Modeling and Operations Research

This chapter contains the report of the Subpanel on Modeling and Operations Research of the CUPM Panel on a General Mathematical Sciences Program, reprinted with minor changes from Chapter V of the 1981 CUPM report entitled RECOMMENDATIONS FOR A GENERAL MATHEMATICAL SCIENCES PROGRAM.

Experience in Applications

This chapter is concerned with mathematical modeling and associated interactive and experience-oriented approaches to teaching mathematical sciences. Mathematical modeling attempts to involve students in the more creative and early design aspects of problem formulation, as well as provide them with a more complete exposure to how mathematics interfaces with other activities in solving problems arising outside of mathematics itself. Model building is a major ingredient of operations research and the contemporary uses of mathematics in the social, life and decision sciences. In addition to being important in their own right, these newer uses of mathematics provide a rich source of suitable materials for interaction and modeling which complement the many modern and classical applications of mathematics in the physical sciences and engineering.

This chapter is intended to assist mathematics faculty in implementing the main panel's recommendation that mathematical sciences majors should have substantial experience with mathematical modeling. Subsequent sections discuss the modeling process in some detail; provide specific suggestions for conducting student projects, applications-experience-related courses and other such programs, along with general recommendations concerning modeling courses at different levels; explain the field of operations research and the requirements for graduate study. The final two sections present outlines for four courses in operations research and modeling, and a compendium of resources and references for modeling courses.

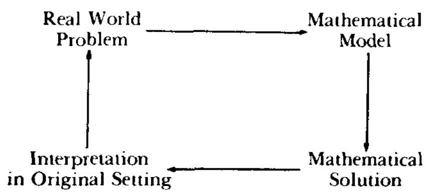
Learning and doing mathematics is a rather individualized and personal activity. The typical classroom lecture in which students are passive spectators has obvious limitations. Students need supervised hands-on experience in problem solving and constructing rigorous proofs. A large variety of alternate teaching techniques and special programs have been developed in attempts to meet this need. These include problem solving approaches using materials from pure and applied mathematics, such as the methods of G. Pólya and R.L. Moore. Problem solving teams for competitions such as the Putnam contest and special departmental practica exist in many colleges. Special courses or seminars on modeling, case studies, and project-oriented activity are becoming more common, as are mathematics clinics and consulting bureaus. Co-op and work-study programs, summer internships, and various other student exchanges have been successfully implemented at some institutions.

The Modeling Subpanel believes that applications and modeling should be included in a nontrivial way in most college-level mathematical sciences courses. Concern with applications has been an important historical force and a major cultural ingredient in the development of all mathematics. Further, the Modeling Subpanel strongly recommends that all mathematical sciences students should obtain first-hand experience with realistic applications of mathematics from the initial stage of model formulation through interpretation of solutions. This can be done in a project-oriented modeling course in one of the alternate out-of-class modes mentioned above. Such an experience yields insight into the place of mathematics in the larger realm of science. It provides an appreciation for the need for interdisciplinary interaction and the limits of specialization. It offers a chance for individuals to make use of their own intuition and creative abilities, to sense the great joy of personal accomplishment, and to develop the confidence to confront similar problems after graduation. Finally, such experience may assist students in choosing careers and fields for future study.

Mathematical Modeling

Modeling is a fundamental part of the general scientific method and is of primary importance in applied mathematics. A model is a simpler realization or an idealization of some more complex reality created for the purpose of gaining new knowledge about a real situation by investigating properties and implications of the model. Models may take many different forms, from physical miniatures to pure intellectual substitutes. Study of a model will hopefully provide understanding and new information about real phenomena which are too complex, excessively expensive, or impossible to analyze in their original setting.

We tend to take the amazing effectiveness of models for granted today. The reader should give a moment's thought to the following examples. One can learn a great deal about a proposed aircraft from wind tunnel experiments before building a costly prototype, and one can learn much about flying an existing airplane from a computer-aided cockpit simulator. Simple computer simulations can provide insights into the complex flow or queueing behavior of traffic in a transportation system. Theoretical studies about elementary particles have provided new insight into fundamental physical laws and have guided subatomic experimentation.



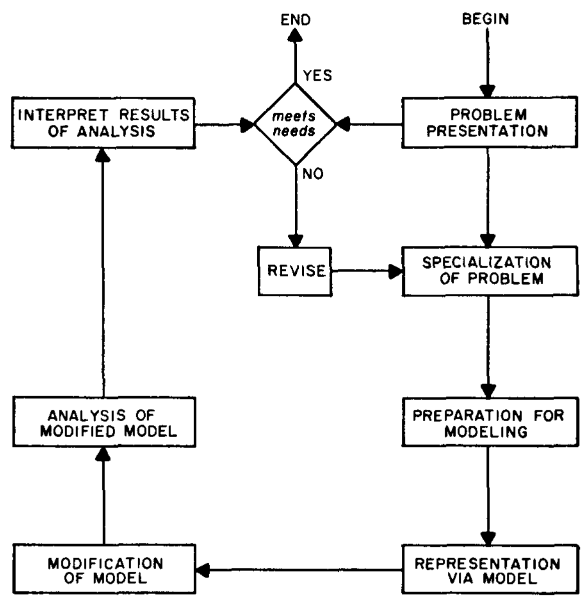
A Simple Model of Mathematical Modeling

Figure 1

The process of mathematical modeling can be simply represented with the diagram in Figure 1. One begins with a problem which arises more or less directly out of the "real world." One builds an abstract model for purposes of analysis, and this frequently takes a mathematical form. The model is solved in this abstract setting. The solution is then interpreted back into its original context. Finally, the analytical conclusions are compared with reality. If they fall short of matching the real situation, then modifications of the model may be called for, and one proceeds around this cycle again. One often proceeds back and forth within a cycle and makes successive iterations about this figure many times before arriving at a satisfactory representation of the real world.

The creation of new knowledge via this modeling route is at the heart of theoretical science and applied mathematics. We will use the word "modeling" to describe the complete progress illustrated in Figure 1. Frequently this term is used only for the model formulation step (the top arrow in the figure). A full discussion of the four steps in this modeling paradigm follow. Additional steps refining the modeling process are sometimes inserted; for example, see Figure 2.

First consider the downward pointing arrow on the right side from "mathematical model" to "mathematical solution." This is the deductive activity of finding solutions to well-formulated mathematical problems. It is usually the most logical, well-defined and straightforward part of modeling, although not necessarily the easiest. It is often the most immediately pleasing, elegant, and intellectual part. This "side" of the "modeling square" is the one covered best in standard applied mathematics courses. Unfortunately, most teaching of applied mathematics is confined to discussing just model-solving mathematical techniques, with superficial treatment of the other three sides of the square, whereas these other sides often involve much more creativity, interaction with other disciplines, and communication skills.



A Refined Model of Mathematical Modeling

Figure 2

The bottom arrow in Figure 1 is concerned with translating or explaining a purely mathematical result in terms of the original real world setting. This involves the need to communicate in a precise and lucid manner. (Inexperience in this skill, according to many employers, is a serious shortcoming in mathematics graduates). This aspect of a mathematical scientist's training should not be left to courses in other sciences or to on-the-job learning after graduation.

In describing the meaning of a mathematical solution one must take great care to be complete and honest. It is dangerous to discard quickly some mathematical solutions to a physical problem as extraneous or having no physical meaning; there have been too many historical incidences where "extraneous" solutions were of fundamental importance. Likewise, one should not select out just the one preconceived answer which the "boss" is looking for to support his or her position. A decision maker frequently does not want just one optimal solution, but desires to know a variety of "good" solutions and the range of reasonable options available from which to select.

There is an old adage to the effect that bosses do not act on quantitative recommendations unless they are communicated in a manner which makes them understandable to such decision makers. This communication can often be a difficult task because of the technical nature of the formulation and solution, and also because large quantities of data and extensive computation may need to be compressed to a manageable size for the layman to understand in a relatively limited time. If mathematical education gave more attention to this aspect of mathematical modeling, there might be wider recognition and visibility of mathematicians in society beyond the academic world!

A major step in real world modeling is to validate models critically and to check out solutions against the original phenomena and known results. This step, represented by the left upward arrow in Figure 1, may involve experimentation, verifying, and evaluating. Two major criteria for evaluating a model are simplicity and accuracy of prediction. Questions about the range of validity, sensitivity of parameters errors resulting from approximations, and such should be investigated. In many cases, a modeling project will simply confirm from another perspective properties that are already believed to be true. The real gain from modeling activity occurs when the modeling leads to discovery of new knowledge (which subsequently is confirmed by other methods).

Modern mathematics education rarely involves itself with this left hand side of the modeling process, except perhaps for an occasional "eyeballing" of an answer or in projects undertaken by a mathematics or statistics consulting clinic. By omitting this activity, mathematical education misses an opportunity to become involved with real-world decision making, judgmental inputs, the limitations of its mathematical tools, and other more human aspects of science, as well as the reward of witnessing the acceptance of a new theory.

Finally, consider the top arrow in Figure 1 which represents the heart of the modeling activity. The construction of an abstract model from a real situation is the really creative activity and an important component of all theoretical science. Building models involves translating into mathematics, maintaining the essential ingredients while filtering out a great amount of excess baggage, and arriving at realistic and manageable intellectual limitations. The three basic elements of a model are:

1. A logical mathematical structure such as calculus, probability, or game theory;
2. An appropriate interpretation of the variables in that structure in terms of the given problem; and
3. A characterization with the structure of all laws and constraints pertinent to the problem.

To build such a mental construct, one must conceptualize, idealize and identify properties precisely. A model builder must carefully balance the tradeoffs between coarse simplifications and unnecessary details—often the effects of such tradeoffs are not apparent until subsequent validation (three steps later in the modeling process).

This initial part of modeling is clearly the most essential and valuable part of the whole process. It is usually the most difficult part. Eddington said "I regard the introductory part of a theory as the most difficult, because we have to use our brains all of the time. Afterwards we can use mathematics." Model building is an art, and must be taught as such.

An Undergraduate Modeling Course

This section discusses various approaches to designing mathematical sciences courses concentrating on the modeling process. The resources listed at the end of this chapter contain a wealth of additional information on models, the modeling process and specific modeling courses as well as references to supplementary materials which the reader may find useful in course design.

Practitioners in the physical or social sciences or engineering have an instinctive feeling of what the modeling process is all about, even if they are not able to articulate it well. Modeling is an important part of their work-a-day activity. For the most part, however, they prefer to leave the analysis and structure of the modeling process itself to workers in other disciplines, like mathematics, or to philosophers of science who are trying to understand the abstract theories underlying these results and how scientists get their results.

How does one go about acquiring experience in real-world modeling? The wrong place to start is looking at big models in the scientific literature which are broad in scope and the epitome of their kind. Indeed, one could probably learn more about sculpting by looking at the pieces that Michelangelo discarded than by looking at the Pieta. The mathematical techniques with which one is familiar will be a primary limiting factor in understanding models. Another factor is that real-world problem areas have their own peculiar "empirical laws" and "principles" which are commonly known to specialists in an area but are not easily accessible to the casual reader.

Apprentice modelers need some help and guidance in selecting model areas for study which will build their modeling skill without discouraging progress. The ideal way to do this within the college curriculum is to begin the modeling process as early as possible in the student's career and reinforce modeling over the entire period of study. That is, the modeling process should be an integral part of the curriculum. Most mathematics departments, for a variety of reasons, are not prepared to give modeling such a major emphasis. For them, a more reasonable approach is to design a course specifically around the modeling process.

Efforts to emphasize the modeling process in undergraduate courses on a broad scale began in the 1960's and were promoted mainly by engineers, operations researchers and social scientists. Extensive discussions of modeling in mathematics courses developed later. The modeling process has been brought into the classroom in many ways but two particular approaches are worth describing in some detail.

First there is the case study approach in which the modeling process is described in a series of examples that are more-or-less self-contained. The examples selected by the instructor are designed to bring out the basic features of the modeling process as well as to inform the students about basic models within a discipline. An excellent early example is *You and Technology: A High School Case Study Text* developed by the engineering departments of the PCM Colleges (Chester, PA), edited by N. Damaskos and M. Smyth.

The second approach applies "hands-on" experience to problems that may only be vaguely described. This approach is sometimes called "open-ended" or "experiential," because it is not clear at the outset what kind of a model will be successful in analyzing a problem, or indeed whether a particular problem is well-posed in any sense. An interesting sidelight on this approach to teaching the modeling process is that the models proposed by students for a particular problem depend not only on the students' breadth of knowledge but, as much as anything else, on time constraints and computer (and other) resources available. Engineers popularized the experiential approach in the early sixties with the high school program *Man Made World,* mostly as a means of exposing students at an early stage to engineering as a profession (a text of the same name was written for this program by J. Truxal, et al., McGraw-Hill publisher).

A range of courses emphasizing the modeling process is clearly possible between the case study approach and the experiential approach.

It is important to note that the scope of the engineering approach to modeling is much broader than just the technical aspects of the problem at hand. In designing a solution to a problem, engineers must take into account time constraints and build into their models prescribed economic and other technical constraints as well as consideration of the impact of their design on society. Engineers do not build elaborate models to explain the fundamental workings of nature nor do they seek the best possible solution to a problem in the absence of the proposed application of that solution. In spite of these differences, there is obviously a large overlap between the engineering and mathematical approaches to modeling.

We now characterize the components of a modeling course in a way that readers should find useful in designing a course to fit their own local needs. The Table on pp. 46-47 organizes much of this information for easy reference. There are six basic aspects of teaching modeling that must be considered:

1. Prerequisites. For whom is the course intended?
2. Effort level. How long—a few weeks, a semester, a year?
3. Course format. Experiential or case study approach? Team or individual work? Instructor's role. Communication skills used.
4. Resources available. Computer system, remote access, good software packages (students should become familiar with using some major software package). Access to expertise in fields considered. Appropriate handouts to keep students progressing.
5. Source of problems. Real-world or contrived? Open-ended or can student answer all questions by looking them up in the literature?
6. Technical thrust. What technical areas should the course emphasize, or avoid? Continuous or discrete models? Deterministic or stochastic? Role of computer programming.

We now expand a little on two of these components, effort level and course format. The level of effort devoted to a modeling course can range from "mini-projects," using a team approach to short projects within an established course, to major projects which last an entire year. The mini-project format requires a great deal of organization and preparation to make it work. See Borrelli and Busenburg "Undergraduate Classroom Experiences in Applied Mathematics" (*UMAP Journal,* Volume 1, 1980) for one approach to structuring a mini-project program, together with its pro's and con's. The one-semester case study course, judging from its popularity, is the best understood and trusted of modeling courses. There are good textbooks and a great many modules written for use in such a course (see list at end of chapter).

While most case studies texts on mathematical modeling are designed for upper-level courses, the text *You and Technology* (mentioned above), supplemented with modules, can easily be adapted for use in a freshman case studies course. Such a course might also present an opportunity for students to see the fundamental differences between engineering and mathematical approaches to modeling (this issue is treated nicely in *You and Technology*). An extensive outline is provided below for a special custom-made, lower-level modeling course.

Experiential modeling courses are not used as often as case study courses. Since the experiential approach is typically used on open-ended problems where the outcome is difficult to predict in advance, this approach is especially risky for a mathematics instructor who is teaching a modeling course for the first time. Nevertheless, experiences of various colleges over the last several years show that the experiential approach is feasible and that, whatever happens, students and instructors find it a rewarding experience. Several successful formats for experiential modeling courses have emerged. All seem to use the team approach with occasional guidance by consultants, as needed. It should be noted that many industrial employers treat such experiential modeling as job-related experience in assessing a student's job qualifications. References at the end of this chapter contain descriptions of the well-known Mathematics Clinics in Claremont and other experiential modeling courses (interested readers can write directly to Harvey Mudd College for first-hand advice).

We close this Section with some important general points to keep in mind when designing any modeling course.

- To encourage initiative and independent work, students should have access to, and be responsible for using, support resources such as documentation of software and previous student projects.
- If high standards are imposed on writing of reports, then these reports deserve some exposure; they should not just be shoved in filing cabinets and forgotten. Instructors should encourage students to seek publication of a paper based on their reports, if warranted, or an article in the campus newspaper. Abstracts of recent reports should be made available to students early in a modeling course. When students know their work will get exposure, they are motivated to write good reports.
- It is valuable to integrate the modeling process into the curriculum as widely as possible and not just as an add-on special course with no connection to any other mathematical sciences course.
- A problem with most modeling courses is that the material in them quickly becomes dated. When students discover that they are working on the same projects or models as their classmates did last year, they lose enthusiasm. What is needed is a format for automatically updating the material. A constant flow of real-world problems, as come into a mathematics consulting clinic, is a great advantage.

Operations Research

Operations research is a mathematical science closely connected to mathematical modeling. Although some notable contributions were made prior to 1940, operations research grew out of World War II. The analysis of military logistics, supply and operational problems by scientists from many different disciplines generated the techniques and approaches that evolved into modern operations research. This subject studies complex systems, structures and institutions with a view towards operating such multiparameter systems more efficiently within various constraints, such as scarce resources. Operations research analyses are used to optimize current activities and predict future feasibility. The complexity of its problems has made operations research heavily dependent on high-speed digital computers. It is now used in fields in which decisions were traditionally made on the basis of less quantitative approaches, such as "experience" or mere hunches. There is frequently a major concern with "people" as well as "things," and the man-system interface in a complex social activity. Major national concerns such as productivity, environmental impact and energy supply have a large operations research component.

The approach in operations research is multidisciplinary in nature, and uses common sense, data, and substantial empiricism (heuristics) combined with new, as well as repackaged traditional, mathematical methodologies. The principal mathematical theories of operations research are mathematical programming and stochastic processes. Major topics in these theories are mentioned in the operations research course contents in the next section. Operations research has major overlap with the fields of industrial engineering, management science, mathematical economics, econometrics and decision theory.

Anatomy of a Modeling Course

Ingredients	Background and Source Material	Remarks
PREREQUISITES:		
Lower Division. Single variable calculus, a science course with lab, some computing.	Case study approach most likely. See, e.g., "You and Technology" or suitable UMAP modules.	If the team approach is selected then there can be some flexibility in these prerequisites.
Upper Division. Multivariable calculus, linear algebra, computation and some computer programming, basic prob/stat., some diff. eqns., a science course with lab.	For experiential approach and case study approach consult appropriately noted reference.	If modeling course is not required, then some thought must be given as to how students can be attracted to such a course: descriptions in registration packets, posters, note to advisor, etc.
EFFORT LEVEL:		
Partial Course. Recommended minimum of 2 weeks out of a 3 hour course preceded by a tooling up period.	Mini-projects are a possibility here. See Borelli and Busenberg. Format of mini-projects can be effectively structured. See Becker, *et al.*, "Handbook for Projects."	Important that mini-project work *not* be simply added to standard load of the host course—it should replace some required work; e.g., an exam.
Full Course. May be designed to fit into special options, either to give job-related training or introduction to modeling process with important models in a discipline.	Many possibilities exist for modeling courses for a full semester—see items below. For a discussion of pros and cons, see Borelli and Busenberg.	Format of instruction can seriously affect the student's interest as well as his capacity for effective work—see "Format" section below for possibilities.
COURSE FORMAT:		
Case Study. The modeling process presented via examples that are more-or-less self-contained.	Material selected from modules, textbooks, conference proceedings, or journals.	Advanced students can be asked to lecture on material that is well enough organized.
Experiential. Hands-on approach to open-ended projects incorporating the modeling process. Some possibilities are:		Internships, work-study programs not appropriate for inclusion here.
1. Problem-centered Course. Class divided into teams to work on a sequence of projects and share experience.	Needs highly experienced instructor to select and present the projects and watch over progress of the teams. Class size limited by instructor's energy. See Borelli and Busenberg for more details.	Oral presentation and written reports are emphasized. Most demanding of instructor's time.
2. Mathematics Clinic, Consulting Group. Intensive, industry-supported team effort on a single project, usually for one year.	Composition of team is critical. See Claremont Clinic Articles for details. Because of time constraints, able support staff must be readily available.	Team communication skills highly emphasized in Clinic program and is crucial to success. Team has main responsibility for work, instructor advises. Student handbook at Claremont Clinic (by Handa) available on request.
3. Research Assistance. Students aid faculty in research work.	MIT has a highly organized program which does this. Mostly, however, it's catch-as-catch-can. The Institute of Decision Science, Claremont Men's College, has developed a classroom approach to such work.	A danger here is that the success of the faculty member's research may take precedence over the impact on the students' education. Students' needs could get lost in the shuffle.
4. Mini-projects. Team approach on short projects within an established course.	See Borelli and Busenberg.	Emphasizes writing skills, highly structured activity; see "Handbook for Projects" by Becker, *et al.*

Anatomy of a Modeling Course

Ingredients	Background and Source Material	Remarks
RESOURCES AVAILABLE:		
Computer. Good access to a high level computer (preferably with time-sharing capability) having good software packages is very important for the success of most modeling courses.		Computer graphics capabilities and knowledgeable (and accessible) consultants at the computer center add not only a professional touch but also help teams live within their time constraints.
Experienced Consultants. Access to knowledgeable colleagues, experts in local industrial firms, and talented computer center personnel are all helpful in keeping a team's progress from faltering.	A successful, long-term program depends to a large extent on the Director's ability to secure *willing* assistance from able consultants.	Be sure that consultants help is acknowledged by the students in all written reports, even if it is only of a casual nature.
Supplemental Materials. Handouts on how to work in a team on projects, or where to go for help, etc., lessen the student's feeling of abandonment when working on projects.	For project work, see the Handbooks by Becker, *et al.*, Handa, Seven and Zagar, and for computer graphics, Saunders, *et al.* (all were developed at Harvey Mudd College and are available on request).	
SOURCE OF PROBLEMS:		
Real World. Open-ended problems submitted by local industrial firms or government agencies which are of current interest to them, or problems from current research of colleagues.	See Borelli and Spanier for a description of one effective method of recruiting sponsored projects from industry. MIT has a highly organized way of advertising current research of its faculty and laboratories and whether undergraduates can play a role or not.	Used only in experiential type modeling courses.
Contrived. Open-ended problems pulled from a variety of sources: from technical journals, suggestions from colleagues, books, etc.	The modeling books in the references are good sources of problems.	Used mostly in experiential type modeling course.
Case Studies. Reasonably well self-contained descriptions of completed projects or problems.	Good sources in modules, proceedings of conferences on case studies and books.	Used only for case study type of modeling course.
TECHNICAL THRUST:		
Discrete-OR. Problems whose models involve discrete structures, programming, or optimization within discrete settings. Also interpolation with finite structures in continuous settings.		Deterministic and stochastic methods are both possibilities here.
Continuous. Problems whose models involve differential or integral equations, continuous probabilities, or optimization within continuous setting.		
Computer. Problems with main goal the production of software either at the systems level or solvers for a class of equations in continuous settings, along with error analysis of same. For DEC users, the IMSL package is a good all-around one to have available on the system.		

There are many opportunities for mathematical sciences majors to pursue graduate studies or find employment in operations research and related fields. Industrial mathematicians in all fields find themselves faced with operations research problems from time-to-time. Thus it is important for mathematical sciences students to have some exposure to operations research and its applications, and also knowledge of its career possibilities. This classroom exposure to operations research can occur in conjunction with undergraduate modeling experience or in a specific course on operations research. The current relevance and naturalness of this subject are immediately clear to students, and realistic projects at various levels of difficulty are readily available. An interesting article by D. Wagner about operations research appeared in the *American Mathematical Monthly* (82, p. 895). Students should also be referred to the booklet *Careers in Operations Research,* available from the Operations Research Society of America, 428 Preston Street, Baltimore, MD 21202.

A student interested in graduate work in operations research should have a solid preparation in undergraduate core mathematics: calculus, linear algebra, real analysis, plus courses in probability, introductory computer science and modeling. A course in operations research itself is more important as a way to learn if one likes the field than as a prerequisite for graduate study. A substantial minor in a relevant area outside mathematics (as recommended for all mathematical sciences majors in the first chapter, "Mathematical Sciences") is important. This outside work should include a sampling of quantitative courses in the social sciences, business, or engineering (if available). Experience solving some problems involving substantial computer computation and an exposure to nontrivial algorithms are also desirable.

At some institutions, mathematics departments are now preparing to offer an operations research course for the first time, while other institutions may have many operations research courses offered in mathematics, economics, business, industrial engineering and computer science. In either extreme and situations in between, mathematical sciences students are best served by some form of interdepartmental cooperation, or at least coordination of offerings. If a mathematics department is planning to offer an operations research course when none previously existed at the institution, mathematics should work closely with other interested departments.

In planning this first course, mathematicians could seek contacts with local industry to obtain practitioners as visiting lecturers. On the other hand, an introductory operations research course can be taught by most college mathematics professors with appropriate attitudes if they are willing to undertake some self study. Indeed, faculty without formal operations research training who are going to teach such a course should be strongly encouraged to learn about the field by attendance at short courses, participation in a department seminar on the subject, or by sabbatical leave (or other released time) at universities or industrial laboratories with operations research activities.

Course Descriptions

Four sample courses on operations research and modeling are described below. Only more general remarks are given for the courses in operations research and stochastic processes since these have become fairly standardized in recent years. More specific details are provided for an elementary-level modeling course using discrete mathematics and for a more advanced modeling course using continuous methods. These are merely illustrations of the wide variety of different sorts of modeling courses which can be taught. The 1972 CUPM *Recommendations on Applied Mathematics* contain a detailed description of a physical-sciences oriented modeling course. Such a modeling course continues to be very valuable and in no way should be considered dated. Many basic intermediate-level courses in the physical sciences are also excellent modeling courses, from the point of view of a mathematical sciences major.

Introductory Operations Research

Much of the material in an introductory operations research course for undergraduates has become fairly standard. The course covers primarily deterministic methods. Most publishing companies have good introductory operations research texts (the text title may be Linear Programming, the course's main topic). The level of this course can vary depending on the prerequisites and student maturity. It is normally an upper-level offering with a prerequisite or corequisite of linear algebra. Calculus and probability should be required if stochastic models are also included.

An operations research course can be a "pure mathematics" course which stresses the fundamental properties of systems of linear inequalities, basic geometry of polyhedra and cones, discrete optimization and complexity of algorithms. Most operations research courses, however, emphasize the many applications which can be solved by linear programming and related techniques of combinatorial optimization. Such courses usually devote some time to efficient algorithms and practical numerical methods (to avoid roundoff errors), as well as basic notions of computational complexity. While problem solving and modeling are important, a first operations research course should cover some topic in reasonable depth and not be merely a collection of simple techniques and routine applications.

COURSE CONTENT

The course should start with a brief discussion of the general nature, history and philosophy of operations research. Some of the older texts such as *Introduction to Operations Research* by C. Churchman, R. Ackoff and E. Arnoff, Wiley, 1957, and *Methods of Operations Research* by P. Morse and G. Kimball, Wiley, 1951, devote extensive space to history. The instructor should not spend much time on history at the beginning of a course but instead should weave it into discussions throughout a course.

The first half of the course in usually devoted to linear programming: its theory, the simplex algorithm, and applications. The course then continues on to a series of special linear programming problems, such as optimal assignment, transportation, trans-shipment, network flow, minimal spanning tree, shortest path, PERT methods and traveling salesperson, each with its own algorithms and associated theory. Basic concepts of graph theory are normally introduced in conjunction with some of the preceding problems. If time permits, elementary aspects from decomposition theory, dynamic programming, integer programming, or nonlinear programming may be included.

It is difficult to find space in an introductory operations research course for even a small sampling of probability or stochastic models. If possible, it is better to include this material in a second course. Similarly, there is usually little time available to discuss game theory, except possibly for showing that two-person, zero-sum games are equivalent to a dual pair of linear programs. Game theory is probably best treated in a separate "topics" course.

Elementary Modeling Course

The following course on mathematical modeling and problem solving is intended for freshmen and sophomores with a solid preparation in high school mathematics. The primary objective is to provide lower-level students with a first-hand experience in forming their own mathematical models and discovering their own solution techniques. A secondary goal is to introduce some of the concepts from modern finite mathematics and illustrate their applications in the social sciences. The instructor might supplement these main themes with brief discussions of some important recent mathematical developments and indicate the current relevance of mathematics to contemporary science and policy making.

The course should maintain an open-minded and questioning approach to problem solving. Much of the class time should be devoted to student discussions of their models and how to improve them. Students should be asked to make formal oral and written expositions. Many of the topics covered are also suitable, with proper adjustments, for upper-level courses or for lower-level "mathematics appreciation" courses. (Readers interested in the latter courses should consult the 1981 Report of the CUPM Panel on Mathematics Appreciation, reprinted later in this volume.) Not all of the topics mentioned below can be covered in any one course, and frequent changes in course content are necessary to maintain the originality of problems.

No one current textbook appears appropriate for this course, although a simpler "prepackaged" version of this course could use the high-school-oriented text *You and Technology* with supplementary modules. The course described below is an example of how various sources can be assembled (as handouts or on library reserve) to form a modeling course, in this instance emphasizing modeling in the social sciences.

COURSE CONTENT

Overview and Patterns of Problem Solving. Introduction to the nature of modeling and problem solving. The role of science, engineering and social sciences in making and implementing new discoveries. The nature of applied mathematics and the interdisciplinary approach to problems. Illustrations of problems solved by quick insight rather than by involved analysis. Many books have chapters on modeling and problem solving; also see *Patterns of Problem Solving* by M. Rubinstein, Prentice-Hall, 1975, or "Foresight-Insight-Hindsight" by J. Frauenthal and T. Saaty, in *Modules in Applied Mathematics*, vol. 3 (W. Lucas, editor), Springer-Verlag.

Graph and Network Problems. A large variety of problems related to undirected and directed graphs and network flows can be assigned and discussed at the outset with no hint of any theory or technical terms. At a later stage, a lecture can be devoted to theory to develop a common vocabulary. The language and general approach of systems analysis can be developed. The four-color theorem can be discussed. References are *Applied Combinatorics*, by F. Roberts, Prentice-Hall, 1984, *Graphs as Mathematical Models* by G. Chartrand, Prindle, Weber and Schmidt, 1977, and *Applied Combinatorics* by A. Tucker, Wiley, 1980.

Some lecture time can be spent illustrating how graphs are applied: to simplify a complex problem, such as Instant Insanity (Chartrand, p. 125 or Tucker, p. 355), or the more difficult Rubik's Cube (*Scientific American,* March, 1981); for purely mathematical purposes, such as to prove Euler's formula $V - E + F = 2$ and use it in turn to prove the existence of exactly five regular polyhedra; or to examine R. Connelly's flexing (nonconvex) polyhedra (*Mathematical Intelligencer,* Vol. 1, No. 3, 1979). The analogy between transportation, fluid flow, electric and hydraulic networks can be illustrated (see G. Minty's article in *Discrete Mathematics and Its Applications* Proceedings of a Conference at Indiana University, ed. M. Thompson, 1977).

Enumeration Problems. (Tucker, 2nd ed., Chapter 5 or Roberts, Chapter 2.) Some practical uses can be covered briefly, e.g., to probability problems or the Pigeonhole Principle. Computational complexity and its application to hard-to-break codes can be discussed.

Value and Utility Theory. Expected utility versus expected value; St. Petersburg paradox; construction of a money versus utility curve: axioms for utility; assessing Coalitional Values (see module by W. Lucas and L. Billera in *Modules in Applied Mathematics,* vol. 2, W. Lucas, editor, Springer-Verlag).

Conflict Resolution. Some three-person cooperative game experiments and analysis; the Prisoner's Dilemma for two or more persons (H. Hamburger in *Journal of Math. Sociology* 3, 1973); illustrations of equilibrium concepts; two-person zero-sum games, e.g., batter versus pitcher (*Economics and the Competitive Process* by J. Case, NYU Press, 1979, p. 3; also see *The Game of Business* by John McDonald, Doubleday, 1975, Anchor paperback, 1977, and *Game Theory: A Nontechnical Introduction* by M. Davis, Basic Books, 1970).

A Discrete Optimization Problem and an Algorithm. Possible topics are the complete and optimal assignment problems (UMAP module 317 by D. Gale), or the marriage problem (D. Gale and L. Shapley, *American Mathematical Monthly* 69, 1962, p. 9).

Simulation. See chapters on simulation in many books and "Four-Way Stop or Traffic Light? An Illustration of the Modeling Process" by E. Packel (in *Modules in Applied Mathematics,* vol. 3, W. Lucas, editor, Springer-Verlag). Additional ideas from Inventory Theory, Scheduling Theory, Dynamic Programming, and Control Theory, e.g., lunar landing, can be included.

Projects and Mini-projects. At least one significant project type activity should be pursued over several weeks by the whole class by means of a sequence of graded exercises and class discussions. Some of the topics listed above can be treated in this mode. Other suitable topics are: the Apportionment Problem (Fair Representation by M. Balinski and H. Young, Yale Press); measuring power in Weighted Voting situations (W. Lucas in *Case Studies in Applied Mathematics* MAA, 1976); Cost Analysis (C. Clark in same *Case Studies* on harvesting fish or forests); some simple topics from statistics such as Asking Sensitive Questions, module by J. Maceli (in *Modules in Applied Mathematics,* vol. 2, W. Lucas, editor, Springer-Verlag); and Social Choice Theory and Voting (*Theory of Voting* by R. Farquharson, Yale, 1960).

In addition to the class project, teams of two or three students can spend a few weeks on a mini-project. Many of the topics above can be applied to a local practical problem. Scheduling, inventory and optimal allocations are good topics, as are gaming experiments, simulations and elementary statistical studies. More theoretical topics, ranging from walking versus running in the rain to designing the inside mechanism of the Rubik's Cube are also possible. Some attempt at discussing possible implementation of a mini-project result, e.g., with a campus administrator, is encouraged in order to show the practical difficulties of implementing mathematically optimal procedures.

Introductory Stochastic Processes

The purpose of this course is to introduce the student to the basic mathematical aspects of the theory of stochastic processes and its applications. This course can equally well be offered under such alternate titles as Applied Probability or Operations Research: Stochastic Models. Stochastic processes is a large and growing field. This course will lay background for further learning on the job or in graduate school.

The prerequisite for this course is at least the equivalent of a full course of post-calculus probability including the following topics: random variables, common univariate and multivariate distributions, moments, conditional probability, stochastic independence, conditional distributions and means, generating functions, and limit theorems. Such a course is fairly traditional now, but if most students have had just the integrated statistics and probability course suggested by the Statistics Subpanel, then the beginning of the stochastic processes course would have to be devoted to completing the needed probability background. It is also desirable for students to have some experience with basic matrix algebra and with using computer terminals.

The course should slight neither mathematical theory nor its applications. It is better to cover few topics with a full discussion of both theory and applications to survey theory alone or to cover only applications. The course emphasizes ***problem solving*** and develops an acquaintance with a variety of models that are widely used. Stochastic modeling and ***problem formulation*** are different activities that should be treated in a modeling course. If many students do not subsequently take a modeling course, then the instructor should consider allocating some time (assuming course time did not also have to be devoted to probability) to a module on stochastic modeling in business or government (see list of modules below) or to a real problem at the local college, e.g., modeling the demand for textbooks in the bookstore or utilization of campus parking spaces.

Computers should be used in this course in two ways:

- As a computational aid to perform, for example, matrix calculations needed in Markov chain theory; and
- As a simulation device to exhibit the behavior of random processes.

Understanding randomness is difficult for undergraduates and discussion of data accumulated in simulation studies can help overcome students' deterministic biases.

COURSE CONTENT

Bernoulli process; Markov chains (random walks, classification of states, limiting distributions); Poisson process (as limit of binomial process and as derived via axioms); Markov processes (transition functions and state probabilities, Kolmogorov equations, limiting probabilities, birth-death processes).

These basic topics have numerous applications that should be an essential feature of the course. In addition, some applied topics can be covered such as quality control, social and occupational mobility, Markovian decision processes, road traffic, reliability, queueing problems, population dynamics or inventory models. Instructors can find these and other applications in the many good texts on stochastic processes. Also see the modules and modeling texts listed at the end of this chapter.

Continuous Modeling

A primary goal of a continuous modeling course is to present the mathematical analysis involved in scientific modeling, as for example, the derivation of the heat equation. The course should also give an introduction to important applied mathematics topics, such as Fourier series, regular and singular perturbations, stability theory and tensor analysis. A few advanced topics can be chosen from boundary layer theory, nonlinear waves and calculus of variations. The course should give a solid motivation for more advanced courses in these topics. A (non-original) paper on a topic of interest to the students serves the dual purpose of developing communication skills and introducing pedagogical flexibility.

A course on continuous modeling usually has as a prerequisite a course in differential equations, although the modeling can be taught concurrently or integrated in one course, using a book such as Martin Braun's ***Differential Equations and Their Applications*** (second edition), Springer-Verlag, 1978. Continuous modeling problems frequently involve concepts from natural sciences. In this case, it is important that either an appropriate background is required of students or the technical essentials are adequately introduced in the course.

The texts by Lin and Segal and by Haberman (see below) are well suited for this course. Selections from the two-volume Lin and Segal text can be used to provide a solid basis for physics and engineering modeling using both classical subjects, such as fluids, solids and heat transfer, and modern subjects, such as fields of biology. The text's broad coverage probably includes an introduction to an area of expertise of the instructor to which he or she can bring personal research insights.

A course which requires a little less sophistication can be designed around Haberman's book. This text's topics in population dynamics, oscillations, and traffic theory require less scientific background than topics in mechanics and mathematical biology, but still provide an excellent basis for modeling discussions. For example, population dynamics provide a good introduction to dynamical systems. Topics in regular and singular perturbation theory can be presented in the context of oscillations. Traffic theory provides a vehicle for introducing continuum mechanical modeling in which the processes are readily appreciated by students. Here the "microscopic" processes involve cars and drivers, and interesting models are obtained by car-following theory. Traffic flows also involve partial differential equations and shock waves.

References on Modeling

Modules

A. MODULE WRITING PROJECTS

Claremont Graduate School (Department of Mathematics):

- A Fractional Calculus Approach to a Simplified Air Pollution Model for Perturbation Analysis.

- Continuous-system Simulation Languages for DEC-10.
- Free Vibrations in the Inner Ear.
- Modeling of Stellar Interiors.
- Subsurface Areal Flow Through Porous Media.
- Variance Reduction for Monto Carlo Applications Involving Deep Penetration.
- Voting Games and Power Indices.

Mathematical Association of America's Committee on the Undergraduate Program in Mathematics Project, Case Studies in Applied Mathematics (designed especially for open-ended experiential teaching).

- Measuring Power in Weighted Voting Systems.
- A Model for Municipal Street Sweeping Operations.
- A Mathematical Model of Renewable Resource Conservation.
- Dynamics of Several-species Ecosystems.
- Population Mathematics.
- MacDonald's Work on Helminth Infections.
- Modeling Linear Systems by Frequency Response Methods.
- Network Analysis of Steam Generator Flow.
- Heat Transfer in Frozen Soil.

Mathematical Association of America Summer 1976 Module-writing Conference (at Cornell University Department of Operations Research):

- About sixty modules covering virtually all areas of application, such as biology, ecology, economics, energy, population dynamics, traffic flow, vibrating strings, and voting.
- Selected modules from this conference along with MAA applied mathematics case studies (ii) above were published by Springer-Verlag (New York, 1983) in four volumes, edited by William Lucas.

Rensselaer Polytechnic Institute (Department of Mathematical Sciences), published in *Case Studies in Mathematical Modeling,* by W. Boyce, Pitman, Boston, 1981:

- Herbicide Resistance.
- Elevator Systems.
- Traffic Flow.
- Shortest Paths in Networks.
- Computer Data Communication and Security.
- Semiconductor Crystal Growth.

State University of New York at Stony Brook (Department of Applied Mathematics and Statistics):

- A Model for Land Development.
- A Model for Waste Water Disposal, I and II.
- A Water Resource Planning Model.
- Man in Competition with the Spruce Budworm.
- Smallpox: When Should Routine Vaccination be Discontinued.
- Stochastic Models for the Allocation of Fire Companies.

B. MODULES DEVELOPED BY INDIVIDUALS

- Undergraduate Mathematics Application Project (UMAP): UMAP has several hundred modules covering all areas of application. Selected modules appear in the *UMAP Journal* (four issues a year), published by Birkhauser-Boston. UMAP catalogue available by writing to: UMAP, Educational Development Center, 55 Chapel Street, Newton, MA 02160.

C. PROCEEDINGS OF MODELING CONFERENCES

1. Discrete Mathematics and Its Applications, Proceedings of a Conference at Indiana University, ed. M. Thompson, 1976.
2. Mathematical Models in the Undergraduate Curriculum, Proceedings of Conference at Indiana University, ed. D. Maki and M. Thompson, 1975.
3. Proceedings of Summer Seminar on Applied Mathematics, ed. M. Thompson, Indiana University, 1978.
4. Mathematical Models for Environmental Problems, Proceedings of the International Conference at the University of Southampton, 1976.
5. Proceedings of Conference on Environmental Modeling and Simulation, Environmental Protection Agency, 1976.
6. Proceedings of a Conference on the Application of Undergraduate Mathematics in the Engineering, Life, Managerial and Social Sciences, ed. P. Knopp and G. Meyer, Georgia Institute of Technology, 1973.
7. Proceedings of the Pittsburgh Conferences on Modeling and Simulations, Vols. 1-9 (1969-78), Instrument Society of America.
8. Proceedings of the Summer Conference for College Teachers on Applied Mathematics, University of Missouri-Rolla, 1971.
9. Information Linkage Between Applied Mathematics and Industry, ed. P. Wang, Academic Press, 1976.

Articles on Teaching Modeling

1. J. Agnew and M. Keener, A Case-study Course in Applied Mathematics Using Regional Industries, *American Mathematical Monthly* 87 (1980).
2. R. Barnes, Applied Mathematics: An Introduction Via Models, *American Mathematical Monthly* 84 (1977).
3. C. Beaumont and R. Wieser, Co-operative Programmes in Mathematical Sciences at the University of Waterloo, *Journal of Co-operative Education* 11 (1975).

4. J. Becker, R. Borrelli, and C. Coleman, *Models for Applied Analysis,* Harvey Mudd College, 1976 and revised annually.
5. R. Borrelli and J. Spanier, The Mathematics Clinic: A Review of Its First Seven Years, *UMAP Journal* 2 (1981).
6. R. Borton, *Mathematical Clinic Handbook,* Claremont Graduate School, 1979.
7. J. Brookshear, A Modeling Problem for the Classroom, *American Mathematical Monthly* 85 (1978).
8. E. Clark, *How To Select a Clinic Project,* Harvey Mudd College, 1975.
9. C. Hall, Industrial Mathematics: A Course in Realism, *American Mathematical Monthly* 82 (1975).
10. L. Handa, *Mathematics Clinic Student Handbook: A Primer for Project Work,* Harvey Mudd College, 1979.
11. J. Hachigian, Applied Mathematics in a Liberal Arts Context, *American Mathematical Monthly* 85 (1978).
12. E. Rodin, Modular Applied Mathematics for Beginning Students, *American Mathematical Monthly* 84 (1977).
13. R. Rubin, Model Formulation Using Intermediate Systems, *American Mathematical Monthly* 86 (1979).
14. M. Seven and T. Zagar, *The Engineering Clinic Guidebook,* Harvey Mudd College, 1975.
15. D. Smith, A Seminar in Mathematical Model-building, *American Mathematical Monthly* 86 (1979).
16. J. Spanier, The Mathematics Clinic: An Innovative Approach to Realism Within an Academic Environment, *American Mathematical Monthly* 83 (1976).

Books on Mathematical Modeling

For further references, see Applications section of *A Basic Library List,* Mathematical Association of America, 1976.

A. GENERAL MODELING

1. J. Andrew and R. McLone, ed., *Mathematical Modeling,* Butterworth, 1976.
2. R. Aris, *Mathematical Modeling Techniques,* Pitman, 1978.
3. E. Beltrami, *Mathematics for Dynamic Modeling,* Academic Press, 1987.
4. E. Bender, *An Introduction to Mathematical Modeling,* Wiley, 1978.
5. G. Carrier, *Topics in Applied Mathematics,* Vol. I and II, MAA summer seminar lecture notes, Mathematical Association of America, 1966.
6. C. Coffman and G. Fix, ed., *Constructive Approaches to Mathematical Models,* Academic Press, 1980.
7. R. DiPrima, ed., *Modern Modeling of Continuous Phenomena,* American Mathematical Society, 1977.
8. C. Dym and E. Ivey, *Principles of Mathematical Modeling,* Academic Press, 1980.
9. B. Friedman, *Lectures on Applications-oriented Mathematics,* Holden-Day, 1969.
10. F. Giordano and M. Weir, *A First Course in Mathematical Modeling,* Brooks/Cole, 1985.
11. P. Lancaster, *Mathematics Models of the Real World,* Prentice Hall, 1976.
12. D. Maki and M. Thompson, *Mathematical Models and Applications,* Prentice Hall, 1976.
13. F. Roberts, *Discrete Mathematical Models,* Prentice Hall, 1976.
14. T. Saaty, *Thinking with Models,* AAAS Study Guides on Contemporary Problems No. 9, 1974.

B. MODELING IN VARIOUS DISCIPLINES

Mathematical modeling is such an integral part of physics and engineering that any text in these fields is implicitly a mathematical modeling book.

1. P. Abell, *Model Building in Sociology,* Shocken, 1971.
2. R. Aggarwal and I. Khera, *Management Science Cases and Applications,* Holden-Day, 1979.
3. R. Atkinson, et al., *Introduction to Mathematical Learning Theory,* Krieger Publishing, 1965.
4. D. Bartholomew, *Stochastic Models for Social Processes,* Wiley, 1973.
5. M. Bartlett, *Stochastic Population Models,* Methuen, 1960.
6. R. Barton, *A Primer on Simulation and Gaming,* Prentice Hall, 1970.
7. S. Brams, *Game Theory and Politics,* The Free Press, 1975.
8. C. Clark, *Mathematical Bioeconomics,* Wiley, 1976.
9. J. Coleman, *Introduction to Mathematical Sociology,* Free Press, 1964.
10. P. Fishburn, *The Theory of Social Choice,* Princeton University Press, 1973.
11. J. Frauenthal, *Introduction to Population Modeling,* UMAP Monograph, 1979.
12. H. Gold, *Mathematical Modeling of Biological Systems,* Wiley, 1977.
13. S. Goldberg, *Some Illustrative Examples of the Use of Undergraduate Mathematics in the Social Sciences,* Mathematical Association of America, CUPM Report, 1977.

14. M. Gross, *Mathematical Models in Linguistics,* Prentice Hall, 1972.
15. R. Haberman, *Mathematical Models, Mechanical Vibrations, Population Dynamics and Traffic Flow,* Prentice Hall, 1977.
16. F. Hoppensteadt, *Mathematical Theories of Populations: Demographics and Epidemics,* SIAM, 1976.
17. J. Kemeny and L. Snell, *Mathematical Models in the Social Sciences,* MIT Press, 1973.
18. C. Lave and J. March, *An Introduction to Models in the Social Sciences,* Harper and Row, 1975.
19. C. Lin and L. Segal, *Mathematics Applied to Deterministic Problems in the Natural Sciences,* Macmillan, 1974.
20. D. Ludwig, *Stochastic Population Theories,* Springer, 1974.
21. J. Maynard-Smith, *Models in Ecology,* Cambridge University Press, 1974.
22. B. Noble, *Applications of Undergraduate Mathematics to Engineering,* Mathematical Association of America, 1976.
23. M. Olinik, *An Introduction to Mathematical Models in the Social and Life Sciences,* Addison Wesley, 1978.
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25. H. Pollard, *Mathematical Introduction to Celestial Mechanics,* Mathematical Association of America, 1977.
26. J. Pollard, *Mathematical Models for the Growth of Human Populations,* Cambridge University Press, 1973.
27. D. Riggs, *The Mathematical Approach to Physiological Problems,* Macmillan, 1979.
28. T. Saaty, *Topics in Behavioral Mathematics,* MAA summer seminar lecture notes, Mathematical Association of America, 1973.
29. H. Scarf, et al., *Notes on Lectures on Mathematics in the Behavioral Sciences,* MAA summer seminar lecture notes, Mathematical Association of America, 1973.
30. C. von Lanzenauer, *Cases in Operations Research,* Holden Day, 1975.
31. H. Williams, *Model Building in Mathematical Programming,* Wiley, 1978.

Subpanel Members

WILLIAM LUCAS, CHAIR, Cornell University.
ROBERT BORRELLI, Harvey Mudd College.
RALPH DISNEY, VPI & SU.
DONALD DREW, RPI.
SAM GOLDBERG, Oberlin College.