$$\int_a^b k_x(t,x)P(x)\,dx = P(c)\int_a^b k_x(t,x)\,dx = P(c)[k(t,b)-k(t,a)],$$

where $a<c<b$. Hence

$$\int_a^b k(t,x)p(x)\,dx = k(t,b)[P(b)-P(c)] + k(t,a)[P(c)-P(a)],$$

$$\left|\int_a^b k(t,x)p(x)\,dx\right| \leqslant M|P(b)-P(c)| + M|P(c)-P(a)|$$

$$= M\left|\int_c^b p(x)\,dx\right| + M\left|\int_a^c p(x)\,dx\right|.$$

Since $p(x)$ is integrable, we can make the integrals on the right as small as we please (independent of t) by choosing a sufficiently large. Hence the convergence is uniform in t.

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THE TEACHING OF MATHEMATICS

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MINIMAL MATHEMATICAL COMPETENCIES FOR COLLEGE GRADUATES*

1. Introduction. Too many people know too little mathematics. Even those who are well informed in other ways often cannot appreciate, much less participate in, some major currents of modern life because of their ideas and feelings about mathematics. In a relatively severe but all too common form, ignorance of mathematics amounts to a form of "functional illiteracy."

Along with the recent revival of interest in general education, "core" curricula, and minimal competencies, this problem has naturally led to the question: What mathematics should every graduate of an American college or university know?

At its January, 1978, meeting, the Association's Committee on the Undergraduate Program in Mathematics (CUPM) established a panel to study the question and make appropriate recommendations. The members of the panel are: Gerald L. Alexanderson (University of Santa Clara), Robert J. Bumcrot (Hofstra University), Edwin H. Spanier (University of California, Berkeley), Juanita J. Peterson (Laney College), and Donald W. Bushaw (Washington State University; chairman).

Some of the work of the panel is described in an Appendix to this document, which is a report from the panel.

*Report of a CUPM Panel whose members are listed in the third paragraph. The report has been approved by CUPM itself.

2. Recommendations. The leading lesson the panel learned from its surveys (see the Appendix) is that American colleges and universities are so diverse that it is impossible to describe either an approximately standard practice or an everywhere attainable goal. A set of minimal competencies that might be woefully inadequate for specialized or selective universities can be a hopeless ideal for others. To perform its task realistically, the panel has therefore felt obliged to interpret the word "minimal" in a really minimal way. *The recommendations listed below accordingly refer to a bare minimum of mathematical competencies for all college graduates*. The panel hopes that individual institutions will go as far beyond these recommendations as local conditions allow. Similarly, how the requirements should be met is left open, for that depends not only on the requirements themselves but also on local policies, traditions, and resources.

The following recommendations result from the panel's studies and deliberations. In preliminary form, they have been reviewed by numerous mathematicians and nonmathematicians, and have been considerably modified in light of comments received. In this sense they represent the collective judgment of a group much larger than the panel itself.

Recommendation A. All college graduates, with rare exceptions, should be expected to have demonstrated reasonable proficiency in the mathematical sciences. Every college or university should therefore formulate, with adequate concreteness, what this "reasonable proficiency" should mean for its students; define how students should demonstrate this proficiency; and establish this demonstration as a degree requirement.

Competence in arithmetic and some facility in making applications in everyday life might be a reasonable graduation requirement for two-year college students in terminal and vocational programs.

Four-year colleges and universities should normally require—perhaps on entrance—not only these but elementary algebra and elementary geometry. They should also expect graduates to understand and be able to use some elementary statistical ideas, to be aware of the place of mathematics in society generally, and to appreciate the nature and societal significance of computing. This applies also to two-year college students in university parallel curricula.

Recommendation B. Whether or not stipulated proficiency is tested by examination, courses should be made available in which it may be acquired. These courses should be taught by effective instructors, and should be designed to be appealing and significant to the students.

Recommendation C. In particular, one or more courses of a remedial nature should be available where there is a need. Such courses, by definition, ordinarily present precollege material, but it should be presented in a way suited to the clientele. In institutions where it is considered improper or impossible to offer remedial courses, mastery of the mathematics should be assured either by entrance requirements or by referring students to other schools where remedial courses can be taken. Two-year colleges have made a large contribution in this role and may be expected to continue to do so.

Is college credit appropriate for remedial courses? On this point we will only quote the statement approved by the MAA Board of Governors on August 20, 1979: "College credit granted for work in mathematics must be carefully controlled. It should not be granted for distinctly high school level work. Mathematics courses offered in college should be examined to determine the extent of their overlap with high school mathematics, and where that overlap is substantial the course should not provide credit toward college graduation; but the students should be graded on their work, and the results should be included in computing grade point averages."

Recommendation D. While almost all undergraduate courses in mathematics should give attention to applications and to historical and philosophical aspects of the subject, there should be one or more courses that concentrate on these aspects while remaining accessible to students with little mathematical background.

Recommendation E. Individual interests often lead students to take a considerable amount of post-secondary mathematics in conventional courses. These students should also be able to take a course of the kind described in Recommendation D, but presupposing more mathematical background.

The MAA Committee on Improving Remediation Efforts in the Colleges, chaired by Professor Joan Leitzel, has gathered information about the effective remedial programs and has made its own recommendations. A separate CUPM panel, chaired by Professor Jerome Goldstein, is at the same time formulating recommendations on "mathematics appreciation" courses of the kind described in recommendations D and E and in Section 4 below. The Minimal Competencies Panel has worked in liaison with both groups and sees no conflict among the various recommendations.

Nevertheless, each of these two main matters will be discussed further in the remaining sections of this report. These discussions are intended primarily to clarify the panel's recommendations, but partly as a way of passing along some of the good ideas it has collected. The separation of the two matters is certainly not intended to imply that remedial courses should do nothing to convey an appreciation of mathematics, or that techniques are out of place in mathematics appreciation courses.

3. Mathematics for coping with life. The idea that all college graduates should be expected to have acquired a certain familiarity with mathematics rests in part on the well-founded belief that such a familiarity is necessary for effective functioning in contemporary life, and certainly for life in those spheres college graduates are most likely to enter. Indeed, it may be argued convincingly—and has been argued many times—that a modest acquaintance with mathematics is necessary for the successful functioning of almost *any* member of modern society. But any prerequisite for contemporary life in general ought to be, *a fortiori*, something one has a right to expect of all college graduates.

Unfortunately many students manage to enter college without having learned the mathematics needed for coping with everyday life, and a deplorable fraction of them leave college in the same condition. The panel's recommendations—most explicitly Recommendation C—suggest that for such students there should be at least one course where basic mathematical deficiencies may be repaired.

Students entering college with mathematical deficiencies have presumably had opportunities to learn the mathematics, and for them those opportunities did not work. Therefore, *the college remedial course should not be a mere rehash, and certainly not an accelerated one, of the traditional secondary or even primary course.* Courses that cover the same old ground in much the same old way tend to be just as uninspiring and unintelligible for these students as the originals, and therefore even less likely to succeed. Students should be able to find even remedial courses fresh, interesting, and significant.

Many courses of this type are being offered, and new ideas are being tested all the time. Several approaches have been described in print (see, for instance, the CUPM booklet *A Course in Basic Mathematics for Colleges*, reprinted in *A Compendium of CUPM Recommendations*, 1, 256-313), and other reports will surely appear. Here there will be only a sketch to illustrate the type of course that might be considered.

The goals of the course would be to impart mathematical knowledge needed for dealing with most common situations in which deductive reasoning or calculation is needed, and to provide some motivation and preparation for a second course in mathematics that could help the students become educated men and women. It is *not* a goal of the course to teach, once and for all, high school mathematics in its entirety, or to provide background for some standard courses in mathematics or other scientific subjects. (The problem of preparing students for mathematics courses required in their fields is discussed at length in the report of the Committee on Improving Remediation Efforts in the Colleges.)

Students in the course would typically have studied no mathematics for three or four years, and have been bored, mystified, or discouraged by past experiences with mathematics courses. They should be in their *first two* years of college. There should be *no* formal prerequisites.

The course should be relatively brief (twenty to thirty meetings), and should be managed in such a way that students participate actively and receive frequent personal attention. To facilitate this, there should be approximately a fifth as many student assistants as there are students. The first few times the course is offered, the assistants might be mathematics or science majors; later, they should be students who have succeeded in this and at least one further mathematics course.

Equipment might include identical calculators for the students, the assistants, and the instructor. The calculator should have the four basic arithmetical operations, sign changes, squares, square roots, floating decimal, a one-word memory, and very little else. A device for projecting the face of the instructor's calculator on a screen would be useful. There should also be a large collection of advertisements, newspaper and magazine articles, sales and credit agreements, and so on, the interpretation or use of which would require some of the topics listed below. These might be complemented by *reasonable* imaginary examples, but the illustration of no topic should depend entirely on artificial applications. If no genuine examples can be found, why should the topic be included? In some topics, however, a step should be taken beyond the evidently practical.

Students should be supplied with a single page of formulas, sufficient for the whole course.

The grading policy should be compassionate but firm. Tests should be frequent and repeatable at least once. They should be straightforward, but only high scores should be considered passing. Mastery should be recognized irrespective of the number of attempts needed to show it, within limits, but outstanding performance should be recognized. If possible, permanent records of students who need to repeat the course should not show the unsuccessful tries.

One list of topics for such a course is given below. Additions and modifications should be made in response to real-world needs and experience in offering the course.

1. Positive decimals; conversion of fractions to decimals with the calculator.
2. Pencil and paper arithmetic with signed whole numbers.
3. Pencil and paper arithmetic with signed fractions. (There should be no three-or-more digit numerators or denominators, except powers of ten.)
4. Calculator arithmetic with signed decimals.
5. Rounding off.
6. Estimation; orders of magnitude.
7. Scientific notation.
8. Units of measurement; elements of the metric system.
9. Percent.
10. What is a formula? What is a function?
11. Times, distance, and rates.
12. Area and volume.
13 What is an algorithm? Flowcharting.
14. Statistics and its dangers.
15. When is an argument correct?
16. Compound interest.
17. Exponential change.

This list should not give rise to hideous visions of workbooks filled with drill exercises. Games, problems of obvious everyday interest, opportunities for creativity, and occasional attention to general problem-solving strategies should contribute to a cheerful and progressive atmosphere and a positive experience.

4. Mathematics appreciation. While the panel does not insist that a knowledge of the cultural side of mathematics should be required of all college students, its Recommendations D and E above suggest that attractive and accessible courses dealing especially with that aspect should be offered. This section of the report contains some reasons for this position and some comments on how it might be realized.

Mathematics has played a central role in the development of modern civilization. It has been essential not only to the growth of science and technology, but has had profound effects on philosophy and other forms of thought as well.

There was certainly no doubt in past centuries that every college graduate, to be an educated person, had to know some mathematics. In medieval times, for example, four of the seven traditional liberal arts were largely or wholly mathematical. The importance attached to mathematics was evident in courses of study in the nineteenth century, and this carried over into the twentieth. Now, however, it is possible to graduate from many colleges without any contact with mathematics beyond the most elementary high-school courses.

While high-school mathematics is important, it does tend to emphasize development of skills. The same, unfortunately, may be said of most college courses whose mission is primarily remedial or preprofessional. But an educated, well-informed person should know something about mathematics beyond skills.

To many, the distinction between mathematicians and accountants is not clear. People who are alert and informed about many things, even colleagues in a university, sometimes assume that mathematicians are constantly doing arithmetic and are surprised to hear that there is such a thing as mathematical research. Their experiences with school mathematics left them with the impression that mathematics is ancient and immutable, and consists of rules and formulas for unfortunate school children to memorize.

The great mathematicians do not occupy their rightful place in the public consciousness. In his *New Yorker* article on mathematics (February 19, 1972), Alfred Adler rightly observed that " ...it would be astonishing if the reader could identify more than two of the following names: Gauss, Cauchy, Euler, Hilbert, Riemann. It would be equally astonishing if he should be unfamiliar with the names of Mann, Stravinsky, de Kooning, Pasteur, John Dewey. The point is not that the first five are the mathematical equivalents of the second five. They are not. They are the mathematical equivalents of Tolstoy, Beethoven, Rembrandt, Darwin, Freud. The geometry of relativity—the work of Riemann—has had consequences as profound as psychoanalysis has..."

Many college graduates know a great deal of mathematics; most of them have had to take mathematics in preparation for their work. But how many of these, or how many mathematics majors, for that matter, could tell much about Abel or Jacobi? More important, how many of them could comment plausibly on the relation of mathematics to other disciplines?

The point here is not that mathematics and mathematicians should be glorified but that a reasonable perspective on the place of mathematics in the human enterprise should be more widely shared.

A course designed specifically to improve this perspective would ideally give some idea of what sorts of problems mathematicians consider and how such problems are attacked. The object would be to promote mathematical literacy, interpreted to include an awareness among future colleagues in colleges and universities, in business, in industry, in government, and in many other callings of what mathematics is, why it is important, and how it might serve them. Some history should be covered along the way, but a straight course in the history of mathematics is not recommended for this purpose; it can have meaning only if the students already have some understanding of the mathematical ideas whose development is traced.

The course could include, for example, a discussion of the Euler formula for polyhedra—and the names of Euler, Descartes, and Cauchy already would have entered the discussion. An account of non-Euclidean geometry would be appropriate, and provide an occasion for introducing Gauss and Riemann as well as Bolyai and Lobachevski, and for commenting on the element of arbitrariness in mathematical modeling of reality. Neither of these topics requires any high level of algebraic skill. A discussion of the insolubility of the quintic equation might involve more algebra but would refer to the work of Lagrange, Galois, and Abel—and the important idea of mathematical impossibility would have arisen. There are many other topics that bring up important mathematical ideas and events but do not require much background.

Axiomatics, though obviously important, should not be overemphasized. Axiomatic systems should not be presented in detail unless one obtains by their use some interesting results that were not intuitively obvious from the start. Elementary graph theory offers some nice opportunities here, as well as a great variety of easily understood applications. Laborious efforts to prove the obvious can convince people that the whole endeavor is silly.

Applications are appealing to many students and should be included. There are convenient sources of authentic applications of mathematics at every level of difficulty. Applications, however, should not be allowed to upstage the real star of the show, mathematical thought itself. Calculators and computing might have their place in the course, and some time could profitably be spent on the place of computers in modern society. Serious study of computer science, however, is probably best left to other courses.

The course should give students copious evidence that mathematics has not only played a great part in human history, but continues to thrive in the service of other fields and as an independent source of intellectual excitement and aesthetic appeal. Mathematical "current events," such as the solution of the four color problem and the discovery of new large primes should be mentioned. Something might be said about Hilbert's problems and the Fields medals. Carefully selected readings from *Scientific American*, *The Mathematical Intelligencer*, and similar publications can help.

The choice of faculty for an appreciation course is critical. It is an extraordinary teaching assistant who would have the experience and breadth of outlook to teach such a course. It should usually be taught by senior faculty, and if appropriate faculty cannot be found, the course should not be taught at all. And it is better that it be taught by the right faculty in larger sections than by reluctant or inept instructors in small ones.

The course mentioned in Recommendation E offers further opportunities. It is still too easy for mathematics and science majors to complete their programs without knowing that research is done in mathematics, that mathematics has deep and productive relationships with many fields, and that mathematics has a rich and fascinating history. A mathematics appreciation course for students with good technical proficiency in mathematics can do much to take care of this and be a memorable experience for all concerned.

As has already been said in Recommendation D, *these observations about separate mathematics appreciation courses should apply*, *to some extent*, *to all mathematics instruction*, even remedial. In a perfect world every mathematics course would be a mathematics appreciation course. The world, however, is not perfect.

5. Conclusion. The recommendations and other ideas set forth in this report will surely not be the last word on the subject. Many intelligent people will be giving further thought to it, and future experience should certainly be allowed and expected to affect our outlook on the whole matter. Accordingly, the panel extends a standing invitation for comments on this report or on the entire question. They should be addressed to the panel in care of Prof. D. W. Bushaw, Dept. of Mathematics, Washington State University, Pullman, Washington 99164.

Appendix. *What the panel did.* The panel began by consulting the pertinent literature; officers of organizations represented in the Council of Scientific Society Presidents or the Conference Board of Mathematical Sciences; and a sample of mathematicians drawn at random from the 1978–1979 *Combined Membership List*. Summaries of the results may be obtained from the chairman of the panel.

A general announcement and appeal for information and ideas also appeared in *Notices of the American Mathematical Society*, *Change Magazine*, *The Mathematics Teacher*, *The Chronicle of Higher Education*, *The Two-Year College Mathematics Journal*, *SIAM News*, and this MONTHLY.

From the first two surveys mentioned, the panel learned not much more than that no national organization in this country, the MAA itself not excepted, has ever taken a position on what college graduates *in general* should know of mathematics.

The survey based on the *Combined Membership List* (CML) and the appeal in periodicals, though more productive, did not provide as much unambiguous guidance as the panel had hoped to get. The CML survey yielded 335 usable responses from a thousand questionnaires. 226 were from persons at colleges and universities. Of these, 105 (39.5%) were from institutions where a mathematics requirement for graduation was in force. These 105 respondents were asked about the nature of the requirement, whether they favored it, and whether they thought it was effective. In the great majority of cases (91 or 86.7%) the requirement could be satisfied by one or more courses. Seven of these respondents reported that the requirement could be satisfied by examination; five others said both courses and an examination were required.

One hundred (95.2%) of the 105 said they favored the requirement, and 75 (71.4%) said they thought it was at least partially effective.

The median course requirement, where one existed, was between 3 and 4 semester hours. A specific course or sequence of courses was seldom required; indeed, acceptable courses were remarkably diverse.

The 161 respondents in colleges and universities which had no general mathematics requirement were asked whether they favored such a requirement. In reply, 148 expressed a preference, and of these 104 (70.3%) favored some kind of a requirement.

When the two groups are combined, one finds that 204 of 253 (80.6%) of those college- or university-affiliated mathematicians in the sample who expressed any preference favored some general graduation requirement in mathematics. The panel did not expect this fraction to be so high. (Unfortunately, the questionnaire did not ask for reasons for the preferences expressed.)

All respondents, academic or not, were asked to mark in a forty-item list of mathematical topics those they thought should be required of all college graduates. The following topics were marked by at least half of the respondents:

basic arithmetic skills (94.6%)
area and volume of common figures (76.4%)
linear equations (71.3%)
algebraic manipulations (63%)
elementary statistics (55.5%)
graphing of elementary functions (54.9%)
integer and fractional exponents (54.3%)
elementary plane geometry (51.9%)

Next in order were: elementary probability (49%), general problem-solving skills (heuristic) (49%), quadratic equations (47.5%), mathematics in business (46.9%), and radicals (43.9%). Computer programming was marked by 33.1%, just after elementary logic (35.5%) and systems of equations (35.2%).

The question about what standard courses should be required elicited a wide variety of answers, many of which were in fact far from standard. College algebra (mentioned by 51 respondents) led the list, and was followed by probability and statistics (47), calculus (45), elementary or intermediate algebra (44), and computer programming or appreciation (30).

About 45% of the respondents accepted an invitation to comment further. Many merely expanded on earlier answers, but some submitted careful statements of their views. These statements, though not easy to summarize, were carefully studied by the panel.

Responses to the appeal in periodicals were interesting too, but they are even less reducible to a brief summary.

The panel met three times and also conducted a voluminous correspondence within itself and with others. It completes this report with high respect for the complexity of the problem, but hopes that its proposals will be of some use in finding solutions.