Calculus

This chapter contains the report of the Subpanel on Calculus of the CUPM Panel on a General Mathematical Sciences Program, reprinted with minor changes from Chapter II of the 1981 CUPM report entitled RECOMMENDATIONS FOR A GENERAL MATHEMATICAL SCIENCES PROGRAM.

Rationale

The Calculus Subpanel was charged with examining the traditional calculus sequence of the first two years of college mathematics: two semesters of single-variable calculus; one semester of linear algebra; one semester of multivariable calculus. In approaching this task, the subpanel considered syllabi through which this sequence is implemented at various colleges and universities, the syllabus for the Advanced Placement Program in Calculus, and alternatives to calculus as the entry-level course in the mathematical sciences, for example, finite mathematics or discrete methods.

The subpanel eventually came to the conclusion that the rationale for certain parts of the traditional calculus sequence remains valid, although some restructuring and increased flexibility are warranted to reflect the differing mathematical requirements of the social and biological sciences and, increasingly, of computer science. The general recommendations of the subpanel are thus:

1. To make no substantive changes in the first semester of calculus;
2. To restructure the second semester around modeling and computation, although leaving it basically a calculus course;
3. To branch to three independent courses in the second year:
   a. Applied Linear Algebra,
   b. Multivariable Calculus (in dimensions 2 and 3),
   c. Discrete Methods.

Descriptions of the first and second semesters of calculus, applied linear algebra, and multivariable calculus are given below. The discrete methods course is discussed in the first chapter, "Mathematical Sciences."

The subpanel views its recommendations as conservative. Tony Ralston has argued, for example, that calculus need not be the entry-level course in the mathematical sciences and that a course in discrete methods is a reasonable alternative, better serving some areas such as computer science (see "The Twilight of the Calculus," which appeared under the title "Computer Science, Mathematics, and the Undergraduate Curricula in Both" in the *American Mathematical Monthly*, 88:7 (1981) 472-485). In his view, to ignore discrete methods, even in the first two years of college mathematics, would be absurd in this day.

The subpanel does not disagree with the general sense of this position. On the other hand, the subpanel feels that the language, spirit, and methods of traditional calculus still permeate mathematics and the natural and social sciences. To quote Ralston himself, "The calculus is one of man's great intellectual achievements; no educated man or woman should be wholly ignorant of its elements." Perhaps the time is not far off when calculus will be displaced as the entry-level course, but it has not arrived yet.

The place for rigor. The subpanel believes strongly that, in the first two years, theorems should be *used* rather than *proved*. Certainly correct statements of theorems such as the Mean Value Theorem or l'Hôpital's Rule should be given; but motivation, as long as it is recognized as such, and usage are more important than proofs. The place for theoretical rigor is in later upper-level courses. In this regard, the subpanel agrees with the program philosophy outlined in the first chapter, "Mathematical Sciences."

First Semester Calculus

The first semester of calculus, especially, contains a consensus on essential ideas that are important for modeling dynamic events. This course has evolved through considerable effort in the mathematical community to present a unified treatment of differential and integral calculus, and it serves well both general education and professional needs. It is historically rich, is filled with significant mathematical ideas, is tempered through its demonstrably important applications, and is philosophically complete. Most syllabi for its teaching cover the usual topics:

A. *Limits and continuity.*
B. *Differentiation rules.*
C. *Meaning of the derivative.* Applications to curve sketching, maximum-minimum problems, related rates, position-velocity-acceleration problems.
D. *Antidifferentiation.*

E. *The definite integral and the Fundamental Theorem of Calculus.*
F. *Trigonometric functions.*
G. *Logarithmic and exponential functions.* Including a brief exposure to first-order, separable differential equations (with emphasis on $y' = ky$).

The first (and second) calculus courses should be 4- or 5-credit hour courses. If less time is available, topics will have to be pushed later into the calculus sequence, with some multivariate calculus material left for an analysis/advanced calculus course. Mathematics courses should not rush trying to cover unrealistic syllabi.

It might be desirable to add more non-physical sciences examples to C (e.g., a discussion of the use of the word "marginal" in economics), although serious modeling examples should be postponed to the second semester. Integration as an averaging process can be included in E, but applications and techniques (numerical or algebraic) of integration are better left to the second semester. Exponential growth and decay are important concepts that must be emphasized in G.

Second Semester Calculus

There does not appear to be much slack or fat in the first semester of calculus. It is in the second semester, therefore, when numerical techniques, models, and computer applications can be introduced. Unlike the first semester of calculus, the second semester does not enjoy the same consensus on either its central theme or its content. It tends to be a grab bag of "further calculus topics"—further techniques of integration, more applications of integration, some extension of techniques to the plane (parametric equations), sequences and infinite series, and more differential equations. Each of these topics is, in isolation, important at some stage in the training of scientists and mathematicians. But it is less clear that packaging them in this way and having them occupy this critical spot in the curriculum is justified today, given the pressing needs of computer science and the non-physical sciences.

From time to time it has been urged that multivariable calculus should be started during the second semester. But few institutions have implemented this suggestion. And the subpanel believes that, in the meantime, higher priorities for the second course have materialized in the form of applications and computing.

The subpanel considered recommending branching in the curriculum after the first semester of calculus, with students advised to take courses more directly relevant to their career goals. But it finally concluded that there are still substantial reasons for keeping students in one "track" through the first two courses. In most American colleges, a "choice" in the second course would require most students to be thinking seriously about career goals within a few weeks of arriving on campus as freshmen. This does not strike us as realistic nor in the best interests of liberal education. Moreover, we continue to feel that many of the ideas and technical skills arising in the second calculus course are reasonable to include at this point in the curriculum. Thus, the final conclusion is that a restructuring and change of emphasis in the second semester calculus course is preferable to its replacement.

The Calculus Subpanel recommends the following changes in the second calculus course:

A. *An early introduction of numerical methods.* Implemented through simple computer programs. Solving one (or a system of two) first-order differential equation(s).

B. *Techniques of integration.* General methods such as integration by parts, use of tables, and techniques that extend the use of tables such as substitutions and (simple) partial fraction expansions; less emphasis should be placed on the codification of special substitutions.

C. *Numerical methods of integration.* Examples where numerical and "formal" methods complement each other, e.g., evaluating improper integrals where substitutions or integration-by-parts make the integral amenable to efficient numerical evaluation.

D. *Applications of integration.* Illustrate the "setting up" of integrals as Riemann sums. The emphasis should be on the modeling process rather than on "visiting" all possible applications of the definite integral.

E. *Sequences and series.* These topics should have substantially changed emphasis:
   1. Sequences should be elevated to independent status, defined not only through "closed formulas" but also via recursion formulas and other iterative algorithms. Estimation of error and analysis of the rate of convergence should accompany some of the examples.
   2. Series should appear as a further important example of the idea of a sequence. Power series, as a bridge from polynomials to special functions, should figure prominently. Specialized convergence tests for series of constants can be deemphasized.
   3. Approximation of functions via Taylor series, and estimation of error, accompanied by im-

plementation of such approximations on a computer.

F. *Differential equations.* Should be treated with less (but not zero) emphasis on special methods for solving first-order equations and constant coefficient linear equations (especially the non-homogeneous case). More valuable would be: vector field interpretation for first-order equations, numerical methods of solution, and power series methods for solving certain equations. Applications should arise in mathematical modeling contexts and both "closed form" and "numerical" solutions should be illustrated.

The new second course in calculus does not differ radically in content from the traditional second semester course. It is a conservative restructuring that can be taught from existing textbooks and based on modest modifications of many existing syllabi. But the intended change in "flavor" and emphasis should be more dramatic. About twelve lectures (of the usual 40 lectures) must be modified substantially to achieve the desired computer emphasis. Numerical algorithms will thus figure prominently, along with the formal techniques of calculus. Concepts not usually in a calculus course such as error estimation, truncation error, round-off error, rate of convergence, and bisection algorithms will be included. The theme for the course will be "calculus models." Consideration of even a few UMAP-type models would be enough to change the nature of the course significantly and to provide the intended "tying together" of the traditional calculus topics that are included in the course.

A syllabus for the course could be constructed by starting with the second calculus course described in the CUPM report, *A General Curriculum for Mathematics in Colleges* (revised 1972), or with the Advanced Placement BC Calculus Syllabus. Topics to be diminished or omitted include: emphasis on special substitutions in integrals, l'Hôpital's rule except as it arises naturally in connection with Taylor series, polar coordinates, vector methods, complex numbers, non-homogeneous differential equations and the general treatment of constant-coefficient homogeneous linear differential equations. Many of these topics will appear in examples but will not be emphasized in themselves.

Intermediate Mathematics Courses

Although the Calculus Subpanel recommends retaining a single track for students during their first year, it just as strongly recommends that three different courses be available from which students choose (with advising) their intermediate mathematics courses. Two of these courses, whose descriptions follow, are Applied Linear Algebra and Multivariable Calculus. The third, Discrete Mathematics, is described in the first chapter, "Mathematical Sciences."

Applied Linear Algebra

For a large part of modern applied mathematics, linear algebra is at least as fundamental as calculus. It is the prerequisite for linear programming and operations research, for statistics, for mathematical economics and Leontief theory, for systems theory, for eigenvalue problems and matrix methods in structures, and for all of numerical analysis, including the solution of differential equations. The attractive aspect about these applications is that they make direct use of what can be taught in a semester of linear algebra. The course can have a sense of purpose, and the examples can reinforce this purpose while they illustrate the theory.

A number of major texts have arrived at a reasonable consensus for a course outline. Their outlines are well matched with the needs of both theory and application. Applications can include such topics as systems of linear differential equations, projections and least squares. But the subpanel strongly recommends that more substantial applications to linear models should be a central part of the construction of the course. Many different applications of this kind are accessible and can be found in the texts mentioned. Thus, no rigid outline is required. The development of the subject moves naturally from dimension 2 to 3 to n, and although that is an easy and familiar step, it nevertheless represents mathematics at its best. The combination of importance and simplicity is almost unique to linear algebra. Linear programming is an excellent final topic in the course. It brings the theory and applications together.

The changes in this course are ones of emphasis that recognize that the course must be more than an introduction to abstract algebra. Abstraction remains a valuable purpose, and linearity permits more success with proofs than the epsilon-delta arguments of calculus. However, the main goal is to emphasize applications and computational methods, opening the course to the large group of students who need to *use* linear algebra.

TEXTS

1. Hill, Richard, *Elementary Linear Algebra,* Academic Press, New York, 1986.
2. Kolman, Bernard, *Introductory Linear Algebra with Applications,* Macmillan, New York, 1979.

3. Rorres, Chris and Anton, Howard, *Applications of Linear Algebra,* John Wiley & Sons, New York, 1979 (paperback supplementary text).
4. Strang, Gil, *Linear Algebra and Its Applications, 3rd Edition,* Harcourt Brace Jovanovich, San Diego, 1988.
5. Tucker, Alan, *A Unified Introduction to Linear Algebra,* Macmillan, New York, 1988.
6. Williams, Gareth, *Linear Algebra with Applications,* Allyn and Bacon, Boston, 1984.

Multivariable Calculus

This is the traditional multivariable calculus course at many colleges and universities. It is not a new course, but for many schools it would represent a movement in the direction of "concrete" treatment of multivariable calculus rather than the more recent elegant treatments making heavy use of linear transformations and couched in general (high dimensional) terms. The course begins with an introduction to vectors and matrix algebra. Topics include Euclidean geometry, linear equations, and determinants. The remainder of the course is an introduction to multivariable calculus, including the analytic geometry of functions of several variables, definitions of limits and partial derivatives, multiple and iterated integrals, non-rectangular coordinates, change of variables, line integrals, and Green's theorem in the plane.

Differential Equations

The Calculus Subpanel has considered the place of differential equations in the curriculum. It recommends that the topic be treated at two levels:

1. Through methods and examples involving differential equations, spiraled through the calculus sequence, and
2. Through a substantial course in differential equations, available to students upon completion of the first-year calculus sequence and applied linear algebra.

We note here topics in differential equations that are part of the preceding courses:

- Solutions of $y' = ky$ occur in the first semester of calculus. Exponential growth and decay are discussed.
- Solution of second order linear differential equations are included in the second semester of calculus. Oscillating solutions occur as examples. In addition, geometrical interpretations (direction field), numerical solutions and power series solutions are included.
- Applied Linear Algebra includes the solution of linear constant coefficient systems of differential equations using eigenvalue methods.

Although the Calculus Subpanel has *not* recommended a full course in differential equations in the calculus sequence of the first two years, it has suggestions for a subsequent course. Such a course should not be a compendium of techniques for solving in closed form various kinds of differential equations. Libraries are full of cookbooks; one hardly needs a course to use them. What is important is to develop carefully the models from which differential equations spring. Modeling obviously means more than an application such as:

> According to physics, the displacement $x(t)$ of a weight attached to a spring satisfies $x''-bx'+kx = 0$. Solve for $x(t)$ given that $b = 2, k = 3, x(0) = 1, x'(0) = 0$.

For a more serious approach to applications, we refer to the art forgery problem at the beginning of Braun (see below) or indeed almost any of the models discussed in the suggested texts.

The meaning of the word "solution" must be scrutinized. Different viewpoints must be introduced—numerical, geometric, qualitative, linear algebraic and discrete.

A possible syllabus for a differential equations course is:

A. *First-order equations.* Models; exact equations; existence and uniqueness and Picard iteration; numerical methods.

B. *Higher-order linear equations.* Models; the linear algebra of the solution set; constant coefficient homogeneous and non-homogeneous; initial value problems and the Laplace transform; series solutions.

C. *Systems of equations and qualitative analysis.* Models; the linear algebra of linear systems and their solutions; existence and uniqueness; phase plane; nonlinear systems; stability.

Since some of these topics will have already been introduced in courses from the calculus sequence, there may be time for a brief discussion of partial differential equations and Fourier series. Existence and uniqueness theorems are included here only because of the light they or their proofs might shed on methods of solution (e.g., Picard iteration).

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The course can be taught using any of the many reasonable differential equations texts with a modest amount of applications, supplemented by:

Braun, Martin, *Differential Equations and Their Applications, Second Edition,* Springer-Verlag, New

York, 1978.

Braun remains the only text to build extensively on applications, but it has the serious drawback that it is based on single-variable calculus and avoids linear algebra.

A somewhat radical alternative is a theoretical course involving more qualitative or topological analysis emphasizing systems of equations. The subpanel does not suggest a syllabus, but refers instead to V.I. Arnold, *Ordinary Differential Equations,* MIT Press, Cambridge (paperback), 1978.

This course would have applied linear algebra and multivariable calculus as prerequisites and could be taken as early as the second semester of the sophomore year if the two prerequisites were taken concurrently the previous semester.

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