing Mt. Everest? or How much colder is it likely to be on the top of some local peak than it is at ground level?

Mathematical ideas: in addition to Jacobians, etc., some simple ordinary differential equations.

References. Review of thermodynamics: [Y] appendix I, possibly truncated and treated in a mathematically more sophisticated manner. Ideal gases: [Y] appendix I and [P] I:5. Heat or energy equation: [Y] II:8. Applications from hydrostatics: [P] I:6,7,10.

2. The more simplified models

Geometrical or physical parameters needed to specify a problem completely lead to characteristic dimensionless parameters (e.g., Mach number, Reynolds number) whose smallness or largeness in particular cases indicate the usefulness of simplified models (e.g., incompressible or inviscid flow). In the discussion of simplified models, emphasis should be shared between their general properties (e.g., Kelvin's circulation theorem) and careful consideration of the extent to which the simplified model is in fact relevant. In particular, the prevalence of nonuniform convergence in going over to the simplified model and the kinds of additional considerations required in the regions of nonuniformity ("boundary layers") should be brought out, at least qualitatively. In assessing relevance, it is probably best to include with the general discussion a number of applications of the basic models to concrete situations. Simple and familiar cases which emphasize the two points mentioned under 1c) in connection with hydrostatics should be considered where possible.

a) Ideal irrotational flow and surface waves on water. (5 lectures)

Here there are a number of opportunities for introducing important mathematical ideas and techniques, for instance: i) some general properties of harmonic functions; ii) solution of boundary value problems for Laplace's equation by superposition of wave solutions (i.e., "separation of variables" or use of Fourier representations); iii) free waves--phase and group velocity; iv) forced waves, e.g., the linear wave-maker problem (radiation condition at infinity, Sturm-Liouville equations, and eigenfunction expansions for boundary value problems).

If the students have not seen a proof of the Fourier series theorem, the instructor might like to insert a lecture on this topic, proving the theorem for piecewise continuously differentiable periodic functions.

References. General properties of the inviscid flow model, dimensionless form of general equations, and inviscid flow as idealization for large Reynolds number: [Y] II:5,6. Kelvin circulation theorem: [Y] III:1,2 and [P] II:2.8. Bernoulli theorems: [Y] III:8,9 and [P] II:2.3,4. Ideal irrotational flow: [Y] IV:1,2,3,4 and [P] II:2.9. Surface waves: [P] II:2.15 and [Y] V:1,2.1,2.2,2.3. Standing waves and group velocity: [Y] V:2.4,2.5,2.6,2.11. The wave-maker problem: [Y] V:2.13.

b) Linear shallow-water theory. (4 lectures)

This provides another simplified model and gives opportunity for further discussion of Sturm-Liouville eigenvalue problems. The relationship with variational techniques can be brought out here in the estimation with trial functions or comparison theorems of the resonant frequencies of soup bowls, swimming pools, harbors, and lakes.

Some properties of the wave equation, for instance the significance of characteristics, can also be included. (Nonlinear shallow-water theory, its analogy with compressible flow and shock waves might be discussed, but probably there will not be sufficient time for this.)

References. Linear shallow-water theory: [L] pp. 169-72, 189. Forced motion and normal modes: [L] pp. 176-79. Tides: [L] pp. 180-82, 198-200. Variable depth or section: [L] pp. 185-86, 191-93.

c) Ideal flow past bodies. (5 lectures)

Flow past circles and spheres gives simple problems in potential theory which can be tied in with Fourier series and spherical harmonics, notably by considering flow past near-circles or near-spheres; ideas of regular perturbation theory enter here as well.

D'Alembert's "paradox" provides a striking example of the failure of a simplified model when interpreted too literally, combined with its rescue and continued usefulness when the main source of the difficulty (flow separation) is identified and appropriately modeled. The elementary theory of airfoils and drag estimates via dynamic pressure arguments could be discussed with questions like: Why do sailplanes have very long slender wings? How big should a parachute be? How much air resistance is a car subject to?

References. Examples: [Y] IV:7.4, [P] II:2.9, and [Y] IV:18. Flow past a near-sphere: [Y] IV:13, possibly generalized and with further discussion of spherical harmonics. (Yih's discussion is perhaps too brief and formalistic, and the fact that surface harmonics are to spheres what sines and cosines are to circles is rather obscured.) Perhaps another mathematical digression could be added here: students are too often so put off by excessive emphasis on associated Legendre functions that they never seem to realize that the rotation group is behind it all. Two-dimensional flows with circulation: [P] II:10. Blasius and Kutta-Joukowski theorems: [Y] IV:22,22.1. Airfoils: [P] III:16,17.

d) Inviscid flow with vorticity. (3 lectures)

Some interesting phenomena of this sort can be studied without too much complication by considering linearized flow in rotating systems. Unfortunately, a more complete picture of the applications of hydrodynamic theory in meteorology and oceanography probably involves too many other considerations to be feasible in this course.

References. Geostropic flow and the Taylor-Proudman theorem: [G] I and [Y] III:12,12.1,12.2. Effect of the earth's rotation on atmospheric and oceanic flows: [P] V:8. Motion of parallel rectilinear vortex filaments: [L] pp. 154-55.

e) Viscosity. (4 lectures)

References. Couette and Poiseuille flow: [Y] VII:2.1,2.2,2.3, 2.10,2.11 and [P] III:1.9b. Ekman and Stokes-Rayleigh boundary layers: [Y] VII:2.12,3.4; [P] V:9; and [G] 2.3. Secondary flow: [P] III:8, [G] 2.4, and [P] V:9. The boundary layer: [P] III:3 and [Y] VII:5,6.

f) Instabilities. (2 lectures)

(Why does water run out of an inverted glass even though the atmospheric pressure can support the weight of a 30-foot column of water--and why does it not similarly run out of a narrow tube?) Kelvin-Helmholtz instability, although an over-simplified model, can be related to wave generation by wind.

References. Gravitational instability: [Y] IX:2.1,2.3 and [P] V:16. Kelvin-Helmholtz instability: [Y] IX:8.1, [L] p. 232, [Y] IX:6.1.

References

The books referred to in the outline are:

[G] Greenspan, H. P. The Theory of Rotating Fluids. New York, Cambridge University Press, 1968. A rather advanced text or research monograph.

[L] Lamb, H. Hydrodynamics. New York, Dover Publications, Inc., 1945 (reprint). The classic work in the field. Its rather old-fashioned mathematical style and extensive character,

combined with a certain tendency to present important results without adequate identification, make it sometimes rather difficult for the novice. As with some other classics, results given almost parenthetically in Lamb continue occasionally to be rediscovered (and published!).

[P] Prandtl, L. Essentials of Fluid Dynamics. New York, Hafner Publishing Company, 1952. An excellent and interesting book from the physical point of view, with clear discussions of many scientific and engineering applications. Most of the less elementary mathematical aspects, however, have (intentionally) been left aside.

[Y] Yih, Chia-Shun. Fluid Mechanics. New York, McGraw-Hill Book Company, 1969. A good graduate-level textbook, emphasizing the theoretical side. Quite a few exercises.

Some other books which the instructor may find useful to have on hand are listed below.

Additional References

Batchelor, G. K. An Introduction to Fluid Mechanics. New York, Cambridge University Press, 1967.

Landau, L. D. and Lifschitz, E. Fluid Mechanics. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1959.

Rouse, H. Fluid Mechanics for Hydraulic Engineers. New York, McGraw-Hill Book Company, 1938.

Von Arx, W. S. Introduction to Physical Oceanography. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1962.

The books by Batchelor and by Landau and Lifschitz are both good; Landau and Lifschitz is written perhaps more from the physicist's point of view, Batchelor from the applied mathematician's.

Also, a good engineering text such as the book by Rouse and, in connection with 2d), the book by Von Arx, may be found helpful.

There are a number of interesting 8mm. film strips on topics in fluid mechanics, as well as some longer films, prepared by the National Committee for Fluid Mechanics Films and available from Encyclopaedia Brittanica Films. They do not on the whole contribute much on the mathematical side but may well add interest and appreciation for the physics. Some which might be found useful in connection with the course outlined above are:

FM-3: Shear Deformation of Viscous Fluids [continuity equation]

FM-14A and B: Visualization of Vorticity with Vorticity Meter [continuity equation, conservative body forces and the mechanical energy equation, vorticity equation]

FM-13: The Bathtub Vortex [general properties of the inviscid flow model, two-dimensional flows with circulation, effect of the earth's rotation on atmospheric and oceanic flows]

FM-10: Generation of Circulation and Lift for an Airfoil [airfoils]

FM-11: The Magnus Effect [secondary flow]

FM-6: Boundary Layer Formation [the boundary layer, Ekman and Stokes-Rayleigh boundary layers]

FM-31: Instabilities in Circular Couette Flow [instabilities, Couette and Poiseuille flows]