COMMENTARY ON

"A GENERAL CURRICULUM IN MATHEMATICS

FOR COLLEGES"

A report of

The *ad hoc* Committee on the Revision of GCMC

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I. PREAMBLE--THE NEED FOR REAPPRAISAL

In 1965 the Committee on the Undergraduate Program in Mathematics (CUPM) published a report entitled A General Curriculum in Mathematics for Colleges (GCMC); this report has had an extensive influence on undergraduate mathematics programs in U. S. colleges and universities. Earlier CUPM reports had recommended specific undergraduate programs in mathematics for a variety of careers (teaching; mathematical research; physics and engineering; biological, management, and social sciences; and computer science). In contrast, the GCMC report undertook to identify a central curriculum beginning with calculus that could be taught by as few as four qualified teachers of mathematics (or four full-time equivalents) and that would serve the basic needs of the more specialized programs as well as possible. The extent to which the GCMC report achieved its purpose is indicated by the large number of colleges that have revised their course offerings in directions indicated by that report. Indeed, its influence has been widespread in spite of its stringent, self-imposed restrictions.

Many departments offer courses in addition to those mentioned in the GCMC report, such as a mathematics appreciation course for students in the arts and humanities, courses for prospective elementary school teachers, courses for students whose high school preparation is seriously deficient in mathematics, and specialized courses for most of the careers mentioned above. Thus the four-man "department" of the GCMC report often consists of four full-time equivalents within a much larger department having 10, 15, or even more members.

Numerous conferences of collegiate mathematicians have been held, both by the Sections of the Mathematical Association of America (MAA) and by CUPM, to discuss the GCMC report and to identify difficulties in following its suggestions. Although the response has been generally favorable, two criticisms have been made repeatedly: (a) The pace of some course outlines is unrealistically fast and in particular leaves no time for applications. (b) Many of the colleges for which the GCMC report was intended have substantial commitments to programs that are not discussed in the GCMC report, and they would welcome assistance with their problems.

For these reasons CUPM felt that the GCMC report should be reviewed. Such a re-evaluation of the entire program has been in progress for two years, and this commentary is the result of these deliberations.

During this review of the GCMC report, many problems have been considered by CUPM, of which three central ones are briefly mentioned below. Aspects of the first two are subjects of other CUPM studies (see Section II). We hope that these problems will be considered by individual departments of mathematics in the light of their local conditions.

1) The Evolving Nature of Mathematics Curricula. During the recent past, mathematics has been growing at a phenomenal rate, both internally and in its interconnections with other human activities. The subject continues to grow, and its influence continues to broaden beyond the traditional boundaries of pure mathematics and classical applied mathematics to include statistics, computer science, operations research, mathematical economics, mathematical biology, etc. When thinking about undergraduate education, therefore, is it not now more appropriate to speak of the mathematical sciences in a braod sense rather than simply mathematics in the traditional sense? Although large universities may have separate departments for the various aspects of the mathematical sciences, this alternative is not feasible at most colleges. Even in institutions where separate departments exist, how can one coordinate the various course offerings to take advantage of the impact that each branch of the mathematical sciences has upon the others and on related disciplines?

A closely related question is whether the "core" of pure mathematics that all departments should offer is now the same as it was presumed to be a few years ago. As new fields develop, some older fields seem less relevant, and today some mathematicians even question the assumption that calculus is the basic component of all college mathematics.

However, we wish to emphasize that no matter what changes occur in the undergraduate mathematics curriculum, one of the desirable alternatives will surely include basic calculus and algebra courses closely akin to Mathematics 1, 2, 3, 4, and 6 of the GCMC report.

2) The Service Functions of Mathematics. Mathematically educated people are needed in many kinds of work. It is therefore pertinent to ask whether the present undergraduate curriculum is sufficiently broad, especially in the freshman and sophomore years, to meet the mathematical needs of students interested in preparing for a variety of careers.

The traditional mathematics curriculum was heavily weighted toward analysis and its applications to physical sciences. One of the major innovations of the program in the original GCMC report was the introduction of linear algebra in the sophomore year and probability in the freshman year, thus exposing a large number of undergraduates to a wider range of mathematical topics. But because of its limited scope, the 1965 GCMC program is necessarily a single-track system, or essentially so. Should a college de-emphasize calculus and offer a variety of entrances and exits in its lower-level mathematics program, assuming that it has adequate staff? If so, what options should be available, and what advanced work should follow these courses? What service courses should be given? How should courses be taught in the light of the availability of computers? How should students be introduced to the mathematics needed for modern applications in the behavioral, biological, and engineering sciences?

3) The Initial Placement of Students. Although increasing numbers of college freshmen arrive with mathematical preparation that qualifies them for advanced placement, there is a simultaneous need for a greater variety of precalculus courses; the latter problem is especially critical at colleges having a policy of open admission. Does the mathematics curriculum provide suitable points of entry and exit for all students? Are placement procedures and policies in mathematics sufficiently flexible?

Thus, for a variety of reasons, it is no longer clear that there should be a single general curriculum in mathematics. Several alternative curricula in mathematics are emerging, and colleges with limited resources will soon have to make difficult choices from among these alternatives.

II. THE NATURE OF THIS STUDY

The intention of CUPM in establishing a committee to review the GCMC report was to publish a new version, incorporating changes as needed to correct deficiencies in the original study and modifying the curriculum in accordance with new conditions in mathematics and mathematics education. Some of the technical shortcomings of the original course outlines (pace and content) proved to be manageable and are taken up below, whereas other problems mentioned in Section I are more difficult, both intrinsically and in their effect on the whole concept of a compact general curriculum. Several of these problems have been considered by other CUPM panels. They include:

(a) Basic mathematics. See A Course in Basic Mathematics for Colleges (1971) and A Transfer Curriculum in Mathematics for Two-Year Colleges (1969).

(b) The training of elementary and secondary school teachers. See Recommendations on Course Content for the Training of Teachers of Mathematics (1971).

(c) A program in computational mathematics. See Recommendations for an Undergraduate Program in Computational Mathematics (1971).

(d) The impact of the computer on the content and organization of introductory courses in mathematics. See Recommendations on Undergraduate Mathematics Courses Involving Computing (1972).

(e) Upper-division courses in probability and statistics. See Preparation for Graduate Work in Statistics (1971).

(f) Lower-division courses in statistics. See Introductory Statistics Without Calculus (1972).

(g) Courses in the applications of mathematics. See Applied Mathematics in the Undergraduate Curriculum (1972).

(h) New teaching techniques and unusual curricula. See Newsletter #7, "New Methods for Teaching Elementary Courses and for the Orientation of Teaching Assistants" (not included in this COMPENDIUM).

Clearly, a definitive restatement of the GCMC report, if possible at all, would have to take into account not only these reports but others that will yet emerge from further study. However, suggestions for improvements in the recommendations of the GCMC report have been developed, and there is no need to defer their publication until a comprehensive reformulation is completed. Accordingly, the present pamphlet gives the current suggestions of CUPM for the half-dozen courses that include a substantial part of the mathematics enrollment in almost all colleges, namely first- and second-year calculus, linear algebra, and the elements of modern algebra. In the next section we shall discuss our proposed changes and our reasons for proposing them. It is entirely possible that when the questions raised in Section I are answered, the needs of large numbers of students will be met more adequately by some completely new selections of courses, rather than by the traditional ones. However, as we stated in Section I, basic calculus and algebra courses like Mathematics 1, 2, 3, 4, and 6 of the GCMC report will surely continue to be taught. Thus, those departments that have made or are making efforts to implement the recommendations of the 1965 GCMC report should continue to do so, with attention to the changes of detail proposed in Section III, changes that do no violence to the basic content of the core program originally proposed.

III. NEW DESCRIPTIONS OF THE BASIC CALCULUS AND ALGEBRA COURSES

As CUPM did in 1965, we use two devices to obtain enough flexibility to accommodate the diversity of achievement and ability of college freshmen. We describe a basic set of semester courses rather than year courses; this arrangement makes it easier for students to take advantage of advanced placement or to leave the mathematics program at a variety of levels. We also suggest that, wherever possible, a college should offer the basic courses Mathematics 1 through 4 every semester. This allows advanced placement students to continue a normal program in mathematics without interruptions. Moreover, students who need to begin with precalculus mathematics can follow it immediately with a calculus sequence.

The following list of basic courses is deliberately given with bare "college catalogue" descriptions, for we do not wish to seem overly prescriptive. In Section IV of this report, however, we include detailed course outlines and commentaries which are meant to identify those topics that we feel are most significant and to convey the spirit in which we recommend that these basic courses be taught.

Mathematics 1. Calculus I. Differential and integral calculus of the elementary functions with associated analytic geometry. [Prerequisite: Mathematics 0 or its equivalent. A description of Mathematics 0 is given in Section VI.]

Mathematics 2. Calculus II. Techniques of integration, introduction to multivariable calculus, elements of differential equations. [Prerequisite: Mathematics 1]

Mathematics 3. Elementary Linear Algebra. An introduction to the algebra and geometry of 3-dimensional Euclidean space and its extension to n-space. [Prerequisite: Mathematics 2 or, in exceptional cases, Mathematics 0]

Mathematics 4. Multivariable Calculus I. Curves, surfaces, series, partial differentiation, multiple integrals. [Prerequisites: Mathematics 2 and 3]

Mathematics 6L. Linear Algebra. Fields, vector spaces over fields, triangular and Jordan forms of matrices, dual spaces and tensor products, bilinear forms, inner product spaces. [Prerequisite: Mathematics 3]

Mathematics 6M. Introductory Modern Algebra. The basic notions of algebra in modern terminology. Groups, rings, fields, unique factorization, categories. [Prerequisite: Mathematics 3]

(More upper-division courses are described in Section V, and outlines for them can be found in Section VI.)

A reader who is familiar with the 1965 GCMC report will notice at once that some significant changes are being proposed here.

In the first place, that document sketched only the broad outlines of a curriculum, giving for each course a (rather ample) college catalogue description. Those who accepted the broad outlines immediately had to face the specific details of implementation: What is a reasonable rate at which to cover new material for the average student? What specific topics can be included if this rate is to be achieved?

CUPM has now attempted to answer these questions by means of commentaries on the course outlines. We have tried to develop a sense of what is meant by "the average student," taking account of the changing capabilities and preparation of the students in most undergraduate courses. Because of the frequent objection that the rate apparently suggested by the 1965 GCMC report was unreasonably fast, we have made a special effort to be realistic about the material that can be covered and to offer suggestions about the pace and style of its presentation. The course outlines are intended as existence proofs rather than as prescriptive recommendations; they represent solutions that CUPM feels are feasible, but we are aware that these are not the only possible solutions. In fact, we encourage others to devise different and more effective ways of achieving the same ends.

The commentaries accompanying the course outlines attempt to convey some specific ideas about the manner of presentation that CUPM feels is appropriate. The suggested pace has been indicated by assigning a number of hours to each group of topics and, in many cases, by more detailed suggestions of what to omit, what to mention only briefly, what to stress. Since a standard semester contains 42 to 48 class meetings, we arbitrarily allowed approximately 36 hours for each one-semester course, representing class time mainly devoted to the discussion and illustration of new material; thus the assignment of, say, six hours to a topic is a guide to the relative proportion of time to be spent on the topic. CUPM hopes that the commentaries are sufficiently detailed to show that the suggested material, in the recommended spirit, can actually be covered in 36 hours. The slack time that we have left provides for tests, review, etc. CUPM feels that a department that wishes to cover additional topics, or to provide deeper penetration of the topics listed, should not attempt to crowd such material into the course as outlined, but rather should either move to courses of four semester-hours or lengthen the program.

The structure of the calculus sequence. The 1965 GCMC report envisioned a program extending over four semesters to cover the traditional subject matter of calculus courses augmented by elementary linear algebra. The present study, on the other hand, seeks to return to the tradition of a basic two-semester calculus course serving both as an introduction to further work in calculus and as a unit for students who will end their study at this point. What is not traditional is that this course (Mathematics 1 and 2) should be a self-contained introduction to the essential ideas of calculus of both one and several variables, including the first ideas of differential equations. Students who stop at the end of a year generally need calculus as a tool rather than as an end in itself or as preparation for a heavily mathematical subject like physics, and they ought to encounter all the main topics, at least in embryo. The present arrangement was suggested in 1965 only as an alternative to a more conventional arrangement. The arguments given above for the present arrangement seem so compelling that now CUPM does not wish to suggest any alternative for the first year of calculus.

We have, however, preserved the feature of GCMC which makes the first semester (Mathematics 1) a meaningful introduction to the major ideas of calculus (limit, derivative, integral, Fundamental Theorem) in a single-variable setting.

To achieve the aims both of Mathematics 1 in this spirit and of Mathematics 1 and 2 as set forth above, a very intuitive treatment is necessary. The course should raise questions in the minds of students rather than rush to answer questions they have not asked. We consider such a treatment to be the right one in any case. It serves the needs of the many students who are taking calculus for its applications in other fields. It is also appropriate for mathematics majors.

Although the recommended treatment is intuitive, it is not intended to be careless. Theorems and definitions should be stated with care. Proofs should be given whenever they constitute part of the natural line of reasoning to a conclusion but are not technically complicated. Those proofs that require detailed epsilon-delta arguments, digressions, or the use of special tricks or techniques should be consciously avoided. Every theorem should be made plausible and be supported by pictures when appropriate and by examples exhibiting the need for the hypotheses. It is often the case that such preparation for a theorem falls short of a proof by only a little. In such cases the proof should be completed. However, stress should always be placed on the meaning and use of the theorem. The following examples should clarify these ideas.

(1) A student may get along, at least for a while, without the formal definition of a limit. But limits, and all other concepts of calculus, should be taught as <u>concepts</u> in some form at every stage. For example, the Fundamental Theorem of Calculus involves <u>two</u> concepts: the "limit" of a sum and the antiderivative. The theorem states that if f is continuous and if

$$\int_a^b f(x)\,dx$$

has been defined by approximating sums, then

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where $F' = f$. There is, to begin with, no obvious relation between the two sides of this equation, and an effort is required to make it credible. One natural approach depends on proving that if

$$G(x) = \int_a^x f(t)\,dt,$$

then $G' = F' = f$, whence G and F differ by a constant which can only be $F(a)$. Thus a simple test to determine whether a student understands the Fundamental Theorem is to ask him to differentiate

$$G(x) = \int_0^x \sqrt{1+t^8}\,dt.$$

If he does not know how, he does not understand the theorem. It is dishonest to conceal the connection between the two concepts by conditioning the student to accept the formalism without his being aware that the concepts are there. On the other hand, to give the student only the concepts without making him fully aware of the formalism is to lose sight of the aspect of calculus that makes it such a powerful tool in applications as well as in pure mathematics.

(2) A "cookbook" course might teach the students to find the maximum of a function by setting its derivative equal to zero, solving the equation, and perhaps checking the sign of the second derivative; it might not discuss other kinds of critical points. A thoroughly rigorous course, on the other hand, might demand careful proofs of the existence of a maximum of a continuous function, Rolle's theorem, and so on. What we suggest for the first calculus course is a clear statement of the problem of maximizing a function on its domain, a precise statement of such pertinent properties as the existence of the maximum, and examples to indicate that the maximum, if it exists, may occur either at endpoints, points where the derivative equals zero, or points where the derivative does not exist.

The commentaries on Mathematics 1 through 4, given in Section IV, may also be consulted for a more detailed presentation of what we have in mind.

The computational aspects of calculus should be the center of attention in Mathematics 1 and 2. This means both the techniques of differentiation and integration and the numerical-computational methods that go along with them. Many people believe that a computer should be used, if possible, to supplement the formal procedures and reinforce their teaching. Guidelines on the use of computers in calculus courses appear in the report of the Panel on the Impact of Computing in Mathematics Courses (<u>Recommendations on Undergraduate Mathematics Courses Involving Computing</u>).

Finally, Mathematics 1 and 2, and indeed all the courses discussed here, should include examples of applications to other fields--the more concrete, the better.

The introduction of Mathematics 3 (Elementary Linear Algebra) was suggested in the 1965 GCMC report for the following reasons:

> Our arguments for placing a formal course in linear algebra in the first semester of the second year are more concerned with the values of the subject itself and its usefulness in other sciences than with linear algebra as a prerequisite for later semesters of calculus. Let us first consider prospective mathematics majors. Their official commitment to major in mathematics is usually made before the junior year of college. It is desirable that this decision be based on mathematical experience which includes college courses other than analysis. For these students linear algebra is a useful subject which

involves a different and more abstract style of reasoning and proof. The same contrasts could be obtained from other algebraic or geometric subjects but hardly with the same usefulness that linear algebra offers.

The usefulness of linear algebra at about the stage of Mathematics 3 is becoming more and more apparent in physics and engineering. In physics it is virtually essential for quantum mechanics, which is now being studied as early as possible in the undergraduate curriculum, especially in crystal structures where matrix formulation is most appropriate. In engineering, matrix methods are increasingly wanted in the second year or earlier for computation, for network analysis, and for linear operator ideas. The basic ideas and techniques of linear algebra are also essential in the social sciences and in business management. Students in these specialties are best served by an early introduction to the material in Mathematics 3.

We think, however, that Mathematics 3 is about the earliest stage at which the subject can profitably be taught to undergraduates generally. It can be taught to selected students in high school, though the high school version of the subject tends to be somewhat lacking in substance. High school students do not have a sufficiently broad scientific or mathematical background to motivate it and have not yet reached the stage of their curriculum when they can use it outside the mathematics classroom.

These reasons seem equally cogent today. However, CUPM is now more persuaded than in 1965 that it is important to have the terminology and elementary results of linear algebra available for the study of the calculus of several variables, and we propose a version of Mathematics 4 that takes as much advantage as possible of what the student has learned in Mathematics 3. How this can be done is explained in some detail in the commentary on Mathematics 4.

The present version of Mathematics 3 is a less demanding course than the Mathematics 3 described in the 1965 GCMC report, which indeed has frequently been criticized as containing too much material. Students who need more linear algebra than can reasonably be included in Mathematics 3 should also take Mathematics 6L.

In 1965 the GCMC report presented a calculus sequence that culminated in Mathematics 5, a course in vector calculus and Fourier methods. This has long been the accepted culmination of the calculus sequence. CUPM no longer feels that this material is to be regarded as basic in the same sense as the material of Mathematics 1 through 4. It is needed for graduate study of mathematics and for physics, but not for many other purposes. In fact, we do not suggest any single sequel to Mathematics 4 as part of the basic program but mention several possible courses at this level, recommending that each college choose one or more of these, or a course of its own design, according to its capabilities and the needs of its students.

Mathematics 6M (Introductory Modern Algebra) introduces the student to the basic notions of algebra as they are used in modern mathematics. We regard this course, or one of similar content, as an essential course that should be available in every college. We also recommend that every college that can do so offer a semester course containing further topics in linear algebra (Mathematics 6L; this is independent of Mathematics 6M). The rationale behind these recommendations is contained in the course descriptions.

IV. NEW OUTLINES FOR THE BASIC CALCULUS AND ALGEBRA COURSES

The following course outlines are intended in part as extended expositions of the ideas that we have in mind, in part as feasibility studies or existence proofs, and in part as proposals for the design of courses and textbooks. They are intended only to suggest content, not to prescribe it; they do, however, convey the spirit in which we believe the lower-division courses should be presented.

Mathematics 1. Calculus I.

[Prerequisite: Mathematics 0] Mathematics 1 is a one-semester intuitive treatment of the major concepts and techniques of single-variable calculus, with careful statements but few proofs; in particular, we think that epsilon-delta proofs are inappropriate at this level. We give a brief outline suggesting the amount of time for each topic; a more detailed commentary follows the outline.

COURSE OUTLINE

1. Introduction. (4 hours) Review of the ideas of function, graph, slope of a line, etc.

2. Limits, continuity. (3 hours) Limit and approximation defined intuitively. Derivatives as examples. Definition of continuity, types of discontinuity, Intermediate Value Theorem.

3. Differentiation of rational functions; maxima and minima. (5 hours)

4. Chain rule. (3 hours) Include derivatives of functions defined implicitly, inverse function and its derivative.

5. Differentiation of trigonometric functions. Higher derivatives. (3 hours)

6. Applications of differentiation. (3 hours) Tangent as "best" linear approximation. Differential, approximations using differentials. Extrema, curve sketching.

7. Intuitive introduction to area. (2 hours)

8. Definite integral. (3 hours)

9. Indefinite integrals, Fundamental Theorem. (4 hours)

10. Logarithmic and exponential functions. (3 hours)

11. Applications of integration. (3 hours)

COMMENTARY ON MATHEMATICS 1

The idea of this course is to provide the student with some understanding of the important ideas of calculus as well as a fair selection of techniques that will be useful whether or not he continues his study of calculus. If all this is to be done, formal proofs must necessarily be slighted. The following comments attempt to bring out the spirit that we have in mind.

1. Introduction. The basic ideas of slope of a straight line and of functions and their graphs can be reviewed in the context of an applied problem leading to the search for an extreme value of a quadratic or cubic polynomial. The ideas of increasing and decreasing functions and of maxima and minima should appear early. The direction of a graph at a point can be introduced as the limiting slope of chords. No formal definition of a limit need be given here: the derivative can be understood as a slope-function, and the vanishing of the derivative can be explored. Alternative interpretations are useful: derivative as velocity, as rate of change in general, and abstractly as $\lim\limits_{x \to a} \frac{f(x)-f(a)}{x - a}$ using the intuitive idea of a limit. Derivatives of the functions $x \to x^2$, the general quadratic function, $x \to 1/x$, $x \to \sqrt{x}$ can be determined. The need for a deeper study of limits can be shown by the attempted computation of $f'(0)$ for $f: x \to \sin x$. Students can use tables or a computer to obtain values of $\frac{\sin x}{x}$ for x near 0.

2. Limits, continuity. We do not intend that this should be a rigorous treatment with ϵ-δ proofs. Rather, the presentation of continuity and the Intermediate Value Theorem should strive to make the definitions and the theorems (and the need for their hypotheses) clear by pictorial means. Limit theorems for sums, products, and quotients should be mentioned and various types of discontinuity illustrated by examples. A discontinuity not of jump type can be illustrated by sketching $\sin(1/x)$ near $x = 0$. The students should be convinced that rational functions are continuous (except at zeros of the denominator).

3. Differentiation of rational functions. The definition of derivative can be repeated with alternative notations:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} .$$

(It is desirable for students to be aware of all the notations that they are likely to meet in other subjects.) Application of limit theorems will yield differentiation formulas for integral powers, sums, products, polynomials; products and quotients; higher derivatives. Calculation of maxima and minima furnishes an immediate application. The distinction between local and global extrema needs to be made here. For curve-sketching, one can make good use of the proposition that a continuous function is monotone between successive local extrema. This is intuitively clear from a diagram and is easily proved.

4. Chain rule. Composite functions can be thought of as compositions of mappings from a line to a line. If the derivative is thought of geometrically as a local magnification, the chain rule then expresses the result of two successive magnifications.

It is worth exploring the geometrical interpretation of the derivative for the inverse function in terms of reflection in the line $y = x$.

5. Differentiation of trigonometric functions. An appropriate argument for demonstrating the value of $\lim_{x \to 0} \frac{\sin x}{x}$ is the geometrical argument using areas (which can be more readily justified than the one using lengths).

6. <u>Applications of differentiation</u>. The Mean Value Theorem is needed here. When this theorem is discussed, the student should see its pictorial representation and should understand that the conditions placed on the function (continuity on the closed interval, differentiability on the open interval) are no more, and no less, than is necessary.

The phrase "tangent as 'best' linear approximation" is intended to suggest the geometric meaning of the formula

$$f(x) = f(a) + f'(a)(x-a) + E,$$

where $E/(x-a) \to 0$ as $x \to a$.

7. <u>Intuitive introduction to area</u>. What is intended is a presentation along such lines as the following: Properties of area (e.g., $A(S) \geqq 0$, $S \cap T = \emptyset \implies A(S \cup T) = A(S) + A(T)$). Area of rectangle accepted from geometry. Area within closed curve expressed in terms of areas under the graphs of functions. Approximation from above and below by sums of areas of rectangles. Idea of area as a limit by squeeze between upper and lower estimates. Error estimates (pictorially obtained) for monotone functions.

8. <u>Definite integral</u>. No formal proofs of the existence and properties of the integral are expected. A possible outline is as follows: Integral as a formal generalization of the idea of area--a number approximated by upper and lower sums formed for any function regardless of sign. Integral as limit of Riemann sums. Interpretation of integral as signed area. Integral of af and of $f + g$. Reversal of order of limits of integration. $\int_a^b f + \int_b^c f = \int_a^c f$. If $f(x) \leqq g(x)$, then, for $a < b$, $\int_a^b f \leqq \int_a^b g$.

Improper integrals: a 15-minute introduction to the idea, with some simple illustrative examples.

9. <u>Indefinite integrals, Fundamental Theorem</u>. Integral as a function of the upper endpoint, $F(x) = \int_a^x f(t)\,dt$. Intuitive discussion of the derivative of this function for continuous f; one can geometrically motivate the inequalities $\min_{[x,x+h]} f(t) \leqq \frac{F(x+h) - F(x)}{h} \leqq \max_{[x,x+h]} f(t)$ and then apply the squeezing or

pinching principle. The student should have some practice in the use of simple substitutions to evaluate integrals by the use of the Fundamental Theorem, including integrals of trigonometric functions. A brief introduction to tables of integrals is desirable at this point, to be continued in the next section when more functions are available.

10. Logarithmic and exponential functions. The definition of the logarithm as an integral is recommended.

One can give a heuristic argument for the formula for differentiating the logarithmic function: from assumed differentiability of the exponential function $f: x \to a^x$ $(a > 0)$, obtain $f'(x) = f'(0)a^x = Cf(x)$; hence, for the inverse function $g: x \to \log_a x$, note that $g'(x) = 1/(Cx)$. This is one way of suggesting the definition of the logarithmic function as an integral.

The Fundamental Theorem can be used to derive some basic rules for logarithms. For example, using $D(\log ax) = \frac{1}{x} = D(\log x)$ and integrating from 1 to b, one obtains $\log(ab) - \log(a) = \log(b) - \log(1)$ or $\log(ab) = \log(a) + \log(b)$.

Integration exercises requiring simple substitutions and the use of integral tables may be continued with special emphasis on integrands involving logarithmic and exponential functions.

The discussion of the differential equation $y' = ky$ provides an alternate approach to the definition of the exponential function. One starts with the solution $y = y_o e^{kx}$ for the differential equation with initial condition $y'(0) = y_o$. To show that this initial value problem defines the exponential function, we must prove that the problem has a unique solution. To do this, suppose z is any solution. Let $u = ze^{-kx}$. Then $z = ue^{kx}$ and, since $z' = kz$, it follows that $u' = 0$. Hence $u =$ constant and the initial condition requires $u = y_o$. Hence $z = y$ and the solution is unique. The discussion of the equation $y' = ky$ also leads naturally to a discussion of growth and decay models as in the next section.

Students may be reminded at this point of the basic rules for operations with exponents, and these rules may be justified.

With the derivatives of logarithmic and exponential functions available, it is now possible to justify the expected rule for differentiating general powers and hence to provide more diversified drill problems on differentiation of elementary functions.

Further use of tables of integrals is now possible and is recommended in place of integration by ingenious devices. Of course, students must be able to make simple substitutions in order to use integral tables effectively.

11. Applications of integration. It is very desirable for the students to see applications of integration to as many fields as possible besides geometry and physics. Since such applications do not yet appear in many textbooks, we have included some specific suggestions with references to places where more information can be found.

It is particularly desirable to have some applications of the integral as a limit of Riemann sums, not merely as an antiderivative. Examples like the following can be used: defining volume of a solid by the parallel slice procedure; defining work done by a variable force applied over an interval as an integral over that interval suggested by Riemann sums; defining the capital value of an income stream obtained over time at a given rate and with interest compounded continuously as the limit of a Riemann sum (see Allen, Roy G. Mathematical Analysis for Economists. New York, St. Martin's Press, Inc., 1962).

An intuitive understanding of probability density (perhaps using the analogy with mass density for a continuous distribution of mass on a line) can also supply sufficient background for interesting applications of definite integrals, since if f is the probability density function (pdf) of a random variable X, then $\Pr(a < X < b) = \int_a^b f(x)\,dx$. Such important practical pdf's as the exponential and normal can be introduced, as well as the uniform, triangular, and other pdf's defined on a finite interval, e.g., $f(x) = 3(1 - x)^2$ if $0 \leqq x \leqq 1$, $f(x) = 0$ elsewhere. The normal pdf offers an opportunity to point out a function that cannot be integrated in elementary form and for which tables are available.

At the conclusion of this semester course, one is able to discuss the growth of a population governed by a differential equation of the form $N'(t) = (a - bN)N$. Here $N(t)$ is the size of the population at time t. If $b = 0$, then we have exponential growth with growth coefficient a. If, however, the growing population encounters environmental resistance (due to limited food or space, say), then $b > 0$ and the differential equation model involves a growth coefficient $(a - bN)$ that diminishes with increasing population size. This leads, when the differential equation is solved, to the logistic curve.

This differential equation and the corresponding logistic curve arise in many different contexts: (i) in the study of the phenomenon of diffusion through some population of a piece of information, of an innovative medical procedure, of a belief, or of a new fashion in clothes (see Coleman, James S. Introduction to Mathematical Sociology. New York, Free Press, 1964); (ii) in epidemiology where one studies the spread of a communicable disease (see Bailey, N. T. The Mathematical Theory of Epidemics. New York, Hafner Publishing Company, 1957); (iii) in biological studies of the size of populations of fruit flies as well as in demographic models of the U. S. population (see references in Keyfitz, Nathan. Introduction to the Mathematics of Population. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1968); (iv) in the analysis of autocatalytic reactions in chemistry (see Frost, Arthur A. and Pearson, R. G. Kinetics and Mechanism; A Study of Homogeneous Chemical Reactions, 2nd ed. New York, John Wiley and Sons, Inc., 1961); (v) in studies of individual response and learning functions in psychology and in operations research models of advertising-sales relationships (see references in Rao, A. G. Quantitative Theories in Advertising. New York, John Wiley and Sons, Inc., 1970).

An hour or two spent on this differential equation offers an opportunity for students to review many parts of the course (inverse functions, the Fundamental Theorem, integration of a rational function, relationships between logarithms and exponentials, sketching the graph of a function with special attention to the asymptotic limiting population size $t \to \infty$). But the logistic example enables the instructor also to make other useful points: that mathematics is widely applied in not only the physical sciences and engineering, but also in the biological, management, and social sciences, and that the same piece of mathematics (the logistic differential equation) often makes an appearance in many different disguises and contexts. Finally, one can point out the progression from simple to more complex models (from pure exponential population growth to logistic growth) as one strives to develop mathematical models that better describe real-world phenomena and data, and one may conclude by pointing out that the logistic itself can be significantly improved by generalizations that take account of the age structure of the population and of stochastic and other complicating features of the growth process.

Mathematics 2. Calculus II.

[Prerequisite: Mathematics 1] Mathematics 2 develops the techniques of single-variable calculus begun in Mathematics 1 and extends the concepts of function, limit, derivative, and integral to functions of more than one variable. The treatment is intended to be intuitive, as in Mathematics 1.

COURSE OUTLINE

1. Techniques of integration. (9 hours) Integration by trigonometric substitutions and by parts; inverse trigonometric functions; use of tables and numerical methods; improper integrals; volumes of solids of revolution.

2. Elementary differential equations. (7 hours)

3. Analytic geometry. (10 hours) Vectors; lines and planes in space; polar coordinates; parametric equations.

4. Partial derivatives. (5 hours)

5. Multiple integrals. (5 hours)

COMMENTARY ON MATHEMATICS 2

1. Techniques of integration. The development of formal integration has been kept to the minimum necessary for intelligent use of

tables.

At the beginning of the course the instructor should review briefly the concepts of derivative, antiderivative, and definite integral, and should emphasize the relationships which hold among them (Fundamental Theorem). The importance of the antiderivative as a tool for obtaining values of definite integrals makes it desirable to have a sizable list of functions with their derivatives. This should motivate the study of the inverse trigonometric functions and the further development of integration methods through trigonometric substitutions and integration by parts. We recommend the use of the latter technique to obtain some of the reduction formulas commonly appearing in integral tables.

The instructor should point out that not all elementary functions have elementary antiderivatives and should use this fact to motivate the study of numerical methods for approximating definite integrals (trapezoidal rule, Simpson's rule). If students have access to a computer, they should be required to evaluate at least one integral numerically with programs they have written.

The improper integral with infinite interval of integration should be introduced. Comparison theorems should be discussed informally as there is not enough time for an excursion into theory. If additional time can be spared, the improper integral for a function with an infinite discontinuity in the interval of integration may be considered.

The method of "volumes by parallel slices" from Mathematics 1 should be applied here to find the volume of solids of revolution by the disk method.

2. Elementary differential equations. Solution of differential equations is a natural topic to follow a unit on formal integration, because it extends the ideas developed there and gives many opportunities to practice integration techniques. The coverage recommended below provides only a brief introduction to the subject, and it is intended that examples be simple and straightforward with time allowed for a variety of applications.

a. First-order equations. The notion of tangent field, solution curve. Separable equations. Linear homogeneous equations of first order. Applications: orthogonal trajectories, decay and mixing problems, falling bodies.

b. Second-order linear equations with constant coefficients. Homogeneous case, case of simple forcing or damping function; initial conditions. Applications: harmonic motion, electric circuits. These topics will require a brief discussion of the complex exponential function and DeMoivre's theorem.

3. Analytic geometry. Vectors and vector operations (sums, multiples, inner products; $\vec{i}$, $\vec{j}$, $\vec{k}$) should be introduced at the beginning of this unit because they greatly simplify the analytic geometry of lines and planes in 3-dimensional space. It is desirable to discuss the algebraic laws for vector operations, but proofs should be kept informal. The efficiency of vector notation can be illustrated by proving one or two theorems from elementary geometry by vector methods, e.g., that the three medians of a triangle intersect in a point.

Equations of lines and planes in 3-dimensional space should first be obtained in vector form and then translated into scalar equations. The students should be able to solve problems involving parallelism, orthogonality, and intersections; they should be familiar with the derivation (by vector methods) of the formula for the distance from a point to a plane.

A very brief introduction to polar coordinates is suggested. Students should learn how to draw simple polar graphs and to convert from x,y to r,θ and vice versa; they should be able to compute areas using polar coordinates.

The brief unit on parametric equations should include parametric representation of curves, motion along curves, velocity, acceleration, and arc length.

4. Partial derivatives. This section is intended to provide a basic acquaintance with functions of two or three variables and with the concept of and notation for partial derivatives.

Examples of functions of two or three variables should be given, and methods of representing such functions as surfaces by means of level curves or level surfaces should be shown. The partial derivatives $f_x(a,b)$ and $f_y(a,b)$ should be defined and explained geometrically as slopes of appropriate curves in the planes $y = b$ and $x = a$, respectively. The concept of a tangent plane to a surface at a point should be introduced. In particular, the tangent plane, if it exists, is generated by the tangent lines in the x- and y-directions. Let these be, respectively,

$$z = c + \alpha(x - a), \quad y = b$$

and

$$z = c + \beta(y - b), \quad x = a,$$

where $c = f(a,b)$, $\alpha = f_x(a,b)$, $\beta = f_y(a,b)$. The normal $\vec{N}$ to the plane must therefore be perpendicular to the directions $\vec{i} + \alpha\vec{k}$ and $\vec{j} + \beta\vec{k}$; hence $\vec{N} = - \alpha\vec{i} - \beta\vec{j} + \vec{k}$, and the tangent plane has the equation

$$z = \alpha(x - a) + \beta(y - b) + c.$$

Extremum problems may be treated briefly as follows: At a point (a,b,c) where $z = f(x,y)$ has a maximum or minimum value, the tangent plane, if it exists, must be parallel to the xy-plane. This gives the necessary conditions that $f_x(a,b)$ and $f_y(a,b)$ both vanish at an interior extremum. Examples should be given to show that this condition is not sufficient. The second derivative test for extrema may be stated and illustrated by examples. Applications should be considered, including the method of least squares.

Topics such as the general concept of differentiability, the chain rule, and implicit functions are not included. (If it is possible to spend another hour or two on this section, it would be worthwhile to invest the time in studying the directional derivative for $z = f(x,y)$, noting that the directions of greatest increase of the function are orthogonal to level curves.)

5. Multiple integrals. The notions of double and triple integrals should be introduced through consideration of areas, volumes, or moments. Evaluation of double integrals by means of iterated integrals can be made plausible by calculating the volume of a solid by integrating the cross-sectional areas. Computations in both rectangular and polar coordinates should be included.

Mathematics 3. Elementary Linear Algebra.

This course is an introduction to the algebra and geometry of R^3 and its extension to R^n. Most students electing Mathematics 3 will have studied some calculus, but Mathematics 2 need not be considered a prerequisite.

Since the content and methods of linear algebra are new to most students, this course should begin by emphasizing computation and geometrical interpretation in R^3, to allow the student time to absorb unfamiliar concepts. In the outline below, the first 18 hours are devoted to this phase of the course. During the second half of the course, many of the same ideas are re-examined and extended in R^n, so that theorem-proving techniques can be developed gradually. Classroom experience has shown that the two outlines given for Mathematics 3 in the original GCMC report are too extensive, so the content of this outline has been reduced. Students who need to go further in linear algebra should resume their study of this subject in Mathematics 6L.

In selecting topics for this first course in linear algebra we confirm the judgments of the 1965 GCMC report: (1) the course content should be as geometrical as possible to offset its natural abstractness; (2) the treatment of determinants should be very brief; (3) the next topics to abbreviate under pressure of limited time are abstract vector spaces and linear transformations.

To prepare students adequately for Mathematics 4, this course must provide a knowledge of vectors in R^n, geometry in R^n, linear mappings from R^n into R^m and their matrix representations, matrix algebra, and determinants of small order. These topics, coupled with the solution of systems of linear equations, also provide a very useful course for students in the social and life sciences, and applications to those subjects serve to enliven the course. This much can be accomplished in one semester, but careful planning is required, and the degree of generality attempted in this first course must be controlled. For most classes it will be necessary to defer to Mathematics 6L consideration of such topics as $n \times n$ determinants, eigenvalues and eigenvectors, canonical forms, quadratic forms, orthogonal mappings, and the spectral theorem.

The instructor is expected to use judgment in adjusting the level of this course to the ability of his class by deciding upon a proper balance between concreteness and generality. Not all theorems have to be proved, but all should be motivated convincingly and illustrated amply. Coordinate-free methods should be used for efficiency and generality in definitions, proofs, and derivations, but students should also be required to perform computations with n-tuples. The examples developed early in the course for R^2 and R^3 should be carried along as illustrations in R^n.

COURSE OUTLINE

1. Vector algebra and geometry of R^3. (7 hours) Vector sum and scalar multiple, with geometric interpretations. Basic properties of vector algebra, summarized in coordinate-free form. Linear combinations of vectors; subspaces of R^3. Points, lines, and planes as translated subspaces. Vector and cartesian equations of lines and planes in R^3. Dot product in R^3; Euclidean length, angle, orthogonality, direction cosines. Projection of a vector on a subspace; the Gram-Schmidt process; vector proofs of familiar geometric theorems. Cross product in R^3, interpreted geometrically; the triple scalar product and its interpretation as the volume of the associated parallelepiped.

2. Systems of linear equations. (4 hours) Geometric interpretation of one linear equation in three variables and of a system of m linear equations; geometric description of possible solutions. Systems of m linear equations in n variables; solution by Gaussian elimination. Matrix representation of a linear system. Analysis of Gaussian elimination as the process of reducing the matrix to echelon form by three basic row operations (transposition of two rows, addition of one row to another, multiplication of a row by a nonzero scalar), followed by backward substitution. The consistency condition; use of an echelon form of the matrix of the system to obtain information about the existence, uniqueness, and form of the solution.

3. Linear transformations on R^3. (7 hours) Linear dependence and independence; the use of Gaussian elimination to test for linear independence. Bases of R^3; representation of a vector relative to a chosen basis; change of basis. Linear transformations on R^2 and R^3; matrix representation relative to a chosen basis. Magnification of area by a linear transformation on R^2; 2×2 determinants. Magnification of volume by a linear transformation on R^3; 3×3 determinant expressed as a triple scalar product and as a trilinear alternating form. The algebra of 3×1 and 3×3 matrices, developed as a representation of the algebra of vectors and linear transformations. Extension to $m \times n$ matrices; sum, scalar multiple, and product of matrices.

4. Real vector spaces. (8 hours) R^n as a vector space; subspaces of R^n. Linear independence, bases, standard basis of R^n. Representation of a linear mapping from R^n to R^m by an $m \times n$ matrix relative to standard bases. Range space and null space of a linear mapping from R^n to R^m; vector space interpretation of the solution of a system of linear equations in n variables, homogeneous and nonhomogeneous. Axiomatic definition of a vector space over R. A variety of examples in addition to R^n, such as polynomial spaces, function spaces, the space of $m \times n$ matrices, solutions of a homogeneous s stem of linear equations, solutions of a linear homogeneous differential equation with constant coefficients. Subspaces; linear combinations; sum and intersection of subspaces. Linear dependence, independence; extension of a linearly independent set of vectors to a basis. Basis and dimension; relation of bases to coordinate systems.

5. Linear mappings. (6 hours) Linear mappings of one real vector space into another. Images and preimages of subspaces; numerous examples to illustrate the algebra of mappings. Range space and null space of a mapping and their dimensions. Nonsingularity. Matrix representations of a linear mapping relative to chosen bases; review of matrix algebra and its relation to the algebra of mappings. Important types of square matrices, including the identity matrix, nonsingular matrices, elementary matrices, diagonal matrices. The relation of elementary matrices to Gaussian elimination, row operations, and nonsingular matrices. Rank of a matrix; determination of rank and computation of the inverse of a nonsingular matrix by elementary row operations.

6. Euclidean spaces. (4 hours) Real inner products introduced axiomatically; examples. Schwarz inequality; metric concepts and their geometric meaning in R^n. Orthogonality, projections, the Gram-Schmidt process, orthogonal bases. Proofs of geometric theorems in R^n.

7. Determinants. (optional) If time is available, the properties and geometric meaning of 2×2 and 3×3 determinants may be used to motivate a brief treatment of $n \times n$ determinants. Emphasis should be given to properties of determinants that are useful in

matrix computations.

COMMENTARY ON MATHEMATICS 3

1. Vector algebra and geometry of R^3. The primary objectives of this first section are to develop geometric insight into R^3 and to gain experience in the methods of vector algebra. Vectors should be introduced both as ordered triples and as translations, the latter leading naturally to a coordinate-free interpretation. Algebraic properties of vectors should be stated in coordinate-free form; later in the course they can be taken as axioms for an abstract vector space. The geometry of lines and planes should be stressed, as should the geometric meanings of the dot and cross product. The triple scalar product should be shown to be an alternating trilinear form, later to be called a 3×3 determinant.

2. Systems of linear equations. The problem of determining the subspace spanned by a given set of vectors in R^3 leads directly to a system of m linear equations in three variables. The solutions of such a system can first be interpreted geometrically as intersections of translated subspaces to provide insight for the consideration of $m \times n$ systems. To solve a system of m linear equations in n variables, Gaussian elimination provides an effective algorithm that should be stressed as a unifying computational method of linear algebra. The system $AX = Y$ can be represented by the augmented matrix $(A|Y)$. A succession of elementary row operations can be used to replace the matrix $(A|Y)$ by $(E|Z)$, where E is in row echelon form. The solutions of $AX = Y$ coincide with those of $EX = Z$ and are easily obtained by backward substitution since E is in row echelon form. At this stage the major emphasis should be concrete and computational. Formal representation of elementary row operations by elementary matrices and the concept of row equivalence are considered in Section 5. For some classes it may be appropriate to suggest that the operations discussed above can be carried out with complex numbers as well as with real numbers.

3. Linear transformations on R^3. Linear independence, basis, linear transformations, and matrix representations are introduced

concretely here and then are repeated in the next section for R^n and for the general vector spaces to provide a gradual, spiral development of these important concepts.

Determinants are introduced geometrically for the 2×2 and 3×3 cases. Properties of these determinants should be observed in a way that facilitates generalization to $n \times n$ determinants, perhaps in a later course.

Matrix algebra arises naturally as a representation of the algebra of vectors and linear transformations on R^3 and then is easily generalized to matrices of arbitrary size.

4. Real vector spaces. Consideration of R^n can be motivated by a geometric interpretation of the algebra of $m \times n$ matrices. The basic concepts of linear algebra in R^n should be studied briefly as natural extensions of the same concepts in R^3. The stage is then set for a general study of real vector spaces in coordinate-free form, illustrated amply by a wide variety of familiar examples. Theorems of various degrees of difficulty can now be proved for any finite-dimensional vector space, and students can be expected to prove some of them.

The concepts of linear independence, basis, and dimension need to be illustrated with many examples. The student should understand that questions about linear independence reduce to questions about the solution of a system of linear equations to which Gaussian elimination provides an answer. The same method can be used to express a given vector in terms of a given basis.

A brief mention of complex vector spaces is appropriate for some classes.

5. Linear mappings. Properties of linear mappings, including rank and nullity and their relation to the dimension of the domain space, should now be treated generally. Prove that if R and T are nonsingular, then the rank of RST equals the rank of S. The isomorphism of matrix algebra with the algebra of linear transformations should be exploited. Elementary matrices, one for each of the three types of elementary row operations, can be used to effect row operations on matrices. A matrix is nonsingular if and only if it is the product of elementary matrices. For some nonsingular P, PA

is in echelon form. By observing that the column rank of a matrix in echelon form is the number of nonzero rows, one can show that the row rank and the column rank of any matrix are equal. Elementary row operations should be used to develop a constructive method for computing the inverse of a nonsingular matrix.

6. Euclidean spaces. The coordinate-free formulation of a real inner product as a bilinear, symmetric, positive-definite function from $V \times V$ to R, where V is a vector space over R, can be viewed as a natural abstraction of the dot product in R^3. Its role as a source of all metric concepts should be emphasized. The Schwarz inequality should be derived in coordinate-free from and then interpreted concretely in various inner product spaces to obtain the classical inequalities. The flavor of this section should be strongly geometric.

Mathematics 4. Multivariable Calculus I.

[Prerequisites: Mathematics 2 and 3] This course completes a four-semester introductory sequence of calculus and linear algebra, building on the intuitive notions of multivariable calculus from Mathematics 2 and the linear algebra of Mathematics 3. The four semesters contain all the topics that seem to us to be essential for every student who has only this much time to spend on calculus; subsequently, students with various interest will need different courses.

A considerable advance in conceptual depth should be possible in Mathematics 4, but there is not enough time for full formal proofs of the theorems; these proofs are not needed except by students who are going at least as far as Mathematics 12, and their omission makes it possible to cover more topics here. Since maximum use should be made of Mathematics 3 and since some of the material suggested here is not yet standard, we give a fairly extensive commentary on the outline.

COURSE OUTLINE

1. Curves and particle kinematics. (5 hours)
2. Surfaces; functions from R^m to R^1. (7 hours)
3. Taylor's theorem for $f: R^m \to R^1$. (5 hours)

4. Sequences, series, power series. (6 hours)
5. Functions from R^m to R^n $(m, n \leqq 3)$. (2 hours)
6. Chain rule. (5 hours)
7. Iterated and multiple integrals. (6 hours)

COMMENTARY ON MATHEMATICS 4

1. <u>Curves</u>. A (parametrically represented) curve in R^n is thought of here as the range of a function $f: R^1 \to R^n$ (with principal emphasis on $n = 2, 3$). Set $\vec{x} = (x_1,\ldots,x_n) = f(t)$. The idea of $\lim_{t\to\alpha} f(t) = \vec{a}$ can be introduced through $\lim_{t\to\alpha} |f(t) - \vec{a}| = 0$; this limit is the same as the component-by-component limit. Continuity can be defined via $\lim_{t\to\alpha} f(t) = f(\alpha)$. The derivative of f is associated with the tangent vector. A curve in R^2 or R^3 can be thought of as the path of a particle; the first and second derivatives with respect to time are then interpreted as velocity and acceleration. At this point plane curves should be reviewed with attention to curve tracing and convexity. The present point of view makes it easy to derive the reflection properties of the conic sections: for example, if $\vec{a}$ and $\vec{b}$ are the foci of an ellipse and $\vec{x}$ is a point on the ellipse, then

$$|\vec{x} - \vec{a}| + |\vec{x} - \vec{b}| = k.$$

Differentiate with respect to the parameter t, using

$$\frac{d|z|}{dt} = \frac{1}{|\vec{z}|}\left(\vec{z} \cdot \frac{d\vec{z}}{dt}\right),$$

to obtain

$$\vec{v} \cdot \frac{\vec{x} - \vec{a}}{|\vec{x} - \vec{a}|} = - \vec{v} \cdot \frac{\vec{x} - \vec{b}}{|\vec{x} - \vec{b}|},$$

where $\vec{v}$ = unit tangent vector. This implies that the rays to the foci from a point on the ellipse make equal angles with the tangent at that point.

2. <u>Surfaces</u>. Consider functions $f: R^m \to R^1$ with emphasis on the case $m = 2$, interpreting the graph of such a function as a surface in R^3. The Euclidean norm $|\ldots|$ in R^m is the most useful, but

it is sometimes also useful to have the maximum norm $\|\ldots\|$ and the inequality $\|\vec{x}\| \leqq |\vec{x}|$. The limit of $f: R^m \to R^1$ at $\vec{a}$ should be defined, and continuity should be defined by $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$. The derivative J of f at $\vec{a}$ can be introduced as the linear transformation from R^m to R^1 satisfying

$$f(\vec{x}) = f(\vec{a}) + J(\vec{x} - \vec{a}) + o(|\vec{x} - \vec{a}|)$$

(but the o-notation itself should not be introduced unless there is time to get the students thoroughly used to it). Thus with R^m as a space of column vectors, $f'(\vec{a})$ is a $1 \times m$ matrix (or row vector), also called the gradient. This should be illustrated especially for $m = 2$ and compared with the treatment of the tangent plane in Mathematics 2. Here $J = \text{grad } f|_{\vec{x} = \vec{a}} = (f_1(\vec{a}), f_2(\vec{a}))$, where $f_i(\vec{a}) = \frac{\partial f}{\partial x_i}|_{\vec{x} = \vec{a}}$. The directional derivative is the rate of increase of $f(\vec{x})$ in the direction of a given unit vector $\vec{t}$, namely $\vec{t}\cdot\text{grad } f$. The notation of differentials should be at least mentioned since books on other subjects will presumably continue to use it. From the present point of view, $df = \vec{v}\cdot\text{grad } f$, where $\vec{v}$ is an arbitrary vector, conventionally denoted by $\vec{i}\,dx + \vec{j}\,dy$. The gradient is a vector in the direction of maximal rate of increase and is orthogonal to level lines.

In general, J is the $1 \times m$ matrix (row vector) with components $\frac{\partial f}{\partial x_i}|_{\vec{x} = \vec{a}}$, $i = 1, \ldots, m$, and the idea of the directional derivative and of the gradient extend to the general case.

It is desirable to use the linear approximation also for nongeometric applications, in particular to estimate the effect on the computed value of a function resulting from small errors in the variables (conventionally done in differential notation).

The Implicit Function Theorem for $f(x,y) = 0$ should be treated geometrically. If J is not the zero vector, the level line $z = 0$ of the surface $z = f(x,y)$ defines a function $y = g(x)$ locally so that $f(x,g(x)) = 0$ (this should be treated with a picture, not a proof). The equation $\frac{dy}{dx} = -\frac{\partial f}{\partial x}/\frac{\partial f}{\partial y}$ follows.

3. <u>Taylor's theorem</u>. Begin with $f: R^1 \to R^1$. An easy approach to the the theorem assumes $|f^{(n+1)}(t)| < M$ for $|t - a| < |x - a|$; repeated integration on (a,x) yields

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x - a)^k + R_n(x),$$

where

$$|R_n(x)| < \frac{M|x - a|^{n+1}}{(n+1)!}.$$

Typical examples: binomial, sine, cosine, exponential, logarithm, arctangent. Such examples lead naturally to the idea of convergence of an infinite series.

As an application one can expand $f: R^2 \to R^1$ to second-degree terms, first with respect to x and then with respect to y, and in reverse sequence; assuming continuous third derivatives one then shows that

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

Taylor's theorem can now be derived for $f: R^2 \to R^1$ and applied to extreme value problems. (Extreme value problems for $f: R^m \to R^1$, $m > 2$, should be treated lightly, if at all.)

4. <u>Sequences, series, power series</u>. It is appropriate to introduce the epsilon and neighborhood definitions of limit of sequences and series of constants, but little attention need be paid to conditional convergence; in the context of this course, absolute convergence is the significant idea. The comparison test, ratio test, and integral test can be treated.

For power series it is important to know that there are an interval and a radius of convergence; a useful formula for the radius is $\lim |a_n/a_{n+1}|$, provided the limit exists. The students should know that the differentiated and integrated series have the same interval of convergence as the original series; proofs can be omitted unless there is ample time. Applications; for example, approximate computation of $\int_0^x e^{-t^2} dt$ for small x.

5. Functions from R^m to R^n. Interpret $f: R^m \to R^n$ in various ways, e.g., the graph as a subset of R^{n+m}; representation of the range of f as a hypersurface. Interpretation by vector fields, e.g., stationary field of force or stationary flow ($R^3 \to R^3$); parametric representation of a surface ($R^2 \to R^3$); unsteady plane flow ($R^3 \to R^2$). Limit and continuity of f: $\vec{L} = \lim_{\vec{x}\to\vec{a}} f(\vec{x})$ means

$\lim_{|\vec{x}-\vec{a}|\to 0} |f(\vec{x}) - \vec{L}| = 0$. Note that this is equivalent to taking the limit component by component: setting $\vec{z} = f(\vec{x}) = (z_1,\ldots,z_n)$, f can be considered as an ordered set of n mappings $\varphi_k: \vec{x} \to z_k$ from R^m to R^1, and $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = \vec{b}$ if and only if $\lim_{\vec{x}\to\vec{a}} \varphi_k(\vec{x}) = b_k$, $k = 1, \ldots, n$.

The derivative J of f at $\vec{a}$ is defined as the linear transformation satisfying

$$f(\vec{x}) = f(\vec{a}) + J(\vec{x} - \vec{a}) + o(|\vec{x} - \vec{a}|).$$

As a linear mapping from R^m to R^n, J may be represented as an $n \times m$ matrix, the Jacobian matrix, with elements

$$J_{ki} = \frac{\partial\varphi_k}{\partial x_i}\Big|_{\vec{x} = \vec{a}}.$$

6. Chain rule. Composition of functions $f: R^m \to R^n$, $g: R^n \to R^p$; emphasis on application to change in parametic equations of a surface under a coordinate transformation (either of domain or range space). Lemma (continuity of linear transformation): For each linear transformation L there is a constant K such that $|L\vec{x}| \leq K|\vec{x}|$ for all $\vec{x}$. Proof: Let $\vec{e}_1, \ldots, \vec{e}_m$ be unit coordinate vectors. $L\vec{x} = L(\Sigma\, x_i\vec{e}_i) = \Sigma\, x_i L\vec{e}_i$, whence $|L\vec{x}| \leq \Sigma|x_i L\vec{e}_i| \leq \max|x_i| \cdot \Sigma|L\vec{e}_i| = K \max|x_i| \leq K|\vec{x}|$.

Theorem: If J_f, J_g, J_{gf} are the derivatives (Jacobian matrices) of f, g, and gf, then $J_{gf} = J_g J_f$.

Proof: Set $f(\vec{x}) = f(\vec{a}) + J_f(\vec{x} - \vec{a}) + o(\vec{x} - \vec{a})$,

$g(\vec{z}) = g(\vec{b}) + J_g(\vec{z} - \vec{b}) + o(\vec{z} - \vec{b})$,

$\vec{z} = f(\vec{x})$, $\vec{b} = f(\vec{a})$, and apply the lemma above.

Special cases: $R^1 \to R^1 \to R^1$, etc. Applications in spaces of dimension at most 3, particularly to polar and cylindrical coordinates.

Coordinate transformations; interpretation of the Jacobian determinant det J as a local scale factor for "volume."

7. Iterated and multiple integrals. A more careful and more general treatment than in Mathematics 2. Iterated integrals of functions on R^2 and R^3 (partly review from Mathematics 2). Multiple integrals as limits of sums; evaluation by iterated integration.

Additivity, linearity, positivity of integrals. Application to volumes, etc.

Change of variables of integration; geometrical interpretation as coordinate transformation. Special attention to polar and cylindrical coordinates. Further applications.

Mathematics 6L. Linear Algebra.

[Prerequisite: Mathematics 3] This course is the first course in linear algebra proper, although it assumes the material on that subject taught in Mathematics 3. It contains the usual basic material of linear algebra needed for further study in mathematics except that the rational canonical form is omitted and the Jordan form is given only brief treatment.

We point out that Mathematics 6L and 6M together do not include the following topics in the outline of the course Abstract Algebra given in the 1965 CUPM report Preparation for Graduate Study in Mathematics [page 453]: Jordan-Hölder theorem, Sylow theorems, exterior algebra, modules over Euclidean rings, canonical forms of matrices, elementary theory of algebraic extensions of fields.

COURSE OUTLINE

1. Fields. (4 hours) Definition. Examples: Q, R, C, Q(x), R(x), C(x), $Q(\sqrt{2})$. The fields of 2, 3, 4 elements explicitly constructed by means of addition and multiplication tables. Characteristic of a field.

2. Vector spaces over fields. (9 hours) Definition. Point out that the material of Mathematics 3 and 4 on vector subspaces of

R^n and their linear transformations carries over verbatim to vector spaces over arbitrary fields. Linear dependence. Bases, dimension, subspaces, direct sums. Linear transformations and matrices. Rank, image and kernel. The preceding material is to be thought of as review of the corresponding material in Mathematics 3 and 4. Matrix representation of linear transformations. Change of basis. A transformation is represented by two matrices A and B if and only if there exist nonsingular matrices P and Q so that $A = PBQ$, i.e., if and only if A and B are equivalent. Systems of linear equations. Relation to linear transformations. Existence and uniqueness of solutions in both the homogeneous and nonhomogeneous cases. Two systems have the same solution if their matrices are row equivalent. Equivalence under elementary row operations of equations and matrix, row echelon form, explicit method for calculating solutions.

3. Triangular and Jordan forms. (6 hours) State without proof that C is algebraically closed. Any linear transformation (matrix) over C has a triangular matrix with respect to some basis (is similar to a triangular matrix). Nilpotent matrices and transformations and their similarity invariants, i.e., such a transformation is completely determined by vectors v_i on which it is nilpotent of index q_i, $i = 1, \ldots, r$. Definition of eigenvalue. Jordan form over C via the theorem: If T is a linear transformation on a vector space V over C with $\dim V < \infty$ and if T has eigenvalues λ_i with multiplicities m_i, $i = 1, \ldots, r$, then $V = \bigoplus_1^r V_i$ with $T(V_i) \subset V_i$, $\dim V_i = m_i$, and $T - \lambda_i$ is nilpotent on V_i. Elementary divisors and minimum polynomial. The Cayley-Hamilton theorem.

4. Dual spaces and tensor products. (6 hours) The dual space of a vector space. Adjoints of linear transformations and transposes of matrices. Finite-dimensional vector spaces are reflexive. Tensor products of vector spaces as the solution of a universal problem. Behavior of tensor product with regard to direct sums, basis of a tensor product, change of base fields by means of tensor products.

5. Forms. (5 hours) Definition of bilinear and quadratic forms. Matrix of a form with respect to different bases. A form yields a linear transformation of the vector space into its dual. General theory of symmetric and skew-symmetric forms, forms over

fields where $2 \neq 0$. Diagonalization and the canonical forms, both a form and a matrix approach. The case of the real and complex fields, Sylvester's theorem.

6. Inner product spaces. (6 hours) Definition over R and C. Orthogonal bases, Gram-Schmidt process, Schwarz inequality for the general case. Review of the treatment of Euclidean space in Mathematics 3. Self-adjoint and hermitian linear transformations and their matrices with respect to an orthonormal basis. Eigenvalues and eigenvectors. All eigenvalues of self-adjoint linear transformations are real. The spectral theorem in several equivalent forms both for transformations and for matrices. Applications to classification of quadrics. Relations between quadratic forms and inner products. Positive-definite forms.

COMMENTARY ON MATHEMATICS 6L

At all times a computational aspect must be preserved. The students should be made aware of the constant interplay between linear transformations and matrices. Thus they should be required to solve several systems of linear equations; find the Jordan form, invariant factors, and elementary divisors of numerical matrices; diagonalize symmetric matrices and find the matrix P such that PAP^{tr} is diagonal; and also diagonalize symmetric and hermitian matrices by means of orthogonal and unitary similarity. In Section 6 the concept of tensor product should be exploited in complexifying a real space in order to prove that eigenvalues of self-adjoint transformations are real.

A treatment of the Jordan form along the lines of Section 3 can be found in Halmos, Paul R. _Finite-Dimensional Vector Spaces_, 2nd ed. New York, Van Nostrand Reinhold Company, 1958.

In addition to the definitive treatment of tensor products to be found in Book I, Chapter II of Bourbaki's treatise _Algèbre Linéaire_ (Bourbaki, N. _Éléments de Mathématiques_, Livre I, Chapitre II (_Algèbre Linéaire_), 3ème éd. Paris, Hermann et Cie., 1962) or in MacLane and Birkhoff's _Algebra_ (MacLane, Saunders and Birkhoff, Garrett. _Algebra_. New York, The Macmillan Company, 1967), briefer and perhaps more accessible treatments may also be found in Goldhaber and Ehrlich (Goldhaber, Jacob K. and Ehrlich, Gertrude. Algebra. New York, The Macmillan Company, 1970) and in Sah (Sah, Chih-Han. Abstract Algebra. New York, Academic Press, Inc., 1966).

All theorems dealing with linear transformations should be accompanied by parallel statements about matrices. Thus, for example, the spectral theorem should be stated in the following three forms for real vector spaces:

I(a). Let V be a real finite-dimensional inner product space and let T be a symmetric linear transformation on V. Then V has an orthonormal basis of eigenvectors of T.

I(b). With the same hypotheses as I(a), there exists a set of orthogonal projections $E_1, \ldots, E_r$ of V such that $T = \sum_1^r \lambda_i E_i$ where λ_i are the distinct eigenvalues of T.

II. Let A be a symmetric real matrix. Then there exists an orthogonal matrix P such that $PAP^{-1} = PAP^{tr}$ is diagonal.

The student should understand that these are equivalent theorems and, given T or A, should be able to compute $\lambda_i E_i$ and P explicitly in low-dimensional cases.

In dealing with positive-definite forms, one should point out that these are equivalent to inner products and that yet another form of the spectral theorem asserts:

Let A, B be symmetric real matrices with A positive-definite. Then there is a matrix P such that $PAP^{-1} = I$ and PBP^{-1} is diagonal.

Mathematics 6M. Introductory Modern Algebra.

[Prerequisite: Mathematics 3] This course introduces the student to the basic notions of algebra as they are used in modern mathematics. It covers the notions of group, ring, and field and also deals extensively with unique factorization. The language of categories is to be used from the beginning of the course, but the formal introduction of categories is deferred to the end of the term. In order to make the material meaningful to the student, the instructor must devise concrete examples that will relate to the student's earlier experiences.

We again point out that Mathematics 6L and 6M together do not include the following topics in the outline of the course Abstract Algebra given in the 1965 CUPM report Preparation for Graduate Study in Mathematics [page 453]: Jordan-Hölder theorem, Sylow theorems, exterior algebra, modules over Euclidean rings, canonical forms of matrices, elementary theory of algebraic extensions of fields.

COURSE OUTLINE

1. Groups. (10 hours) Definition. Examples, vector subgroups of R^n, linear groups, additive group of reals, permutation and transformation groups, cyclic groups, groups of symmetries of geometric figures. Subgroups. Order of an element. Theorem: Every subgroup of a cyclic group is cyclic. Coset decomposition. Lagrange's theorem. Normal subgroups. Homomorphisms of groups. The first two homomorphism theorems.

2. Rings and fields. (9 hours) Definitions. Examples: integers, integers modulo m, polynomials over the reals, the rationals, the Gaussian integers, all linear transformations on a vector subspace of R^n, rings of functions. Zero divisors and inverses. Division rings and fields. Domains, quotient fields as solution to a universal problem. Homomorphisms, isomorphisms, monomorphisms. Ideals. Congruences in Z. Tests for divisibility by 3, 11, 9, etc. Fermat's little theorem: $a^{p-1} \equiv 1 \pmod p$, using group theory. Residue class rings.

3. Unique factorization domains. (11 hours) Prime elements in a commutative ring. Reminder of unique factorization in Z. Examples where unique factorization fails, say in $Z[\sqrt{-5}]$. Definition of Euclidean ring, regarded as a device to unify the discussion for Z and $F[x]$, F a field. Division algorithm and Euclidean algorithm in a Euclidean ring; greatest common divisor; Theorem: If a prime divides a product, then it divides at least one factor. Unique factorization in a Euclidean ring. GCD and LCM. Theorem: A principal ideal domain is a unique factorization domain. Gauss' lemma. Theorem: If D is a unique factorization domain, then $D[x]$ is also a unique factorization domain.

4. Categories of sets. (6 hours) The notion of a category

of sets. The categories of sets, groups, abelian groups, rings, fields, vector spaces over the reals. Epimorphisms, monomorphisms, isomorphisms, surjections, injections. Examples to show that epimorphisms may not always be surjections, etc. Exact sequences. Functors and natural transformations. The homomorphism theorems of group and ring theory in categorical language, monomorphisms and epimorphisms in the categories of groups, rings, and fields.

COMMENTARY ON MATHEMATICS 6M

From the beginning of the course the language of category theory should be used. Thus arrows, diagrams (commutative and otherwise), and exact sequences should be defined and used as soon as possible. For example, the first homomorphism theorem for groups should be stated as follows: Let G be a group, f a surjective homomorphism of G onto H, and $N = \ker f$. If $p\colon G \to G/N$ is the natural projection, then there exists a unique homomorphism g which makes the diagram

$$\begin{tikzcd}
 & & & 1 \arrow[d] & \\
1 \arrow[r] & N \arrow[r] & G \arrow[r, "p"] \arrow[dr, "f"'] & G/N \arrow[r] \arrow[d, "g"] & 1 \\
 & & & H \arrow[d] & \\
 & & & 1 &
\end{tikzcd}$$

commutative with exact row and column.

In addition to extensive treatments of categories in such treatises as Mitchell's (Mitchell, Barry. Theory of Categories. New York, Academic Press, Inc., 1965) and MacLane and Birkhoff's (MacLane, Saunders and Birkhoff, Garrett. Algebra. New York, The Macmillan Company, 1967), a brief treatment of this subject can also be found in Goldhaber and Ehrlich (Goldhaber, Jacob K. and Ehrlich, Gertrude. Algebra. New York, The Macmillan Company, 1970).

In the section on groups, the general linear and orthogonal groups should be introduced and based on the material of Mathematics 3. The affine group and its relationship to the general linear group should be discussed. The students should do a considerable number of concrete computations involving groups and counting problems.

In Section 2 there is an opportunity to introduce some elementary number theory: the Euler phi-function counts the number of units of the ring $Z/(n)$; $a^{\varphi(n)} \equiv a$ is a theorem that can be demonstrated by these methods; the divisibility of $2^{32} + 1$ by 641 can easily be asserted using congruences; calendar and time problems can also be introduced to illustrate the notions of congruence and ideals. Again, the homework should include many problems of this kind so that the student gains some familiarity with the notions introduced here. Fields of 2, 4, 3, 9, and p^n elements, p a prime, should be introduced, at least in the exercises.

In Section 3 the Euclidean algorithm should be introduced and used to calculate the greatest common divisor of large integers and of polynomials having degree higher than three. If time permits, Euclidean rings different from Z and $F[x]$ should be introduced in the homework. The integers of certain quadratic number fields are especially suitable for this.

In Section 4 the material of the first three sections must be used to illustrate the definitions at each step; when natural transformations are discussed, the "naturality" of the homomorphism theorems should be underlined and many examples given. The language of categories should be familiar to all students who pursue mathematics beyond this level. This language reveals how much various mathematical disciplines have in common and how different disciplines may be related to each other. By virtue of its generality, category theory is a very valuable source of meaningful conjectures and an effort should be made, even at this level, to emphasize this.

V. A FOUR-YEAR CURRICULUM

In the 1965 GCMC report, CUPM presented a curriculum for four years of college mathematics. It devoted a considerable amount of attention to both upper- and lower-division courses other than basic calculus and algebra, indicating their relationships and their significance for various kinds of students. The 1965 GCMC report is now out of print, but many of its suggestions are still relevant, at least to one very common kind of mathematics curriculum. Consequently, CUPM feels that it will be useful to repeat some of its suggestions of 1965 with modifications prompted by recent developments and to reprint some of the course outlines even though experience has shown they are open to objections such as excessive length.

We have not described a special one-year course in mathematics appreciation for students in liberal arts colleges because we think that it is better for the student to take Mathematics 1 and 2, 1 and 2P, or 1 and 3. (A description of the probability course Mathematics 2P is given in Section VI.) These ways of satisfying a liberal arts requirement open more doors for the student than any form of appreciation course, and they are consistent with our view that mathematics is best appreciated through a serious effort to acquire some of its content and methodology and to examine some of its applications.

A student who has successfully completed Mathematics 1 may select Mathematics 2, 2P, or 3 according to his interests. In particular, many students who are interested in the social sciences will choose Mathematics 2P or 3 in preference to Mathematics 2.

For those students for whom a sequence beginning with Mathematics 1 is not possible or not appropriate, there are several possibilities. In the first place, Mathematics 0 and 1 forms a reasonable year sequence for students whose preparation will not permit them to start with Mathematics 1. In many colleges students have been taking and will continue to take a full year course like Mathematics 0. (A description of Mathematics 0 is given in Section VI.)

Among the students for whom neither Mathematics 0 nor Mathematics 1 is appropriate we recognize a sizable number who are preparing to become elementary school teachers. Their needs should be met by special courses described in the CUPM publication Recommendations on Course Content for the Training of Teachers of Mathematics (1971).

Finally, there is a rather large number of students who need further study of mathematics in order to function effectively in the modern world. Some have never had the usual mathematics courses in high school, whereas others have not achieved any mastery of the topics they studied. These students are older and more mature than high school students, and so they need a fresh approach to the necessary topics, if possible one involving obviously significant applications to the real world. One suggestion is the course Mathematics A, "Elementary functions and coordinate geometry," from the CUPM report A Transfer Curriculum in Mathematics for Two-Year Colleges (1969). For students who are not ready even for Mathematics A, we suggest the less conventional course Mathematics E described in considerable detail in the CUPM report A Course in Basic Mathematics for Colleges (1971).

1. Lower-division courses.

By lower-division courses we mean Mathematics 1, 2, 2P, 3, 4, Mathematics 0, and any other basic precalculus courses that are offered. Mathematics 1, 2, 3, and 4 have already been described in detail. Outlines of Mathematics 0 and of Mathematics 2P appear in Section VI reproduced from the 1965 GCMC report.

2. Upper-division courses.

The following list of typical courses might be offered once a year or, in some cases, in alternate years, to meet the needs of students requiring advanced work in mathematics. At many colleges some of these upper-division courses are combined into year courses. Which of them are offered will depend on the needs of the students and special qualifications of the staff. The order is a rough indication of the level. The course outlines for Mathematics 6L and 6M appear in Section IV and the outlines for the remaining courses appear in Section VI.

Although we describe the upper-division work in terms of semester courses, these advanced subjects may also be treated by independent or directed study, tutorials, or seminars. This is especially appropriate in a small college where it may not be possible to organize classes in every subject.

Mathematics 5. Multivariable Calculus II. This is a calculus course to follow Mathematics 4. Two possibilities are (1) a course in vector calculus and (2) a course consisting of selected topics in analysis. Two examples of the first possibility are quoted from the 1965 GCMC report in Section VI. An example of the second, appropriate not only for statisticians but also for physical scientists and mathematics majors, is quoted from the 1971 CUPM report Preparation for Graduate Work in Statistics.

Mathematics 6L and 6M. Linear Algebra and Introductory Modern Algebra. Mathematics 6M is essential for all mathematics majors including prospective high school teachers. Both courses are essential for students preparing for graduate work in mathematics and are useful for computer science students as well. Many physical science students are now finding both courses important, and social science students often require the material of Mathematics 6L.

Mathematics 7. Probability and Statistics. In place of a one-semester course recommended in the 1965 GCMC report we now recommend the two-semester course in probability and statistics suggested in Preparation for Graduate Work in Statistics (1971) and reproduced in Section VI. This course is essential for students preparing for graduate work in statistics. It is desirable for mathematics majors, for mathematically oriented biology or social science students, for engineering students, particularly in communication fields or industrial engineering, and for theoretical physicists and chemists.

Mathematics 8. Introduction to Numerical Analysis. This course is desirable not only for mathematics majors but also for students majoring in a science that makes extensive use of mathematics. In place of the course outlined in the 1965 GCMC report we now suggest the course outlined in Section VI.

Mathematics 9. Geometry. This course should cover a single concentrated geometric theory from a modern axiomatic viewpoint; it is not intended to be a descriptive or survey course in "college geometry." If the college undertakes the training of prospective secondary school teachers, the essential content of this course is Euclidean geometry. A more widely ranging full-year course in the same spirit is desirable if it is possible. Other subjects which provide the appropriate depth include topology, convexity, projective geometry, and differential geometry. A serious introduction to geometric ideas and geometric proof is valuable for all undergraduates majoring in mathematics.

In Section VI two geometry courses of general appeal are quoted from the CUPM report Recommendations on Course Content for the Training of Teachers of Mathematics (1971).

Mathematics 10. Applied Mathematics. Although this course is not yet a standard part of the curriculum, it is desirable for mathematics majors to become aware of the ways in which their subject is applied. Several versions of such a course--optimization theory, graph theory and combinatorial analysis, and fluid mechanics--are described in the CUPM report Applied Mathematics in the Undergraduate Curriculum (1972) [page 705].

Mathematics 11-12. Introductory Real Variable Theory. Preferably this is a one-year course, but if necessary it may be offered in a one-semester version or combined with complex analysis in a one-year course. The student should learn to prove the basic propositions of real variable theory.

At least one semester is desirable for any mathematics major. Mathematics 11-12 is essential for students preparing for graduate work in mathematics. On completion of Mathematics 12 a student should be ready to begin a graduate course in measure and integration theory or in functional analysis. The topics and skills are basic in such fields of analysis as differential equations, calculus of variations, harmonic analysis, complex variables, probability theory, and many others. We feel an extensive coverage of subject matter, especially in the directions of abstract topologies and functional analysis, should be sacrificed in favor of active practice by the student in proving theorems. For an outline of Mathematics 11-12, see Section VI.

Mathematics 13. Complex Analysis. This course contains standard material in the elementary theory of analytic functions of a single complex variable.

Many prefer to have this course precede Mathematics 11-12. It is important for mathematics majors, engineering students, applied mathematicians, and theory-oriented students of physics and chemistry. For an outline of Mathematics 13 see Section VI.

VI. ADDITIONAL COURSE OUTLINES

Mathematics 0. Elementary Functions and Coordinate Geometry. (3 or 4 semester hours) (Reprinted from the 1965 GCMC report)

1. Definition of function and algebra of functions. (5 lessons) Various ways of describing functions, examples from previous mathematics and from outside mathematics, graphs of functions, algebraic operations on functions, composition, inverse functions.

2. Polynomial and rational functions. (10 lessons) Definitions, graphs of quadratic and power functions, zeros of polynomial functions, remainder and factor theorems, complex roots, rational functions and their graphs.

3. Exponential functions. (6 lessons) Review of integral and rational exponents, real exponents, graphs, applications, exponential growth.

4. Logarithmic functions. (4 lessons) Logarithmic function as inverse of exponential, graphs, applications.

5. Trigonometric functions. (10 lessons) Review of numerical trigonometry and trigonometric functions of angles, trigonometric functions defined on the unit circle, trigonometric functions defined on the real line, graphs, periodicity, periodic motion, inverse trigonometric functions, graphs.

6. Functions of two variables. (4 lessons) Three-dimensional rectangular coordinate system, sketching graphs of $z = f(x,y)$ by plane slices.

Mathematics 2P. Probability. (3 semester hours) (Reprinted from the 1965 GCMC report) [Prerequisite: Mathematics 1]

1. Probability as a mathematical system. (9 lessons) Sample spaces, events as subsets, probability axioms, simple theorems, finite sample spaces and equiprobable measure as special case, binomial coefficients and counting techniques applied to probability problems, conditional probability, independent events, Bayes' formula.

2. Random variables and their distributions. (13 lessons) Random variables (discrete and continuous), probability functions, density and distribution functions, special distributions (binomial, hypergeometric, Poisson, uniform, exponential, normal, etc.), mean and variance, Chebychev inequality, independent random variables, functions of random variables and their distributions.

3. Limit theorems. (4 lessons) Poisson and normal approximation to the binomial, Central Limit Theorem, Law of Large Numbers, some statistical applications.

4. Topics in statistical inference. (7-13 lessons) Estimation and sampling, point and interval estimates, hypothesis-testing, power of a test, regression, a few examples of nonparametric methods.

Remarks:

For students with only the minimum prerequisite training in calculus (Mathematics 1), about six lessons will have to be devoted to additional calculus topics needed in Mathematics 2P: improper integrals, integration by substitution, infinite series, power series, Taylor's expansion. For such students there will remain only about seven lessons in statistical inference. Students electing Mathematics 2P after Mathematics 4 will be able to complete the entire course as outlined above.

Mathematics 5. Multivariable Calculus II. (3 semester hours) (Conventional version of Advanced Multivariable Calculus as printed in the 1965 GCMC report) [Prerequisites: Mathematics 1, 2, 3, 4]

The differential and integral calculus of Euclidean 3-space, using vector notation, leading up to the formulation and solution (in simple cases) of the partial differential equations of mathematical physics. Considerable use can and should be made of the students' preparation in linear algebra.

1. Vector algebra. (4 lessons) Dot and cross product, identities. Geometric interpretation and applications. Invariance under change of orthogonal bases.

2. Differential vector calculus. (8 lessons) Functions from V_m to V_n, continuity. Functions from V_1 to V_3, differential geometry of curves. Functions from V_3 to V_1, scalar fields, directional derivative, gradient. Functions from V_3 to V_3, vector fields, divergence, curl. The differential operator ∇, identities. Expression in general orthogonal coordinates.

3. Integral vector calculus. (15 lessons) Line, surface, and volume integrals. Change of variables. Green's, divergence, and Stokes' theorems. Invariant definitions of gradient, divergence, and curl. Integrals independent of path, potentials. Derivation of the Laplace, heat, and wave equations.

4. Fourier series. (6 lessons) The vector space of square-integrable functions, orthogonal sets, approximation by finite sums, notion of complete orthogonal set, general Fourier series. Trigonometric functions as a special case, proof of completeness.

5. Boundary value problems. (6 lessons) Separation of variables. Use of Fourier series to satisfy boundary conditions. Numerical methods.

Mathematics 5. Multivariable Calculus II. (3 semester hours) (Alternate version of Advanced Multivariable Calculus employing differential forms as printed in the 1965 GCMC report) [Prerequisites: Mathematics 1, 2, 3, 4]

A study of the properties of continuous mappings from E_n to E_m, making use of the linear algebra in Mathematics 3, and an introduction to differential forms and vector calculus based upon line integrals, surface integrals, and the general Stokes theorem. Application should be made to field theory, elementary hydrodynamics, or other similar topics so that some intuitive understanding can be gained.

1. Transformations. (15 lessons) Functions (mappings) from E_n to E_m, for $n, m = 1, 2, 3, 4$. Continuity and implications of continuity; differentiation and the differential of a mapping as a matrix-valued function. The role of the Jacobian as the determinant of the differential; local and global inverses of mappings and the Implicit Function Theorem. Review of the chain rule for differentiation and reduction to matrix multiplication. Application to change of variable in multiple integrals and to the area of surfaces.

2. Differential forms. (6 lessons) Integrals along curves. Introduction of differential forms; algebraic operations; differentiation rules. Application to the change of variable in multiple integrals. Surface integrals; the meaning of a general k-form.

3. Vector analysis. (4 lessons) Reinterpretation in terms of vectors; vector function as mapping into E_3; vector field as mapping from E_3 into E_3. Formulation of line and surface integrals (1-forms and 2-forms) in terms of vectors. The operations Div, Grad, Curl, and their corresponding translations into differential forms.

4. Vector calculus. (8 lessons) The theorems of Gauss, Green, Stokes, stated for differential forms and translated into vector equivalents. Invariant definitions of Div and Curl. Exact differential forms and independence of path for line integrals. Application to a topic in hydrodynamics, or to Maxwell's equations, or to the derivation of Green's identities and their specializations for harmonic functions.

5. Fourier methods. (6 lessons) The continuous functions as a vector (linear) space; inner products and orthogonality; geometric concepts and analogy with E_n. Best L^2 approximation; notion of an orthogonal basis and of completeness. The Schwarz and Bessel inequalities. Generalized Fourier series with respect to an orthonormal basis. Treatment of the case $\{e^{inx}\}$ and the standard trigonometric case. Application to the solution of one standard boundary value problem.

Mathematics 5. Multivariable Calculus II. (3 semester hours)
(Reprint of Selected Topics in Analysis from the 1971 report Preparation for Graduate Work in Statistics)

The Panel on Statistics feels that the course Mathematics 5 presented in the 1965 GCMC report is not particularly appropriate for statistics students, and it has recommended that a course including the special topics listed below be offered in place of Mathematics 5 for students preparing for graduate work in statistics.

The course it recommends gives the student additional analytic skills more advanced than those acquired in the beginning analysis sequence. Topics to be included are multiple integration in n dimensions, Jacobians and change of variables in multiple integrals, improper integrals, special functions (beta, gamma), Stirling's formula, Lagrange multipliers, generating functions and Laplace transforms, difference equations, additional work on ordinary differential equations, and an introduction to partial differential equations.

It is possible that the suggested topics can be studied in a unified course devoted to optimization problems. Such a course, at a level which presupposes only the beginning analysis and linear algebra courses and which may be taken concurrently with a course in probability theory, would be a valuable addition to the undergraduate curriculum, not only for students preparing for graduate work in statistics but also for students in economics, business administration, operations research, engineering, etc. Experimentation by teachers in the preparation of written materials and textbooks for such a course would be useful and is worthy of encouragement.

Mathematics 7. Probability and Statistics. (6 semester hours)
(Reprinted from the 1971 report Preparation for Graduate Work in Statistics)

This key course is a one-year combination of probability and statistics. On the semester system, a complete course in probability should be followed by a course in statistics. If the course is given on a quarter system, it may be possible to have a quarter of probability, followed by two quarters of statistics or by a second quarter of statistics and a third quarter of topics in probability and/or statistics. In any case, these courses should be taught as one sequence.

Prerequisites for this one-year course are Mathematics 1, 2, and 4 (Calculus). Students should also be encouraged to have taken Mathematics 3 (Elementary Linear Algebra). [For detailed course descriptions see Section IV.] All students in this course, whether they be prospective graduate students of statistics, other mathematics majors, or students from other disciplines, should be encouraged to take the full year rather than only the first-semester probability course. Almost all students will have studied the calculus sequence and perhaps linear algebra without interruption during their first two years in college. Although our recommended probability course and Mathematics 2P differ only little in content, our course assumes the additional maturity and ability of students who have successfully completed the three or four semesters of the core curriculum described above.

The probability course should include the following topics:

Sample spaces, axioms and elementary theorems of probability, combinatorics, independence, conditional probability, Bayes' theorem.

Random variables, probability distributions, expectation, mean, variance, moment-generating functions.

Special distributions, multivariate distributions, transformations of random variables, conditional and marginal distributions.

Chebychev's inequality, limit theorems (Law of Large Numbers, Central Limit Theorem).

Examples of stochastic processes such as random walks and Markov chains.

The course in probability should provide a wide variety of examples of problems which arise in the study of random phenomena. With this aim in mind, we recommend that this course be taught so as to maintain a proper balance between theory and its applications.

The time allotted to the probability course will not permit detailed treatment of all topics listed above. We recommend that such topics as the Central Limit Theorem and the use of Jacobians in transformations of random variables be presented without proof. Also, discussion of multivariate distributions should include only a brief description of the multivariate normal distribution. Random walks and Markov chains may serve as useful topics for two or three lectures to illustrate interesting applications of probability theory. Even though the topics of this paragraph are not treated in depth mathematically, we recommend their inclusion to enrich the student's comprehension of the scope of probability theory.

The statistics course can be implemented in a variety of ways, giving different emphases to topics and, indeed, including different topics. Widely divergent approaches are acceptable as preparation for graduate work and are illustrated in the statistics books listed below, selected from many appropriate texts for this course:

Brunk, H. D. Introduction to Mathematical Statistics, 2nd ed. New York, Blaisdell Publishing Company, 1965.

Freeman, H. A. Introduction to Statistical Inference. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1963.

Freund, John E. Mathematical Statistics. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1962.

Hadley, G. Introduction to Probability and Statistical Decision Theory. San Francisco, California, Holden-Day, Inc., 1967.

Hoel, Paul G.; Port, Sidney C.; Stone, Charles J. Introduction to Statistical Theory. Boston, Massachusetts, Houghton Mifflin Company, 1971.

Hogg, Robert V. and Craig, A. T. Introduction to Mathematical Statistics, 3rd ed. New York, The Macmillan Company, 1970.

Lindgren, B. W. Statistical Theory, 2nd ed. New York, The Macmillan Company, 1968.

Mood, Alexander M. and Graybill, F. A. Introduction to the Theory of Statistics, 2nd ed. New York, McGraw-Hill Book Company, 1963.

Despite the diversity of possible approaches, most will include the following topics:

Estimation: consistency, unbiasedness, maximum likelihood, confidence intervals.

Testing hypotheses: power functions, Type I and II errors, Neyman-Pearson lemma, likelihood ratio tests, tests for means and variances.

Regression and correlation.

Chi-square tests.

Other topics to be included in the statistics course will depend on the available time and method of approach. Possible topics include:

Estimation: efficiency, sufficiency, Cramer-Rao theorem, Rao-Blackwell theorem.

Linear models.

Nonparametric statistics.

Sequential analysis.

Design of experiments.

Decision theory, utility theory, Bayesian analysis.

Robustness.

The above list of additional topics for the key course in statistics is much too large to be adequately covered in its entirety. The fact that many topics will have to be omitted or treated superficially gives the statistics course much more flexibility in approach and coverage than is possible in the probability course. The instructor's choice of topics may be influenced by the following factors. Decision theory, Bayesian analysis, and sequential analysis dealing with foundations of inference will appeal to the philosophically inclined students. The Cramer-Rao theorem and the Rao-Blackwell theorem appeal to mathematically oriented students and illustrate statistical theory. In design of experiments and estimation, one has an opportunity to apply techniques of optimization. Nonparametric techniques utilize combinatorial probability and illustrate the high efficiency that can be attained from simple methods. Analysis of variance provides an application of linear algebra and matrix methods and should interest students who have taken Mathematics 3.

Detailed outlines for the probability and statistics courses have not been presented on the assumption that the choice of texts, which is difficult to anticipate, will tend to determine the order of presentation and the emphasis in a satisfactory fashion. It may be remarked that most statistics texts at this level begin with a portion which can be used for the probability course.

To avoid a formal, dull statistics course and to provide sufficient insight into practice, we recommend that meaningful cross-reference between theoretical models and real-world problems be made throughout the course. Use of the computer will help to accomplish this goal. Three reports that are valuable in appraising the potential role of computers in statistics courses are:

<ins>Development of Materials and Techniques for the Instructional Use of Computers in Statistics Courses</ins>. University of North Carolina, Chapel Hill, North Carolina, 1971.

<ins>Proceedings of a Conference on Computers in the Undergraduate Curricula</ins>. The University of Iowa, Iowa City, Iowa, 1970.

<ins>Proceedings of the Second Annual Conference on Computers in the Undergraduate Curricula</ins>. Dartmouth College, Hanover, New Hampshire, 1971.

Mathematics 8. Introduction to Numerical Analysis. (3 semester hours) [Prerequisites: Mathematics 1, 2, 3, 4]

1. Introduction. (1 hour) Number representation on a computer, discussion of the various types of errors in numerical processes, the idea of stability in numerical processes.

2. Solution of a single nonlinear equation. (7 hours) Existence of a fixed point; contraction theorem and some consequences; Ostrowski's point-of-attraction theorem; the rate of convergence for successive approximations; Newton's method: local convergence and rate of convergence, convergence theorem in the convex case; secant methods, including regula falsi; roots of polynomials: Newton-Raphson method, Sturm sequences, discussion of ill-conditioning.

3. Linear systems of equations. (7 hours) Gaussian elimination with pivoting, the factorization into upper and lower triangular matrices, inversion of matrices, discussion of ill-conditioning, vector and matrix norms, condition numbers, discussion of error bounds, iterative improvement, Gaussian elimination for symmetric positive-definitive matrices.

4. Interpolation and approximation. (6 hours) Lagrange interpolating polynomial; Newton interpolating polynomial; error formula for the interpolating polynomial; Chebychev polynomial approximation; least squares approximation: numerical problem associated with the normal equations, the use of orthogonal polynomials.

5. Numerical integration and differentiation. (6 hours) Quadrature based on interpolatory polynomials, error in approximate integration, integration over large intervals, Romberg integration including development of the even-powered error expansion, error in differentiating the interpolating polynomial, differentiation by extrapolation to the limit.

6. Initial value problems in ordinary differential equations. (9 hours) Taylor's series expansion technique; Euler's method with convergence theorem; Runge-Kutta methods; predictor-corrector methods: convergence of the corrector as an iteration, local error bound for predictor-corrector of same order; general discussion of stability using the model problem $y' = Ay$, consistency and convergence; reduction of higher-order problems to a system of first-order problems.

Mathematics 9. Geometry. (3 semester hours) (Reprint of Foundations of Euclidean Geometry from the 1971 report Recommendations on Course Content for the Training of Teachers of Mathematics)

The purpose of this course is two-fold. On the one hand it presents an adequate axiomatic basis for Euclidean geometry, including the one commonly taught in secondary schools, while on the other hand it provides insight into the interdependence of the various theorems and axioms. It is this latter aspect that is of the greater importance for it shows the prospective teacher that there is no one royal road to the classical theorems. This deeper appreciation of geometry will better prepare the teacher to assess the virtues of alternative approaches and to be receptive to the changes in the secondary school geometry program that loom on the horizon.

Courses similar to this have now become commonplace. As a consequence, no great detail should be necessary in this guide. There is a greater abundance of appropriate topics than can be covered in one course, so some selection will always need to be made.

Although enough consideration should be given to 3-space to build spatial intuition, the major emphasis should be on the plane, since it is in 2-space that the serious and subtle difficulties first become apparent. The principal defects in Euclid's Elements relate to the order and separation properties and to the completeness of the line. Emphasis should be directed to clarifying these subtle matters with an indication of some of the ways by which they can be circumvented. The prospective teacher must be aware of these matters and have enough mathematical sophistication to proceed to new topics with only an indication of how they are resolved.

The course consists of six parts, after a brief historical introduction and a critique of Euclid's Elements. The allotment of times that have been assigned for these parts are but suggestions to be used as a guide, because emphasis will vary with the background of the students, the text used, and the tastes of the instructor. Prerequisites for the course are a modest familiarity with rigorous deduction from axioms, for example as encountered in algebra, and the completeness of the real number system.

1. Incidence and order properties. (8 lessons) In this part of the course, after a brief treatment of incidence properties, the inherent difficulties of betweenness and separation are discussed. The easiest, and suggested, way to proceed is in terms of distance. The popular method today is to use the Birkhoff axioms or a modification such as given by the School Mathematics Study Group. In addition, one should give some indication of a synthetic foundation for betweenness such as that of Hilbert. A brief experience with a synthetic treatment of betweenness is enough to convince the student

of the power of the metric apparatus.

Alternatively, one can begin with a synthetic treatment of betweenness and then introduce the metric apparatus. With this approach, metric betweenness is a welcome simplification.

2. Congruence of triangles and inequalities in triangles. (8 lessons) It is recommended that angle congruence be based on angle measure (the Birkhoff axioms). Yet here too some remarks on a synthetic approach are desirable.

The order of presentation of the congruence theorems can depend on the underlying axiom system used. What is perhaps more important is to observe their interrelations. At this point a global view of transformations of the plane should receive attention. Ruler and compass constructions should be deferred, as the treatment is simpler and more elegant after the parallel axiom has been introduced. The triangle inequality and the exterior angle theorem occur here.

3. Absolute and non-Euclidean geometry. (6 lessons) Up to this point there has been no mention of the parallel postulate. It is desirable to explore some of the attempts to prove it. One should prove a few theorems in absolute geometry, in particular ones about Saccheri quadrilaterals. Then some theorems in hyperbolic geometry can be given, among which the angle-sum theorem for angles in a triangle is most important. A model, without proof, for hyperbolic geometry is natural here.

This part of the course can also be taught after Part 4 when Euclid's parallel axiom and consequences of it have been covered.

4. The parallel postulate. (8 lessons) There are many topics, of central importance in high school, that need to be discussed in this part of the course. It is desirable to give here, as well as in Part 3, considerable attention to the history of the parallel axiom. Due to time limitations, it will probably be necessary to omit some topics. Nevertheless, some attention should be given to: parallelograms, existence of rectangles, Pythagorean theorem, angle-sum theorem for triangles, similarity, ruler and compass construction, and an introduction to the notion of area.

5. The real numbers and geometry. (8 lessons) This part is

devoted to matters in which the completeness of the real number system plays a role. Some attention must be given to the completeness of the line and the consequences thereof. Archimedes' axiom arises naturally here. Important topics are: similarity of triangles for the incommensurable case; circumference; area in general and, in particular, area of circles; and, finally, a coordinate model of Euclidean geometry. It is possible to give a coordinate model of a non-Archimedean geometry at this time.

6. Recapitulation. (3 lessons) This part is intended to give perspective on the preceding sections. It should have a strong historical flavor and might well include lectures with outside reading or a short essay.

Mathematics 9a. Geometry. (Reprint of Vector Geometry from the 1971 report Recommendations on Course Content for the Training of Teachers of Mathematics)

There are approaches to geometry other than the classical synthetic Euclidean approach, and several of these are being suggested for use in both the high school and college curricula. Moreover, exposure to different foundations for geometry yields deeper insights into geometry and can serve to relate Euclidean geometry to the mainstream of current mathematical interest. It is this latter reason which underlies much of the discussion about geometry that is now prevalent. There are at least three approaches that merit consideration.

I. The classical approach of Felix Klein, wherein one begins with projective spaces and, by considering successively smaller subgroups of the group acting on the space, one eventually arrives at Euclidean geometry. A course of this nature might be called projective geometry, but it should proceed as rapidly as possible to Euclidean geometry. Besides books on projective geometry, other references are:

Artin, Emil. Geometric Algebra. New York, John Wiley and Sons, Inc., 1957.

Gans, David. Transformations and Geometries. New York, Appleton-Century-Crofts, 1968.

Klein, Felix. Vorlesungen über Nicht-Euklidische Geometrie. New York, Chelsea Publishing Company, Inc., 1959.

Schreier, Otto and Sperner, Emanuel. Projective Geometry of n Dimensions. New York, Chelsea Publishing Company, Inc., 1961.

(Throughout this outline, references are given because of their content, with no implication that the level of presentation is appropriate. Indeed, adjustments will normally be necessary.)

II. The transformation approach, which in some ways is a variant of Klein's, uses the Euclidean group to define congruence and other familiar concepts. As a further variant of this, one finds books which begin with synthetic Euclidean geometry and proceed to the Euclidean group. References are:

Bachmann, F. Aufbau der Geometrie aus dem Spiegelungsbegriff. Berlin, Springer-Verlag, 1959.

Choquet, Gustave. Geometry in a Modern Setting. Boston, Massachusetts, Houghton Mifflin Company, 1969.

Coxford, A. F. and Usiskin, Z. P. Geometry, A Transformation Approach, vol. I, II. River Forest, Illinois, Laidlaw Brothers, 1970.

Eccles, Frank. An Introduction to Transformational Geometry. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1971.

III. The vector space approach, the one suggested for this course, uses vector spaces as an axiomatic foundation for the investigation of affine and Euclidean geometry. Through the use of vector spaces, classical geometry is brought within the scope of the central topics of modern mathematics and, at the same time, is illuminated by fresh views of familiar theorems. Some of the references below contain isolated chapters which are relevant to this approach; in such cases these chapters are indicated.

Artin, Emil. Geometric Algebra. New York, John Wiley and Sons, Inc., 1957.

Artzy, Rafael. Linear Geometry. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1965.

Dieudonné, Jean. Linear Algebra and Geometry. Boston, Massachusetts, Houghton Mifflin Company, 1969.

Gruenberg, K. W. and Weir, A. J. Linear Geometry. New York, Van Nostrand Reinhold Company, 1967.

MacLane, Saunders and Birkhoff, Garrett. Algebra. New York, The Macmillan Company, 1967. (Chapters VII, XI, XII)

Mostow, George; Sampson, Joseph; Meyer, Jean-Pierre. Fundamental Structures of Algebra. New York, McGraw-Hill Book Company, 1963. (Chapters 8, 9, 14)

Murtha, J. A. and Willard, E. R. Linear Algebra and Geometry. New York, Holt, Rinehart and Winston, Inc., 1969.

Snapper, Ernst and Troyer, Robert. Metric Affine Geometry. New York, Academic Press, Inc., 1971.

The course outlined below has as prerequisite an elementary course in linear algebra (Mathematics 3). The main topics are:

1. Affine geometry and affine transformations

2. Euclidean geometry and Euclidean transformations

3. Non-Euclidean geometries

Because of the relative unfamiliarity of this approach to geometry, more details such as definitions and typical results will be included. Also, a brief justification is given.

In Euclidean geometry one considers the notion of a translation of the space into itself. These translations form a real vector space under the operation (addition) of function composition and multiplication by a real number. Thus the "vector space of translations" acts on the set of points of Euclidean space and satisfies the following two properties:

A. If (x,y) is an ordered pair of points, there is a translation T such that $T(x) = y$. Moreover, this translation is unique.

B. If T_1 and T_2 are translations and x is a point, then the definition of "vector addition" as function composition is indicated by the formula

$$(T_1 + T_2)(x) = T_1(T_2(x)).$$

With this intuitive background, the details of the course outline are now given. The definitions and propositions are stated for dimension n since this causes no complication, but the emphasis will be on dimensions 2 and 3.

1. Affine geometry and affine transformations. One defines real n-dimensional affine space as the triple (V,X,μ) where V is a real vector space of dimension n (the vector space of the translations), X is the set of points of the geometry, and $\mu: V \times X \to X$

defined by $\mu(T,x) = T(x)$ is the action of V on X which satisfies properties A and B above. For convenience, the affine space (V,X,μ) is usually denoted simply by X.

Affine subspaces of X are defined as follows. Let $x \in X$ and let U be a linear subspace of V (a subspace of translations). The affine subspace determined by x and U is denoted by $S(U,x)$ and consists of the set of points

$$\{T(x):T \in U\},$$

i.e., $S(U,x)$ consists of all translates of x by a translation belonging to U. The dimension of $S(U,x)$ is defined to be the dimension of U. Then 1-dimensional affine subspaces are called lines, 2-dimensional affine subspaces are called planes, and (n-1)-dimensional affine subspaces are called hyperplanes (n = dimension of V).

Two affine subspaces S and S' are called parallel $(S \parallel S')$ if there exists a translation T such that $T(S) \subset S'$ or $T(S') \subset S$. Parallelism and incidence are investigated, with special emphasis on dimensions two and three. Results such as the following are obtained.

a. Lines ℓ and m in the plane are parallel if and only if $\ell = m$ or $\ell \cap m = \emptyset$.

b. A line ℓ and a plane π in 3-space are parallel if and only if $\ell \subset \pi$ or $\ell \cap \pi = \emptyset$. If $\ell \nparallel \pi$, then $\ell \cap \pi$ is a point.

c. There exist skew lines in 3-space.

d. Planes π and π' in 3-space are parallel if and only if $\pi = \pi'$ or $\pi \cap \pi' = \emptyset$. If $\pi \nparallel \pi'$, then $\pi \cap \pi'$ is a line.

A coordinate system for the affine space X consists of a point $c \in X$ and an ordered basis for V. A point $x \in X$ is assigned the coordinates $(x_1,\ldots,x_n)$ if T is the unique translation such that $T(c) = x$ and T has coordinates $(x_1,\ldots,x_n)$ with respect to the given ordered basis for V. Using these notions, one can study analytic geometry, e.g., the parametric equations for lines, the linear equations for hyperplanes, the relationship between the linear equations of parallel hyperplanes, incidence in terms of coordinate representations, etc.

For each point $c \in X$, there is a natural way to make X into a vector space which is isomorphic to V. Namely, if r is a real number, $x, y \in X$, and T_1, T_2 are the unique translations satisfying $T_1(c) = x$ and $T_2(c) = y$, then one defines

$$x + y = T_2(T_1(c)) \quad \text{and} \quad rx = (rT_1)(x).$$

The vector space X_c with origin c is the tangent space of classical differential geometry. (Affine space is often defined as the vector space V itself; this approach to affine geometry is based on the isomorphism between X_c and V.)

An affine transformation is a function $f: X \to X$ with the following properties:

a. f is one-to-one and onto.

b. If ℓ and ℓ' are parallel lines, then $f(\ell)$ and $f(\ell')$ are parallel lines.

The affine transformations form a group called the affine group which contains the translation group as a commutative subgroup. For each point $c \in X$, the set of affine transformations which leave c fixed form a subgroup of the affine group; moreover, this subgroup is the general linear group of the vector space X_c and is therefore isomorphic to the general linear group of V. Finally, properties of affine transformations are investigated.

Other topics of affine geometry which are studied include orientation, betweenness, independence of points, affine subspace spanned by points, and simplexes.

2. Euclidean geometry and Euclidean transformations. Euclidean space is defined as the affine space (V,X,μ), where V has been given the additional structure of a positive-definite inner product. Thus for each $T \in V$, T^2 is a nonnegative real number. A distance function is introduced on X by defining the distance between an ordered pair (x,y) of points of X to be $\sqrt{T^2}$, where T is the unique translation such that $T(x) = y$. A Euclidean transformation (rigid motion, isometry) of X is a mapping of X which preserves distance.

The Euclidean transformations form a subgroup of the affine group. For each $c \in X$, the Euclidean transformations which leave c fixed form a subgroup of the Euclidean group. In fact, this is the orthogonal group of the vector space X_c (with the inner product induced on it from V through the given isomorphism) and therefore is isomorphic to the orthogonal group of V.

Rotations and reflections are first defined for the Euclidean plane and then for n-dimensional space. The Cartan-Dieudonné theorem becomes an important tool in the investigation of the Euclidean group. It states that every Euclidean transformation of n-space is the product of at most $n + 1$ reflections in hyperplanes. It follows immediately that there are four kinds of Euclidean transformations of the Euclidean plane: translations, rotations, reflections, and glide reflections.

Rotations and reflections of Euclidean 3-space are investigated. From the Cartan-Dieudonné theorem it follows that every rotation of 3-space has a line of fixed points (the axis of rotation). The set of all rotations with a given line ℓ as axis is a subgroup of the rotation group of 3-space. Moreover, this rotation group with axis ℓ is isomorphic to the rotation group of the Euclidean plane, thus giving the classical result that every rotation of 3-space is determined by an axis and a given "angle of rotation."

One now defines a <u>figure</u> to be a subset of X and calls two figures congruent if there is a Euclidean transformation which maps one figure onto the other. Using these concepts, one proceeds to proofs of the classical congruence theorems of plane geometry (S.S.S., S.A.S., A.S.A, H.S.).

Finally, orthogonality and similarity are investigated.

3. <u>Non-Euclidean geometries</u>. The classical method of obtaining a non-Euclidean plane geometry is to replace the parallel postulate by another postulate on parallel lines and thus obtain hyperbolic geometry. Here the approach is different. The positive-definite inner product is replaced by other (nonsingular) inner products. The geometry obtained is non-Euclidean, but the parallel postulate is still valid! This startling result is true because the underlying space is the affine plane (in which the parallel

postulate is valid) and the change of inner product does not disturb the affine structure.

Actually, the investigation of non-Euclidean geometries can be made concurrently with that of Euclidean geometry. For example, the Lorentz plane and the negative Euclidean plane can be defined and investigated at the same time as the Euclidean plane. "Circles" in the Lorentz plane are related to hyperbolas of the Euclidean plane, etc.

One of the major results is Sylvester's theorem, from which one concludes that there are precisely $n + 1$ distinct nonsingular geometries which can be placed on n-dimensional affine space.

Mathematics 10. Applied Mathematics.

Applied mathematics is a mathematical science distinguished from other branches of mathematics in that it actively employs the scientific method. A working applied mathematician is usually confronted with a real situation whose mathematical aspects are not clearly defined. He must identify specific questions whose answers will shed light on the situation, and he must construct a mathematical model which will aid in his study of these questions. Using the model he translates the questions from the original terms into mathematical terms. He then uses mathematical ideas and techniques to study the problem. He must decide upon methods of approximation and computation which will enable him to determine relevant numbers. Finally, he must interpret the results of his mathematical work in the setting of the original situation.

Mathematics 10 was designed to introduce the student to applied mathematics and, in particular, to model building. Courses concentrating primarily on mathematical techniques which are useful in applications do not satisfy the goals set here for Mathematics 10. Rather, it is intended that the student participate in the total experience of applied mathematics from formulating precise questions to interpreting the results of the mathematical analysis in terms of the original situation, and that particular emphasis be given to model building. A number of courses involving different mathematical topics can be constructed which fulfill these goals. In constructing such a course the instructor should have the following recommendations in mind.

First, the role of model building must be made clear and should be amply illustrated. The student should have considerable experience in building models, in noting their strengths and weaknesses, and in modifying them to fit the situation more accurately. Also, he must realize that often there is more than one approach to a situation and that different approaches may lead to different models. He should be trained to be critical of the models he constructs so that he will know what kind of information to expect from the model and what kind not to expect.

Second, the situations investigated must be realistic. Throughout the course the student should be working on significant problems which are interesting and real to him.

Third, the mathematical topics which arise in the course should be worthwhile and should have applicability beyond the specific problem being discussed. The mathematical topics and the depth of treatment should be appropriate for the level at which the course is offered.

Fourth, the mathematical results should always be interpreted in the original setting. Stopping short of this gives the impression that the manipulation of symbols, methods of approximation, techniques of computation, or other mathematical points are the primary concerns of the course, whereas they are only intermediate steps, essential though they are, in the study of a real situation.

Finally, the course should avoid the extremes of (1) a course about mathematical methods whose reference to the real world consists mainly of assigning appropriate names to problems already completely formulated in mathematical terms and (2) a kind of survey of mathematical models in which only trivial mathematical development of the models is carried out.

The 1972 report of the Panel on Applied Mathematics, Applied Mathematics in the Undergraduate Curriculum, offers three outlines as aids to constructing courses of the type recommended here. [See page 705.]

Mathematics 11-12. Introductory Real Variable Theory. (6 semester hours) (Reprinted from the 1965 GCMC report)

FIRST SEMESTER - 39 lessons

1. Real numbers. (6 lessons) The integers; induction. The rational numbers; order structure, Dedekind cuts. The reals defined as a Dedekind-complete field. Outline of the Dedekind construction. Least upper bound property. Nested interval property. Denseness of the rationals. Archimedean property. Inequalities. The extended real number system.

2. Complex numbers. (3 lessons) The complex numbers introduced as ordered pairs of reals; their arithmetic and geometry. Statement of algebraic completeness. Schwarz inequality.

3. Set theory. (4 lessons) Basic notation and terminology: membership, inclusion, union and intersection, cartesian product, relation, function, sequence, equivalence relation, etc.; arbitrary unions and intersections. Countability of the rationals; uncountability of the reals.

4. Metric spaces. (6 lessons) Basic definitions: metric, ball, boundedness, neighborhood, open set, closed set, interior, boundary, accumulation point, etc. Unions and intersections of open or closed sets. Subspaces. Compactness. Connectedness. Convergent sequence, subsequences, uniqueness of limit. A point of accumulation of a set is a limit of a sequence of points of the set. Cauchy sequence. Completeness.

5. Euclidean spaces. (6 lessons) R^n as a normed vector space over R. Completeness. Countable base for the topology. Bolzano-Weierstrass and Heine-Borel-Lebesgue theorems. Topology of the line. The open sets; the connected sets. The Cantor set. Outline of the Cauchy construction of R. Infinite decimals.

6. Continuity. (8 lessons) (Functions into a metric space:) Limit at a point, continuity at a point. Continuity; inverses of open sets, inverses of closed sets. Continuous images of compact sets are compact. Continuous images of connected sets are connected. Uniform continuity; a continuous function on a compact set is uniformly continuous. (Functions into R:) Algebra of continuous functions. A continuous function on a compact set attains its maximum. Intermediate Value Theorem. Kinds of discontinuities.

7. Differentiation. (6 lessons) (Functions into R:) The derivative. Algebra of differentiable functions. Chain rule. Sign of the derivative. Mean Value Theorems. The Intermediate Value Theorem for derivatives. L'Hôpital's rule. Taylor's theorem with remainder. One-sided derivatives; infinite derivatives. (This material will be relatively familiar to the student from his calculus course, so it can be covered rather quickly.)

SECOND SEMESTER - 39 lessons

8. The Riemann-Stieltjes integral. (11 lessons) [Alternative: the Riemann integral] Upper and lower Riemann integrals. [Existence of the Riemann integral: for f continuous, for f monotonic.] Monotonic functions and functions of bounded variation. Riemann-Stieltjes integrals. Existence of $\int_a^b f\, d\alpha$ for f continuous and α of bounded variation. Reduction to the Riemann integral in case α has a continuous derivative. Linearity of the integral. The integral as a limit of sums. Integration by parts. Change of variable. Mean Value Theorems. The integral as a function of its upper limit. The Fundamental Theorem of Calculus. Improper integrals. The gamma function.

9. Series of numbers. (11 lessons) (Complex:) Convergent series. Tests for convergence (root, ratio, integral, Dirichlet, Abel). Absolute and conditional convergence. Multiplication of series. (Real:) Monotone sequences; lim sup and lim inf of a sequence. Series of positive terms; the number e. Stirling's formula, Euler's constant.

10. Series of functions. (7 lessons) (Complex:) Uniform convergence; continuity of uniform limit of continuous functions. Equicontinuity; equicontinuity on compact sets. (Real:) Integration term-by-term. Differentiation term-by-term. Weierstrass approximation theorem. Nowhere-differentiable continuous functions.

11. Series expansions. (10 lessons) Power series, interval of convergence, real analytic functions, Taylor's theorem. Taylor expansions for exponential, logarithmic, and trigonometric functions. Fourier series: orthonormal systems, mean square approximation, Bessel's inequality, Dirichlet kernel, Fejér kernel, localization theorem, Fejér's theorem. Parseval's theorem.

Mathematics 11. Introductory Real Variable Theory. (3 semester hours) (One-semester version) (Reprinted from the 1965 GCMC report)

1. Real numbers. (3 lessons) Describe various ways of constructing the real numbers but omit details. Least upper bound property, nested interval property, denseness of the rationals.

2. Set theory. (4 lessons) Basic notation and terminology: membership, inclusion, union and intersection, cartesian product, relation, function, sequence, equivalence relation, etc; arbitrary unions and intersections. Countability of the rationals; uncountability of the reals.

3. Metric spaces. (4 lessons) Material of topic 4 in Mathematics 11-12, condensed.

4. Euclidean spaces. (4 lessons) R^n as a normed vector space over R. Completeness. Bolzano-Weierstrass and Heine-Borel-Lebesgue theorems. Topology of the line. Outline of the Cauchy construction of R. Infinite decimals.

5. Continuity. (5 lessons) (Functions into a metric space:) Limit at a point, continuity at a point, inverses of open or closed sets. Uniform continuity. (Functions into R:) A continuous function on a compact set attains its maximum. Intermediate Value Theorem.

6. Differentiation. (3 lessons) Review of previous information, including sign of the derivative, Mean Value Theorem, L'Hôpital's rule, Taylor's theorem with remainder.

7. Riemann-Stieltjes or Riemann integration. (5 lessons) Functions of bounded variation (if the Riemann-Stieltjes integral is covered), basic properties of the integral, the Fundamental Theorem of Calculus.

8. Series of numbers. (8 lessons) Tests for convergence, absolute and conditional convergence. Monotone sequences, lim sup, series of positive terms.

9. Series of functions. (3 lessons) Uniform convergence, continuity of uniform limit of continuous functions, integration and differentiation term-by-term.

Mathematics 13. Complex Analysis. (3 semester hours) (Reprinted from the 1965 GCMC report)

This course is suitable for students who have completed work at the level of vector analysis and ordinary differential equations. The development of skills in this area is very important in the sciences, and the course must exhibit many examples which illustrate the influence of singularities and which require varieties of techniques for finding conformal maps, for evaluating contour integrals (especially those with multivalued integrands), and for using integral transforms.

1. Introduction. (4 lessons) The algebra and geometry of complex numbers. Definitions and properties of elementary functions, e.g., e^z, $\sin z$, $\log z$.

2. Analytic functions. (2 lessons) Limits, derivatives, Cauchy-Riemann equations.

3. Integration. (6 lessons) Integrals, functions defined by integrals. Cauchy's theorem and formula, integral representation of derivatives of all orders. Maximum modulus, Liouville's theorem, Fundamental Theorem of Algebra.

4. Series. (5 lessons) Taylor and Laurent series. Uniform convergence, term-by-term differentiation, uniform convergence in general. Domain of convergence and classification of singularities.

5. Contour integration. (3 lessons) The residue theorem. Evaluation of integrals involving single-valued functions.

6. Analytic continuation and multivalued functions. (6 lessons) Analytic continuation, multivalued functions, and branch points. Technique for contour integrals involving multivalued functions.

7. Conformal mapping. (6 lessons) Conformal mapping. Bilinear and Schwarz-Christoffel transformations, use of mapping in contour integral evaluation. Some mention should be made of the general Riemann mapping theorem.

8. Boundary value problems. (3 lessons) Laplace's equation in two dimensions and the solution of some of its boundary value problems, using conformal mappings.

9. Integral transforms. (4 lessons) The Fourier and Laplace transforms, their inversion identities, and their use in boundary value problems.