APPLIED MATHEMATICS

The importance of applications of mathematics to other areas was recognized by CUPM early in its existence. Among the original four panels were a Panel on Mathematics for the Physical Sciences and Engineering and a Panel on Mathematics for the Biological, Management, and Social Sciences, each charged with the task of making recommendations for the undergraduate mathematics program of students whose major interest lay in one of the stated fields.

The Panel on Physical Sciences and Engineering concentrated its efforts on the training of engineers and physicists, issuing its first report (Recommendations on the Undergraduate Mathematics Program for Engineers and Physicists) in 1962. The demand for this document was so great that it was necessary to have it reprinted in 1965. Significant developments which occurred during the mid-sixties prompted the Panel to revise its recommendations and issue a new report in 1967. In the meantime this Panel had also developed CUPM's first definitive statement regarding the role of the computer in undergraduate mathematics. Its 1964 report Recommendations on the Undergraduate Mathematics Program for Work in Computing contained outlines for introductory and technical courses in computer science and a description of a program for mathematics majors planning to enter the field of computing; it is not being reproduced in this COMPENDIUM because it has been superseded by more recent CUPM documents. (See the section on COMPUTING.) Another document, Mathematical Engineering--A Five-Year Program, was issued by the Panel in 1966 to provide a means of alleviating what was then a drastic shortage of engineers having a substantial background in mathematics. Described as "a suggestion, rather than a recommendation," this report gives several outlines for options in operations research, orbit mechanics, and control theory.

The Panel on Mathematics for the Biological, Management, and Social Sciences, confronting problems which were less well defined, issued its Tentative Recommendations for the Undergraduate Mathematics Program for Students in the Biological, Management, and Social Sciences* in 1964. Primarily concerned with the mathematics curriculum for prospective graduate students in those fields, the report was meant to serve as a basis for discussion and experimentation. As a result of several issues raised in reaction to this document, CUPM decided in 1967 to concentrate on individual disciplines and, as a first step, appointed a Panel on Mathematics in the Life Sciences, charged with making recommendations for the mathematical training of the undergraduate life science student, whether or not he goes on to graduate school. The term "life science" here referred to agriculture and renewable resources, all branches of biology, and medicine. This Panel worked closely with the Commission on Undergraduate Education in the Biological Sciences, and its investigations culminated

* Not included in this COMPENDIUM.

in the publication of Recommendations for the Undergraduate Mathematics Program for Students in the Life Sciences--An Interim Report (1970). Although it was anticipated that a final form of this report would eventually be issued, this project was never undertaken due to lack of funds.

Appointed in 1964, the CUPM *ad hoc* Subcommittee on Applied Mathematics was charged with suggesting appropriate undergraduate programs for students planning careers in applied mathematics. The Subcommittee's recommendations for such a program, together with suggestions for implementation and course descriptions, appeared in the 1966 report A Curriculum in Applied Mathematics.* During the years 1967-69 an Advisory Group on Applications kept CUPM informed on current developments in applied mathematics. The extremely rapid development of applications of mathematics, particularly in fields outside the physical sciences, together with a renewed interest in applications among mathematicians, led CUPM to appoint in 1970 a Panel on Applied Mathematics, whose duty was to reconsider some of the questions which the Subcommittee had studied earlier, and to draw up new recommendations in line with the nature and methods of applied mathematics. The Panel's suggestions, which emphasize the role of model building, are given in Applied Mathematics in the Undergraduate Curriculum (1972). This report contains detailed outlines of three options for a course in applied mathematics, each of which utilizes the model-building approach.

* Not included in this COMPENDIUM.

RECOMMENDATIONS ON THE UNDERGRADUATE MATHEMATICS PROGRAM FOR ENGINEERS AND PHYSICISTS

A Report of

The Panel on Mathematics for the Physical Sciences and Engineering

Revised January 1967

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BACKGROUND (1962)

One reason for the current effort on the undergraduate program is the rapid change in the mathematical world and in its immediate surroundings. Three aspects of this change have a particular effect on undergraduate curricula in the physical sciences and engineering. The first is the work being done in improving mathematics education in the secondary school. Several programs of improvement in secondary school mathematics have already had considerable effect and can be expected to have a great deal more. Not only can we hope that soon most freshmen expecting to take a scientific program will have covered precalculus mathematics, but, perhaps more important, they will be accustomed to care and precision of mathematical thought and statement. Of course, not all students will have this level of preparation in the foreseeable future, but the proportion will be large enough to enable us to plan on this basis. Students with poorer preparation may be expected to take remedial courses without credit before they start the regular program.

This improved preparation obviously means that we will be able to improve the content of the beginning calculus course since topics which take time in the first two years will have been covered earlier. More than that, however, it means that the elementary calculus course will have to take a more sophisticated attitude in order to keep the student from laughing at a course in college which is less careful mathematically than its secondary school predecessors.

The second aspect of change in mathematics which confronts us is the expansion in the applications of mathematics. There is a real "revolution" in engineering--perhaps "explosion" is an even better description than "revolution," because, as it turns out, several trends heading in different directions are simultaneously visible. One is a trend toward basic science. The mathematical aspect of this trend is a strengthening of interest in more algebraic and abstract concepts. An orthogonal trend is one toward the engineering of large systems. These systems, both military and nonmilitary, are of ever-increasing complexity and must be optimized with regard to such factors as cost, reliability, maintenance, etc. Resulting mathematical interests are linear algebra and probability-statistics. A further trend, in part a consequence of the preceding two, is a real increase in the variety and depth of the mathematical tools which interest the engineer. In general, engineers are finding that they need to use new and unfamiliar mathematics of a wide variety of types.

A third factor is the arrival of the electronic computer. It is having its effect on every phase of science and technology, all the way from basic research to the production line. In mathematics it has, for one thing, moved some techniques from the abstract to the practical field; for example, some series expansion, iterative techniques, and so forth. Then too, computers have led people to tackle problems they would never have considered before, such as large systems of linear equations, linear and nonlinear programming, and

Monte Carlo methods. Many of these new techniques require increased sophistication in mathematics.

An additional factor entering from another direction must also be mentioned. Mathematicians in the United States have in recent years become much more closely involved with areas adjacent to their own research. Of the many factors which enter here, we may mention the greatly increased interest of mathematicians at all levels in education, the rapid growth of mathematical employment in industry, the spread of research and consulting contracts into the universities, and the development of a number of mathematical disciplines, such as information theory, that have many applications but are not classical applied mathematics. There is thus a real desire among mathematicians and scientists to cooperate in matters of education.

The conclusions above and the recommendations that constitute the body of this report were formulated by the Panel after extensive consultation with mathematicians, physicists, and engineers. In engineering, in particular, representatives of many fields and many types of institutions were consulted, as well as officials of the American Society for Engineering Education.* The recommendations for physicists were drawn up in close collaboration with the Commission on College Physics.

In considering the recommendations which follow, it is crucial to examine what has been our attitude toward certain ideas which inevitably occupy a central position in any discussion of mathematical education. Among these are mathematical sophistication and mathematical rigor, motivation, and intuition. Now it is a fact that mathematical rigor--by which we mean an attempt to prove essentially everything that is used--is not the way of life of the physicist and the engineer. On the other hand, mathematical sophistication--which means to us careful and clear mathematical statements, proofs of many things, and generally speaking a broad appreciation of the mathematical blocks from which models are built--is desired by, and desirable for, all students preparing for a scientific career. How does one choose what is actually to be proved? It seems to us that this is related to the plausibility of the desired result. It is unwise to give rigor to either the utterly plausible or the utterly implausible, the former because the student cannot see what the fuss is all about, and the latter because the most likely effect is rejection of mathematics. The moderately plausible and the moderately implausible are the middle ground where we may insist on rigor with the greatest profit; the great danger in the overzealous use of rigor is to employ it to verify only that which is utterly apparent.

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* Some of the results of a conference with engineers are embodied in four addresses delivered at a Conference on Mathematics in the Engineering Curriculum, held under the auspices of this Panel in March, 1961. These addresses were published in the Journal of Engineering Education, 52 (1961), pp. 171-207.

Let us turn next to the subject of motivation. Motivation means different things to different people and thus requires clarification. One aspect of motivation is concerned with the difference between mathematics and the applications of mathematics, between a mathematical model and the real world. For many engineers and physicists motivation of mathematical concepts can be supplied by formulating real situations which lead to the construction of reasonable models that exhibit both the desirability and the usefulness of the mathematical concept. Thus, motion of a particle or growth of a bacterial culture may be used as physical motivation for the notion of a derivative. It is also possible, of course, to give a mathematical motivation for a new mathematical concept; the geometric notion of a tangent to a curve also leads to the notion of derivative and is quite enough motivation to a mathematician. Since each kind of motivation is meaningful to large groups of students, we feel that both should appear wherever relevant. It is certainly a matter of individual taste whether one or both motivations should precede, or perhaps follow, the presentation of a mathematical topic. In either case, however, it is necessary to be very clear in distinguishing the motivating mathematical or physical situation from the resulting abstraction.

Physical and mathematical examples which are used as motivation, as well as previous mathematical experience, help to develop one's intuition for the mathematical concept being considered. By "intuition" we mean an ability to guess both the mathematical properties and the limitations of a mathematical abstraction by analogy with known properties of the mathematical or physical objects which motivated that abstraction. Intuition should lead the way to rigor whenever possible; neither can be exchanged or substituted for the other in the development of mathematics.

A mathematics course for engineers and physicists must involve the full spectrum from motivation and intuition to sophistication and rigor. While the relative emphasis on these various aspects will forever be a subject for debate, no mathematics course is a complete experience if any of them is omitted.

INTRODUCTION TO THE REVISION (1967)

In the five years that have elapsed since the first publication of these recommendations, several factors have emerged to affect the teaching of mathematics to engineers. The most striking of these is the widespread application of automatic computers to engineering problems. It is now a commonplace that all engineers must know how to use computers and that this knowledge must be gained early in their training and reinforced by use throughout it. We have, accordingly, included an introductory course in computer science as a

requisite for all engineering students and have increased the amount of numerical mathematics in other courses wherever possible.

A second factor is the fairly general acceptance of linear algebra as part of the beginning mathematics program for all students. In the engineering curriculum this is tied in to the expansion in computing, since linear algebra and computers are precisely the right team for handling the large problems in systems analysis that appear in so many modern investigations. Five years ago there were only a handful of elementary texts on linear algebra; now treatments are appearing almost as fast as calculus books (with which they are often combined).

A development of particular interest to these recommendations is the appearance of the CUPM report A General Curriculum in Mathematics for Colleges (1965), referred to hereafter as GCMC. It is too early to judge how widely the GCMC will be adopted, but initial reactions, including those of teachers of engineering students, have been generally favorable. GCMC makes considerable use of material in the first version of these recommendations, and now we, in turn, borrow some of the courses in GCMC.

Minor changes in the content of courses and some rearrangement and changes of emphasis are the result of experience and discussions over the years.

Relatively little change has been made in the program for physicists. The only major one has been the inclusion of Introduction to Computer Science in the required courses. We do this in the conviction that all scientists (if not, indeed, all college graduates) should know something about the powers and limitations of automatic computers.

Applications of Undergraduate Mathematics in Engineering, written and edited by Ben Noble, published in January, 1967, by the Mathematical Association of America and the Macmillan Company, is based on a collection of problems assembled as a joint project of CUPM and the Commission on Engineering Education. The book has five parts: Illustrative Applications of Elementary Mathematics, Applications of Ordinary Differential Equations, Applications to Field Problems, Applications of Linear Algebra, Applications of Probability Theory.

INTRODUCTION TO THE RECOMMENDATIONS

This report presents a program for the undergraduate mathematical preparation of engineers and physicists.

Since obviously no single program of study can be the best one for all types of students, all institutions, and all times, it is important that anyone expecting to make use of the present recommendations understand the assumptions underlying them. The following comments should make these assumptions clear and also explain some other features of the recommendations.

1. This is a program for today, not for several years in the future. Programs somewhat like this are already being given at various places, and the sample courses we outline are patterned after existing ones. We assume a good but not unusual background for the entering freshman.

Five or ten years from now the situation will undoubtedly be different--in the high schools, in research, in engineering practice, and in such adjacent areas as automatic computation. Such differences will necessitate changes in the mathematics curriculum, but a good curriculum can never be static, and it is our belief that the present proposal can be continually modified to keep up with developments. However, the material encompassed here will certainly continue to be an important part of the mathematical education needed by engineers and physicists.

2. The program we recommend may seem excessive in the light of what is now being done at many places, but it is our conviction that this is the minimal amount of mathematics appropriate for students who will be starting their careers four or five years from now. We recognize that some institutions may simply be unable to introduce such a program very soon. We hope that such places will regard the program as something to work toward.

3. Beyond the courses required of all students there must be available considerable flexibility to allow for variations in fields and in the quality of students. The advanced material whose availability we have recommended can be regarded as a main stem that may have branches at any point. Also, students may truncate the program at points appropriate to their interests and abilities.

4. The order of presentation of topics in mathematics and some related courses is strongly influenced by two factors:

a. The best possible treatment of certain subjects in engineering and physics requires that they be preceded by certain mathematical topics.

b. Topics introduced in mathematics courses should be used in applications as soon afterwards as possible.

To attain these ends, coordination among the mathematics, engineering, and physics faculties is necessary, and this may lead to course changes in all fields.

5. The recommendations are, of course, the responsibility of CUPM. In cases where it seems of interest and is available, we have indicated the reaction of the groups of engineers and physicists who were consulted. For convenience we refer to them as "the consultants."

LIST OF RECOMMENDED COURSES

It is desirable that all calculus prerequisites, including analytic geometry, be taught in high school. At present it may be necessary to include some analytic geometry in the beginning analysis course, but all other deficiencies should be corrected on a non-credit basis.

The following courses should be available for undergraduate majors in engineering and physics:

1. Beginning Analysis. (9-12 semester hours)

As far as general content is concerned, this is a relatively standard course in calculus and differential equations. There can be many variations of such a course in matters of rigor, motivation, arrangement of topics, etc., and textbooks have been and are being written from several points of view.

The course should contain the following topics:

a. An intuitive introduction of four to six weeks to the basic notions of differentiation and integration. This course serves the dual purpose of augmenting the student's intuition for the more sophisticated treatment to come and preparing for immediate applications to physics.

b. Theory and techniques of differentiation and integration of functions of one real variable, with applications.

c. Infinite series, including Taylor series expansion.

d. A brief introduction to differentiation and integration of functions of two or more real variables.

e. Topics in differential equations, including the following: linear differential equations with constant coefficients and first-order systems--linear algebra (including eigenvalue theory, see 2 below) should be used to treat both homogeneous and nonhomogeneous problems; first-order linear and nonlinear equations, with Picard's method and an introduction to numerical techniques.

f. Some attempt should be made to fill the gap between the high school algebra of complex numbers and the use of complex exponentials in the solution of differential equations. In particular, some work on the calculus of complex-valued functions of a real variable should be included in items b and c.

g. Students should become familiar with vectors in two and three dimensions and with the differentiation of vector-valued functions of one variable. This material can obviously be correlated with the course in linear algebra (see below).

h. Theory and simple techniques of numerical computation should be introduced where relevant. Further comments on this point, applying to the whole program, will be found below (under course 3).

We feel that the above comments on beginning analysis sufficiently describe a familiar course. The remaining courses in our list are less generally familiar. Hence the brief descriptions of courses 2 through 12 are supplemented in the Appendix [or elsewhere in this COMPENDIUM] by detailed outlines of sample courses of the kind we have in mind.

2. Linear Algebra. (3 semester hours)

A knowledge of the basic properties of n-dimensional vector spaces has become imperative for many fields of applications as well as for progress in mathematics itself. Since this subject is so fundamental and since its development makes no use of the concepts of calculus, it should appear very early in the student's program. We recommend a course with strong emphasis on the geometrical interpretation of vectors and matrices, with applications to mathematics (see items 1-e and 1-g above), physics, and engineering. Topics should include the algebra and geometry of vector spaces, linear transformations and matrices, linear equations (including computational methods), quadratic forms and symmetric matrices, and elementary eigenvalue theory.

It may be desirable, for mathematical or scheduling purposes, to combine beginning analysis and linear algebra into a single coordinated course to be completed in the sophomore year.

For outlines of a Beginning Analysis sequence, see the courses Mathematics 1, 2, and 4 described in Commentary on A General Curriculum in Mathematics for Colleges, page 44 . The course Mathematics 3 (Elementary Linear Algebra) of the GCMC Commentary (page 55) approximates the linear algebra course described here, but does not contain the recommended material on quadratic forms and elementary eigenvalue theory. This Panel's recommended courses on functions of several variables, functions of a complex variable, real variables, and algebraic structures coincide with those of the GCMC Commentary (Mathematics 5 [alternate version], 13, 11, and 6M, respectively).

3. Introduction to Computer Science. (3 semester hours)

The development of high-speed computers has made it necessary for the appliers of mathematics to know the path from mathematical theory through programming logic to numerical results. This course gives an understanding of the position of the computer along this path, the manner of its use, its capabilities, and its limitations. It also provides the student with the basic techniques needed in order to use the computer to solve problems in other courses.

An even more important part of the path must be provided by the student's program as a whole. All the courses discussed here should contain, where it is suitable and applicable, mathematical topics motivated by the desire to relate mathematical understanding to computation. It is especially desirable that the student see the possibility of significant advantage in combining analytical insight with numerical work. Indications of such opportunities are scattered throughout the recommended course outlines.

4. Probability and Statistics. (6 semester hours)

Basic topics in probability theory, both discrete and continuous, have become essential in every branch of engineering, and in many engineering fields an introduction to statistics is also needed. We recommend a course based on the notions of random variables and sample spaces, including, *inter alia*, an introduction to limit theorems and stochastic processes and to estimation and hypothesis testing. Although this should be regarded as a single integrated course, the first half can be taken as a course in probability theory. For ease of reference we designate the two halves 4a and 4b.

5. Advanced Multivariable Calculus. (3 semester hours)

Continuation of item 1-d. A study of the properties of continuous mappings from E_n to E_m, making use of the linear algebra in course 2, and an introduction to differential forms and vector calculus based on line integrals, surface integrals, and the general Stokes theorem. Application should be made to field theory, elementary hydrodynamics, or other similar topics, so that some intuitive understanding can be gained.

6. Intermediate Ordinary Differential Equations. (3 semester hours)

This course continues the work on item 1-e into further topics important to applications, including linear equations with variable coefficients, boundary value problems, rudimentary existence theorems, and an introduction to nonlinear problems. Much attention should be given to numerical techniques.

7. Functions of a Complex Variable. (3 semester hours)

This course presupposes somewhat more mathematical maturity than courses 5 and 6 and so would ordinarily be taken after them,

even though they are not prerequisites as far as subject matter is concerned. In addition to the usual development of integrals and series, there should be material on multivalued functions, contour integration, conformal mapping, and integral transforms.

8. Partial Differential Equations. (3 semester hours)

Derivation, classification, and solution techniques of boundary value problems.

9. Introduction to Functional Analysis. (3 semester hours)

An introduction to the properties of general linear spaces and metric spaces, their transformations, measure theory, general Fourier series, and approximation theory.

10. Elements of Real Variable Theory. (3 semester hours)

A rigorous treatment of basic topics in the theory of functions of a real variable.

11. Optimization. (3 semester hours)

Linear, nonlinear, and dynamic programming, combinatorics, and calculus of variations.

12. Algebraic Structures. (3 semester hours)

An introduction to the theory of groups, rings, and fields.

13. Numerical Analysis.

14. Mathematical Logic.

15. Differential Geoemtry.

The last three courses are topics that might well be of interest to special groups of students. Their lengths and contents may vary considerably. For a sample outline of a course in Numerical Analysis, see Mathematics 8 (Introduction to Numerical Analysis) in Commentary on A General Curriculum in Mathematics for Colleges, page 83.

The above list of courses is the result of careful consideration by the Panel and the consultants. The brief description given here and the detailed sample outlines found in the Appendix [or elsewhere in this COMPENDIUM], while based on the mathematical structure of the topics themselves, reflect strongly the expressed interests of engineers and physicists. We realize that the nature of the institution and the requirements of other users of mathematics as well as of the mathematics majors may influence the specific offerings.

RECOMMENDED PROGRAM FOR ENGINEERS

A. Courses to be required of all students.

1. Beginning Analysis. This recommendation needs no comments.

2. Linear Algebra. The great majority of the consultants felt that this is important material that all engineers should have during the first two years.

3. Introduction to Computer Science. Developments of the last few years make it clear that engineering is strongly dependent on a knowledgeable use of computers.

4a. Probability. All students should have at least a 3-semester-hour course in probability. The consultants agreed on the value of probability to an engineer, but there was considerable disagreement among the consultants as to the advisability of requiring it of all students. However, the members of our Panel are unanimously and strongly of the opinion that this subject will soon pervade all branches of engineering and that now is the time to begin preparing students for this development.

B. Courses recommended for students intending to go into research and development.

4b. Statistics.

5. Advanced Multivariable Calculus.

6. Intermediate Ordinary Differential Equations.

7. Functions of a Complex Variable.

The consultants agreed to the value of the material in courses 5, 6, and 7, and some preferred that it be completed within the junior year. The Panel is convinced that an adequate presentation requires a minimum of nine semester hours, which could, of course, be taken in one year if desired. The order in which courses 5 and 6 are taken is immaterial except as they may be coordinated with other courses. If they are to be presented to the students in a fixed order, the instructor may wish to adjust the time schedules and choice of topics.

C. Courses which should be available for theoretically minded students capable of extended graduate study.

8. Partial Differential Equations.

9. Introduction to Functional Analysis.

10. Elements of Real Variable Theory.

Presumably a student would take either 9 or 10 but not both; 9 is probably more valuable but 10 is more likely to be available.

11. Optimization.

D. Courses of possible interest to special groups.

12. Algebraic Structures.

13. Numerical Analysis.

14. Mathematical Logic.

15. Differential Geometry.

RECOMMENDED PROGRAM FOR PHYSICISTS

A. Courses to be required of all students.

1. Beginning Analysis.

2. Linear Algebra. Like the engineers, the physicists felt that this material is essential.

3. Introduction to Computer Science.

5. Advanced Multivariable Calculus. This course should be taken in the sophomore year if possible, and in any event no later than the first part of the junior year.

6. Intermediate Ordinary Differential Equations.

B. Additional courses, in order of preference. Students contemplating graduate work should be required to take a minimum of three to nine semester hours of these courses.

7. Functions of a Complex Variable.

9. Introduction to Functional Analysis.

4a. Probability. The value of requiring this course in the undergraduate program of all physicists is not as well established as it is for engineers.

12. Algebraic Structures.

10. Elements of Real Variable Theory.

8. Partial Differential Equations.

Appendix

DESCRIPTION OF RECOMMENDED COURSES

While we feel strongly about the spirit of the courses outlined here, the specific embodiments are to be considered primarily as samples. Courses close to these have been taught successfully at appropriate levels, and our time schedules are based on this experience. Some of these courses are sufficiently common that approximations to complete texts already exist; others have appeared only in lecture form.

2. Linear Algebra. (3 semester hours)

The purpose of this course is to develop the algebra and geometry of finite-dimensional linear vector spaces and their linear transformations, the algebra of matrices, and the theory of eigenvalues and eigenvectors.

The course Mathematics 3 (Elementary Linear Algebra) of Commentary on a General Curriculum in Mathematics for Colleges (page 55) approximates the linear algebra course which this Panel has in mind. Mathematics 3 does not, however, contain the recommended material on quadratic forms and elementary eigenvalue theory.

3. Introduction to Computer Science. (3 semester hours)

This course serves a number of purposes:

(1) It gives students an appreciation of the powers and limitations of automata.

(2) It develops an understanding of the interplay between the machine, its associated languages, and the algorithmic formulation of problems.

(3) It teaches students how to use a modern computer.

(4) It enables instructors in later courses to assign problems to be solved on the computer.

For an outline of such a course, see C1 (Introduction to Computing) in Recommendations for an Undergraduate Program in Computational Mathematics (page 563).

4. Probability and Statistics. (6 semester hours)

This is a one-year course presenting the basic theory of probability and statistics. Although the development of the ideas and results is mathematically precise, the aim is to prepare students to formulate realistic models and to apply appropriate statistical techniques in problems likely to arise in engineering. Therefore new ideas will be motivated and applications of results will be given wherever possible.

First Semester: Probability.

a. Basic probability theory. (4 lectures) Different theories of probability (classical, frequency, and axiomatic). Combinatorial methods for computing probability. Conditional probability, independence. Bayes' theorem. Geometrical probability.

b. Random variables. (5 lectures) Concept of random variable and of distribution function. Discrete and continuous types. Multidimensional random variables. Marginal and conditional distributions.

c. Parameters of a distribution. (4 lectures) Expected values. Moments. Moment-generating functions. Moment inequalities.

d. Characteristic functions. (4 lectures) Definition, properties. Characteristic functions and moments. Determination of distribution function from characteristic function.

e. Various probability distributions. (6 lectures) Binomial, Poisson, multinomial. Uniform, normal, gamma, Weibull, multivariate normal. Importance of normal distribution. Applications of normal distribution to error analysis.

f. Limit theorems. (6 lectures) Various kinds of convergence. Law of Large Numbers. Central Limit Theorem.

g. Markov chains. (4 lectures) Transition matrix. Ergodic theorem.

h. Stochastic processes. (6 lectures) Markov processes. Processes with independent increments. Poisson process. Wiener process. Stationary processes.

Second Semester: Statistics.

a. Sample moments and their distributions. (5 lectures) Sample, statistic. Distribution of sample mean. Student's distribution. Fisher's Z distribution.

b. Order statistics. (4 lectures) Empirical distribution function. Tolerance limits. Kolmogorov-Smirnov statistic.

c. Tests of hypotheses. (5 lectures) Simple hypothesis against simple alternative. Composite hypotheses. Likelihood ratio test. Applications.

d. Point estimation. (5 lectures) Consistent estimates. Unbiased estimates. Sufficient estimates. Efficiency of estimate. Methods of finding estimates.

e. Interval estimation. (6 lectures) Confidence and tolerance intervals. Confidence intervals for large samples.

f. Regression and linear hypotheses. (4 lectures) Elementary linear models. The general linear hypothesis.

g. Nonparametric methods. (5 lectures) Tolerance limits. Comparison of two populations. Sign test. Mann-Whitney test.

h. Sequential methods. (5 lectures) The probability ratio sequential test. Sequential estimation.

5. Advanced Multivariable Calculus. (3 semester hours)

For an outline of this course, see Mathematics 5 (Multivariable Calculus II--alternate version) in Commentary on A General Curriculum in Mathematics for Colleges, page 77.

6. Intermediate Ordinary Differential Equations. (3 semester hours)

The presentation of the course material should include: (1) an account of the manner in which ordinary differential equations and their boundary value problems, both linear and nonlinear, arise; (2) a carefully reasoned discussion of the qualitative behavior of the solution of such problems, sometimes on a predictive basis and at other times in an *a posteriori* manner; (3) a clearly described awareness of the role of numerical processes in the treatment of these problems, including the disadvantages as well as the advantages--in particular, there should be a firm emphasis on the fact

that numerical integration is not a substitute for thought; (4) an admission that we devote most of our lecture time to linear problems because (with isolated exceptions) we don't know much about any nonlinear ones except those that (precisely or approximately) can be attacked through our understanding of the linear ones. Thus, a thorough treatment of linear problems must precede a sophisticated attack on the nonlinear ones.

The distribution of time among items d through f cannot be prescribed easily or with universal acceptability. Only a superficial account of these topics can be given in the available time, but each should be introduced.

a. Systems of linear ordinary differential equations with constant coefficients. (6 lectures) Review of homogeneous and nonhomogeneous problems; superposition and its dependence on linearity; transients in mechanical and electrical systems. The Laplace transform as a carefully developed operational technique without inversion integrals.

b. Linear ordinary differential equations with variable coefficients. (10 lectures) Singular points, series solutions about regular points and about singular points. Bessel's equation and Bessel functions; Legendre's equation and Legendre polynomials; confluent hypergeometric functions. Wronskians, linear independence, number of linearly independent solutions of an ordinary differential equation. Sturm-Liouville theory and eigenfunction expansions.

c. Solution of boundary value problems involving nonhomogeneous linear ordinary differential equations. (7 lectures) Methods using Wronskians, Green's functions (introduce δ functions), and eigenfunction expansions. Numerical methods. Rudimentary existence and uniqueness questions.

d. Asymptotic expansion and asymptotic behavior of solutions of ordinary differential equations. (3 lectures) Essentially the material on pp. 498-500 and pp. 519-527 of Methods of Mathematical Physics by Harold Jeffreys and Bertha S. Jeffreys (third edition; New York, Cambridge University Press, 1956).

e. Introduction to nonlinear ordinary differential equations. (6 lectures) Special nonlinear equations which are reducible to linear ones or to quadratures, elliptic functions (pendulum oscillations), introductory phase plane analysis (Poincaré).

f. Numerical methods. (7 lectures) Step-by-step solution of initial value problems for single equations and for systems. Error analysis, roundoff, stability. Improper boundary conditions, discontinuities, and other pitfalls.

7. Functions of a Complex Variable. (3 semester hours)

For an outline of this course, see Mathematics 13 (Complex Analysis) in Commentary on A General Curriculum in Mathematics for Colleges, page 97.

8. Partial Differential Equations. (3 semester hours)

This course is suitable for students who have completed a course in functions of a complex variable. The emphasis is on the development and solution of suitable mathematical formulations of scientific problems. Problems should be selected to emphasize the role of "time-like" and "space-like" coordinates and their relationship to the classification of differential equations. (It seems very useful to introduce the appropriate boundary conditions motivated by the physical questions and be led to the classification question by observing the properties of the solution.) The student should be led to recognize how few techniques we have and how special the equations and domains must be if explicit and exact solutions are to be obtained; he particularly must come to realize that the effective use of mathematics in science depends critically on the researcher's ability to select those questions which both fill the scientific need and admit efficient mathematical treatment. To accomplish this realization, the instructor should frequently introduce a realistic question from which he must retreat to a related tractable problem whose interpretation is informative in the context of the original question.

a. Derivation of equations. (2 lectures) The derivation of mathematical models associated with many scientific problems. Review of heat conduction to a moving medium, the flow of a fluid in a porous medium, the diffusion of a solute in moving fluids, the dynamics of elastic structures, neutron diffusion, radiative transfer, surface waves in liquids.

b. Eigenfunction expansions. (5 lectures) Eigenfunction expansions in both finite and infinite domains (Titchmarsh).

c. Separation of variables. (7 lectures) The product series solutions of partial differential equation boundary value problems.

Integral transforms such as the Laplace, Fourier, Mellin, and Hankel transforms and their use. Copious illustration of these techniques, using elliptic, parabolic, and hyperbolic problems.

d. Types of partial differential equations. (5 lectures) The classification of partial differential equations, characteristics; appropriate boundary conditions. Domains of influence and dependence in hyperbolic and parabolic problems. The use of characteristics as "independent" coordinates.

e. Numerical methods. (8 lectures) Replacement of differential equations by difference equations; iterative methods; the method of characteristics. Convergence and error analysis.

f. Green's function and Riemann's function. (9 lectures) Their determination and use in solving boundary value problems. Their use in converting partial differential equation boundary value problems into integral equation problems.

g. Similarity solutions. (3 lectures)

h. Expansions in a parameter. (3 lectures) Perturbation methods in both linear and nonlinear problems.

9. Introduction to Functional Analysis. (3 semester hours)

The purpose of this course is to present some of the basic ideas of elementary functional analysis in a form which permits their use in other courses in mathematics and its applications. It should also enable a student to gain insight into the ways of thought of a practicing mathematician and it should open up much of the modern technical literature dealing with operator theory.

Prerequisite to this course is a good foundation in linear algebra and in the concepts and techniques of the calculus of several variables. The material of this course should be presented with a strong geometrical flavor; undue time should not be spent on the more remote and theoretical aspects of functional analysis. Topics should be developed and first employed in mathematical surroundings familiar to the student. It would be very much in keeping with the intention of the course to emphasize the relationship between functional analysis and approximation theory, discussing (for example) some aspects of best uniform or best L^2 approximation to functions, and some error estimates in integration or interpolation formulas.

While some knowledge of measure theory and Lebesgue integration is needed for an understanding of this material, it is not

intended that the treatment be as complete as that in a standard real analysis course. The intended level is that to be found in the treatment by Kolmogorov and Fomin (Kolmogorov, A. N. and Fomin, S. V. Elements of the Theory of Functions and Functional Analysis. Vol. 2: Measure, the Lebesgue Integral, Hilbert Space. Baltimore, Maryland, Graylock Press, 1961.) If there is additional time, students might be introduced to some of the elementary theory of integral equations, or to applications in probability theory, or to the study of a specific compact operator, or to distributions.

For an outline of such a course, see Mathematics Q (Functional Analysis) in A Beginning Graduate Program in Mathematics for Prospective Teachers of Undergraduates, page 125.

10. Elements of Real Variable Theory. (3 semester hours)

For an outline of this course, see Mathematics 11 (Introductory Real Variable Theory) in Commentary on A General Curriculum in Mathematics for Colleges, page 93.

11. Optimization. (3 semester hours)

Attempts to determine the "best" or "most desirable" solution to large-scale engineering problems inevitably lead to optimization studies. Generally, the appropriate methods are highly mathematical and include such relatively new techniques as mathematical programming, optimal control theory, and certain combinatorial methods, in addition to more classical techniques of the calculus of variations and standard maxima-minima considerations of the calculus.

The 3-semester-hour course outlined below is planned to provide a basic mathematical background for such optimization studies. Another outline for a course in optimization, utilizing methods of programming and game theory, can be found in the report Applied Mathematics in the Undergraduate Curriculum, page 722.

a. Simple, specific examples of typical optimization problems. (3 lectures) Minimization with side conditions (Lagrange multipliers, simple geometrical example). Linear program (diet problem). Nonlinear program (least squares under inequality constraints, delay line problem). Combinatorial problem (marriage or network). Variational problem (brachistochrone). Control problem (missile). Dynamic program (replacement schedule).

b. Convexity and n-space geometry. (6 lectures) Convex regions, functions, general definition (homework: use definition

to show convexity [or nonconvexity] in nonobvious cases, such as Chebychev error over simple family of functions). Local, global minima. Convex polyhedra (review matrix, scalar product geometry). Geometric picture of linear programming.

c. Lagrange multipliers and duality. (6 lectures) Classical problem with equality constraints. Kuhn-Tucker conditions for inequality constraints. Linear programs. Dual variables as Lagrange multipliers. Reciprocity, duality theorems.

d. Solution of linear programs--simplex method. (3 lectures)

e. Combinatorial problems. (6 lectures) Unimodular property. Assignment problem (Hall's theorem, unique representatives). Networks (min-cut max-flow).

f. Classical calculus of variations. (7 lectures) Stationarity. Euler's differential equation, gradient in function space. Examples, especially Fermat's principle and brachistochrones.

g. Control theory. (8 lectures) Formulation. Pontryagin's maximum principle (Lagrange multipliers again).

12. Algebraic Structures. (3 semester hours)

For an outline of this course, see Mathematics 6M (Introductory Modern Algebra) in Commentary on A General Curriculum in Mathematics for Colleges, page 68.

MATHEMATICAL ENGINEERING

A FIVE-YEAR PROGRAM

A Report of

The Panel on Mathematics for the Physical Sciences

and Engineering

October 1966

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INTRODUCTION

During a conversation at the mathematics meetings in January, 1964, the late S. S. Wilks of Princeton University pointed out the absence of any special large-scale efforts to provide technical personnel for the nation's space program. As he saw it, there was a continuing need for persons really well-trained in the mathematical sciences and able to apply their fields to the complicated engineering problems of the space effort. His aim was to provide technical manpower specifically prepared for a strong combination of mathematics and engineering--in addition to the more customary converts from a great variety of backgrounds. He felt that CUPM, with its tradition of pervasive concern for all aspects of collegiate mathematical education, might undertake a study of this problem. The present report is, in part, a response to his ideas.

A pair of meetings with representatives of NASA and the space industry confirmed the eminent need for the projected product. At the same time it became clear that many other industries, such as electronics and communication, would have an equal interest in a mathematical engineer with the same basic background but possibly somewhat different specialization. All these industries face innumerable engineering problems with a common need for extensive and sophisticated mathematical analysis. For the solution of such problems it is no longer true that the prime requirement is a good physical intuition; rather, one also needs a well-developed mathematical intuition. Thus the mathematical engineering program must involve a heavy concentration in mathematics, but with a choice of topics that will give a useful basis for applications as well as solid grounding in theory. As the Panel came to grips with this multiplicity of purposes, it developed the notion of a common core, for all mathematical engineers, of material in the basic physical sciences and, more extensively, in the mathematical sciences. The core, in turn, is complemented by a number of options, which are more specialized developments in depth and which assure that the student will be fairly well acquainted with at least one branch of engineering. Orbit mechanics, operations research, and control theory are three options which are developed in detail in the present report.

It turns out that a minimum of five years, rather than four, is needed to carry out the desired sequences in depth. It is not immediately obvious what the student's area of concentration should be called. The dual emphasis on mathematics and engineering makes either field conceivable; in fact, the program comes close to being recognizable as a master's program in applied mathematics. However, the heavy emphasis on the physical sciences, the concern in each option with the building of mathematical models, and the rather heavily prescriptive nature of the program make a realization within the engineering school more suitable. Wherever the program may appear within an institution's offerings, it should involve close cooperation between the mathematicians and the engineers.

A natural question also arises as to the possible fields of further graduate study which a student could enter on completion of the present program. It is our opinion that relatively little, if any, "remedial" work will be necessary to qualify the student for a doctoral program in applied mathematics or, depending on the particular option, in engineering science, or in industrial or electrical engineering. Whether or not a master's degree should be given for the completion of this program is a matter for the offering institution to decide. As remarked above, the content of the program is of the right order of magnitude for this degree. Other considerations (e.g., requirement of a thesis) may be deciding factors.

A number of additional remarks about the program are in order. A most important aspect is the flexibility which would automatically be built into the mathematical engineer. With such a background and with a considerable facility in making connections between the real world and mathematical models thereof, such a man could easily retrain himself, say, from space science to oceanography, if a sudden shift of present national interest should make this desirable. Secondly, the similarity in spirit of this program to the recently introduced engineering physics and engineering science programs is worth noting. The idea of these programs is to give the student a solid background of the kind of physics that would be useful in a wide variety of engineering applications, along with enough engineering subjects to impart some feeling for the kinds of problems he would encounter. There is now no question of the value of such training. In just the same way, mathematical engineering combines a solid foundation in major areas of applicable mathematics with real strength in some particular area of engineering, and experience in connecting the two. Incidentally, it should be remarked that mathematical engineering has existed for some years, much in the spirit of the present report, at several universities in the Netherlands. It seems to be a successful program from the point of view of both employment opportunity and preparation for further graduate work.

DESCRIPTION OF PROGRAM

As remarked above, the program is constructed around a core consisting of a heavy concentration of mathematics and the physical sciences. Attached to the core there may be many options, each providing motivation, application, and extension of the core material to some phase of engineering. The core is fairly well defined and will probably not vary greatly from one institution to another. The options, on the other hand, will necessarily have much local flavor both in their general subject matter and in the particular courses that compose them. The three options that we present here are thus to be regarded as samples of what can be done.

The Core

Modern engineering is built upon a three-part foundation consisting of mathematics, the physical sciences, and automatic computing. The last of these is a newcomer whose precise role and manner of development are still matters of speculation, but there is no question as to its basic importance. These three topics, then, compose the core.

The mathematical portion, which is, for this program, the most extensive, is based on Recommendations on the Undergraduate Mathematics Program for Engineers and Physicists, page 628. The courses recommended there as preparation for graduate work have been modified somewhat and about nine semester hours have been added, much of it as additional work in topics already begun. This gives us the following list (the initial number refers to the course outlines given in the Appendix [or elsewhere in this COMPENDIUM], and "hours" means "semester hours"):

	Calculus and Linear Algebra (12-15 hours)
1.	Functions of Several Variables (3 hours)
2.	Intermediate Ordinary Differential Equations (3 hours)
4.	Numerical Analysis (3 hours)
5.	Probability and Statistics (6 hours)
6.	Complex Variables (3 hours)
7.	Functional Analysis (3 hours)
10.	Partial Differential Equations (6 hours)
11.	Optimization (3 hours)

Discussion of the individual courses is deferred until the whole core has been described.

The resulting 42-45 hours of mathematics in the core is about the magnitude of a good undergraduate major in mathematics, but the emphasis is quite different. This program stresses analysis heavily. Indeed, the minimal treatment of algebra and geometry is perhaps the most vulnerable point of this curriculum. However, for the foreseeable future the topics included are certainly of first importance. Other mathematical topics needed in certain courses can be developed when needed to the extent required.

We recognize full well that the value of these courses depends on the spirit in which they are taught. One must keep in mind that their ultimate purpose is application, either directly or as preparation for more obviously applicable topics. Such courses as Partial Differential Equations and Optimization should lean heavily on applied problems. Computational methods should be stressed throughout. Further, as much interconnection as possible should be built into the whole program. It is planned that both the mathematics and the engineering courses should reinforce one another to an unprecedented degree.

A corollary of this requirement is that the courses should be taught by whoever is capable of doing the best job, regardless of the department he happens to be in. For the more standard and the more theoretical courses the mathematics department would be the natural place to look for teachers, but where applications are heavily stressed the best teacher may well be found in some other field.

Beyond a fairly standard 15- to 18-hour introduction to physics and chemistry, the program calls for 12 more hours in the basic sciences. Six of these are accounted for by a mechanics course, intended to be a coordinated combination of physics, mathematics, and computing. Considerable time is spent on variational methods and continuum mechanics, as well as on the standard mechanics of particles and rigid bodies.

The remaining six hours is divided between electromagnetics and thermodynamics (including statistical mechanics). Of the many possible continuations of the basic material it was felt that these two, because of their fundamental nature, their wide applicability, and their susceptibility to interesting mathematical analysis, are particularly appropriate to this program.

Modern computing facilities and the techniques for using them are still developing with bewildering rapidity, and no program fixed now will give adequate coverage for very long. We are painfully aware of these rapid developments and claim no special powers of prophecy. The proposal delineated here provides two realistic approaches to current problems of computing. A direct approach is the inclusion of two courses devoted to computation, the Numerical Analysis mentioned above and an Introduction to Computer Science [such as the course C1 in Recommendations for an Undergraduate Program in Computational Mathematics]. These give the principles of modern computation, including the use of a programming language, and their basic applications to mathematical problems.

Less direct, but perhaps of more ultimate importance, is the inclusion of computational methods in connection with each appropriate topic in other courses. Such topics occur in almost all the core courses, but particularly in Differential Equations (both Ordinary and Partial), Mechanics, and Optimization. It is expected that significant problems to program and run on a computer will be part of the work in these courses. Only in this way can a real understanding of the power and (especially) of the limitations of modern computing techniques be communicated.

The overall structure of the core can be seen in the table. Here each entry represents a 3-hour course.

Year	Core Courses		Electives*
I	Calculus	Physics Chemistry	6
	Calculus	Physics Chemistry	6
II	Linear Algebra	Physics Computer Science	6
	Calculus	Physics	9
III	1. Functions of Several Variables	3. Mechanics 4. Numerical Analysis 5. Probability	3
	2. Ordinary Differential Equations	3. Mechanics 5. Statistics	6
IV	6. Complex Variables	8. Electromagnetics	9
	7. Functional Analysis	9. Thermodynamics	9
V	10. Partial Differential Equations	11. Optimization	9
	10. Partial Differential Equations		12

* semester hours

Of an assumed total of 150 semester hours for the five years, 45 hours are devoted to mathematics and computing and 30 to basic sciences. This leaves the remaining 75 hours for humanities electives and for additional courses in engineering, mathematics, and science. With a roughly even split this should satisfy normal requirements. The most demanding of our sample options, Control Theory, specifies only 24 hours, leaving, say, 15 hours for basic engineering and technical electives and 36 hours for the humanities.

The first two years of the program are fairly standard. For the mathematics portion we recommend the sequence of courses described in Commentary on A General Curriculum in Mathematics for Colleges, page 33. In addition to linear algebra and the usual elementary calculus of one or more variables this includes an introduction to differential equations.

Introductory physics and chemistry courses are at present under intensive review by professional groups; some radically new physics courses have recently appeared and other experimental programs are underway. We therefore refrain from specifying these courses in any detail but urge the reader to consult the publications of the Commission on College Physics and the Advisory Council on College Chemistry. Widespread adoption of new elementary courses could require great changes in the content, or even in the selection, of the later science courses in the core.

The introductory computer science course should include a discussion of the nature of an automatic computer and the manner in which it solves problems, an introduction to a specific computer language and its role in this process, and some practice in the actual solution of various types of problems. Either by means of this course or from supplementary instruction, students should be able to program and run simple problems early in their sophomore year.

The third year has the heaviest concentration of core courses, foundation for the more advanced material in the core and for the technical applications in the options. These courses are of fairly standard type except for Mechanics, which has been described, and Intermediate Ordinary Differential Equations. The occurrence of considerable material on differential equations in the calculus course justifies the initial adjective in the latter title and permits the course to concentrate on linear equations with variable coefficients, boundary value problems, and special functions. There is also a brief introduction to nonlinear equations.

The report Commentary on A General Curriculum in Mathematics for Colleges outlines three courses in Functions of Several Variables (Mathematics 5. Multivariable Calculus II.). The first of these has a classical vector analysis approach, while the second uses differential forms; the third is particularly suited for students in statistics. We recommend the second course, outlined on page 77, partly because the vector technique is covered in the physics courses but also because the more general approach is a valuable background for fourth-year Functional Analysis.

With the exception of the Probability and Statistics all the third-year work is closely interconnected, and considerable thought should be given to the sequence of topics so as to get the most coordination. In particular, linear algebra, numerical techniques, and the use of a computer to solve problems are ever-recurring themes in the year's work.

The fourth and fifth years of the core are fairly light, since here will come most of the work in the options. In the mathematics courses, in addition to the obvious requirements of Complex Variables and Partial Differential Equations--six hours of the latter is necessary for any sort of coverage, we have included courses in Functional Analysis and Optimization. The first of these provides an introduction to some hitherto abstract topics that are proving useful in a

variety of applications, such as numerical analysis, communication theory, quantum physics, and many branches of mathematics. General metric and linear spaces, operators and functionals, with an introduction to measure theory, are the central topics.

Optimization is another introductory course, but one tied very closely to applications. Based on the notions of compactness, convexity, and Lagrange multipliers, it treats briefly the various types of mathematical programming, some combinatorial problems, and the calculus of variations.

The two science courses in the fourth year, Electromagnetics and Thermodynamics, could vary considerably in content. In any case, however, they should take advantage of the students' exceptional background in analysis, probability, mechanics, and computing to give a considerably more sophisticated treatment than could commonly be contemplated.

The selection and arrangement of the courses comprising the core represent the Panel's best judgment of the curriculum currently needed to develop the kind of highly trained but still flexible engineer described in the Introduction. Local conditions and opinions will undoubtedly suggest some changes, and future developments--for example, an upsurge of biological engineering--may call for a reappraisal of the whole program. But the basic framework of mathematics, science, and computing should still be appropriate for many years to come.

The Options

The role of the option is to provide the student with a solid acquaintance with some branch of engineering, at the same time giving background and applications of many of the subjects treated in the core. In general, serious work in the option will begin in the fourth year, following the heavy load of third-year core courses and some appropriate technical introduction. With this background the work in the option can begin, and proceed, at a higher level of sophistication than is usually possible.

As samples of what might be done we present three options, labeled, for want of better names, Operations Research in Systems Engineering, Orbit Mechanics, and Control Theory. There is nothing special about these; they simply happened to be topics of interest to some of the Panel members and consultants. Other topics of equal suitability might be, for example, Fluid Mechanics, Solid Mechanics, Electronics and Microwaves, Wave Propagation and Plasma Physics, Materials Engineering, and Nuclear Engineering.

For each of the sample options we give here a brief description of the program and an outline of its structure. Detailed syllabi of the courses are given in the Appendix.

OPERATIONS RESEARCH IN SYSTEMS ENGINEERING

Operations Research has been described as the application of mathematical methods to the solution of practical optimization problems in engineering, in business, and in government. The Operations Research Option builds onto the core those features of operations research that pertain especially to the design, development, and production of large-scale engineering systems. These require analysis of the complicated interrelationships among component and system performances, development and production costs, scheduling priorities, available manpower and facilities, and a host of other factors. Such considerations have made necessary the use of various optimization techniques, the application of probabilistic and statistical methods, the development of a highly mathematical reliability theory, Monte Carlo simulation methods, optimal control theory, linear, nonlinear and dynamic programming methods, and queueing theory. These mathematical topics provide the typical tools for operations research studies which find wide applicability in the evaluation (and comparison) of performance, programs, and policies in certain types of engineering and industrial situations.

The Operations Research Option, which has been fleshed out in some detail, represents an attempt to provide suitable training for engineers who have to cope with such problems. Building upon the 6-hour course in Probability and Statistics of the core, it provides a 6-hour course in mathematical methods of reliability engineering. The course includes both probabilistic models of reliability problems and statistical techniques of reliability estimation. The introductory optimization course of the core is supplemented by a further 6-hour course in linear programming techniques, dynamic programming, inventory and scheduling problems, queueing theory, and related topics.

The third course recommended in this option is a 3-hour course in System Simulation, which exploits the use of a computer in carrying out the analysis of such operations research activities.

Additional courses in economics, such as Economic Decision Theory, or in management science would constitute appropriate electives for certain students.

Note that the core course in Optimization has been moved into the fourth year to provide the necessary background for the fifth-year course in Operations Research.

Year	Core Courses	Option Courses	Electives
IV	6. Complex Variables	OR1. Reliability	9
	7. Functional Analysis	OR1. Reliability	
	8. Electromagnetics		
	9. Thermodynamics and Statistical Mechanics		
	11. Optimization		
V.	10. Partial Differential Equations	OR2. Operations Research	15
	10. Partial Differential Equations	OR2. Operations Research	
		OR3. Systems Simulation	

ORBIT MECHANICS

As with each of the options, the aim is to build upon the foundation supplied by the core to provide greater specialization in an aspect of mathematics of central importance in modern engineering, here space science.

The design of space vehicles; prediction, correction, and control of their space flight; transmission and evaluation of information collected in space--all such tasks place unusual new requirements on engineering skills and training. Additional problems arise from the necessity for real-time computations and corrections during space flight. Underlying all these difficulties is the problem of developing a correct physical intuition for the nature of space travel, vehicle control, and environmental conditions in space.

The Orbit Mechanics Option supplements the core courses in mechanics with substantial one-semester courses in celestial mechanics and in orbit theory. The addition of an advanced programming course and an introduction to control theory provides solid grounding for many problems of space vehicle engineering. The course in data smoothing and prediction provides training essential to the successful collection, retrieval, and interpretation of telemetered information.

Additional courses in astronomy or in space physics constitute natural electives for students in such a program.

Since some of Partial Differential Equations is needed for Advanced Numerical Analysis and Celestial Mechanics, the core course 10 must be put in the fourth year. This gives a rather heavy concentration of mathematics in the fourth year, but this could be relieved, if desired, by a further shifting of some of the other courses.

Year	Core Courses	Option Courses	Electives
IV	6. Complex Variables	OM1. Advanced Numerical Analysis	9
	7. Functional Analysis		
	8. Electromagnetics		
	9. Thermodynamics and Statistical Mechanics		
	10. Partial Differential Equations		
	10. Partial Differential Equations		
V	11. Optimization	OM2. Advanced Programming	12
		OM3. Celestial Mechanics	
		OM4. Orbit Theory	
		CT2. Control	
		CT5. Data Smoothing and Prediction	

CONTROL THEORY

The advances in computers and in instrumentation have brought an enormous increase in the sophistication of control systems. The instruments allow us to measure rapidly and precisely many variables which were previously hard to measure, and the computer allows us to

make use of all the data while it is still current. The space program has given a great impetus to control theory by bringing up a number of new problems with very strict requirements. Another aspect of many control problems is that they involve control loops which extend over great distances, thereby creating an interface problem between the control and the communications specialist.

The Control Theory Option starts in the third year with a one-semester course in circuit theory which exposes the student to the modeling problem, to some specific physical devices which he will encounter later, and to basic system concepts in simple physical situations. In the fourth year the control course will furnish the student the basic facts about control systems and the linear systems course will provide the common base for further courses in control, communications, and circuits. The fifth year includes a course on the techniques of optimization, one on advanced control, one on advanced communications, and one on information theory.

Year	Core Courses	Option Courses	Electives
IV	6. Complex Variables	CT2. Control	6
	7. Functional Analysis	CT3. Laboratory	
	8. Electromagnetics	CT4. Linear Systems	
	9. Thermodynamics and Statistical Mechanics	CT5. Data Smoothing and Prediction	
V	10. Partial Differential Equations	CT6. Advanced Control	12
	10. Partial Differential Equations	CT7. Information Theory	
	11. Optimization	CT8. Advanced Communications	

APPENDIX

Sample Outlines of the Courses

The course outlines given in this Appendix or elsewhere in the COMPENDIUM are intended in part as extended expositions of the ideas

that we have in mind, in part as feasibility studies, and in part as proposals for the design of courses and textbooks. They have a wide variety of origins. Some are standard courses now given in universities and some are experiments that have never yet been tried. Most of them, however, are modifications or combinations, more or less radical, of familiar material. They have been prepared by many different persons, with a broad spectrum of interests in mathematics and related fields, and representing industrial as well as academic interests. However, all outlines were carefully scrutinized by the whole Panel and were not accepted until their value to the whole program was clear. For better or worse, this is a committee product.

It will be observed that there is considerable overlapping in some of the content of the courses, for example in Intermediate Ordinary Differential Equations and in Numerical Analysis. This is inevitable in the courses in any modern university, where most courses are taken by a variety of students in different programs and with different backgrounds. If a neater dovetailing of these courses is possible in particular cases, the contents should of course be modified accordingly.

The Core

I, II		Physics	(12)	Calculus and Linear Algebra (12-15)
		Chemistry	(6)	Computer Science (3)
III	1.	Functions of Several Variables		(3)
	2.	Intermediate Ordinary Differential Equations		(3)
	3.	Mechanics		(6)
	4.	Numerical Analysis		(3)
	5.	Probability and Statistics		(6)
IV	6.	Complex Variables		(3)
	7.	Functional Analysis		(3)
	8.	Electromagnetics		(3)
	9.	Thermodynamics and Statistical Mechanics		(3)
V	10.	Partial Differential Equations		(6)
	11.	Optimization		(3)

1. Functions of Several Variables. (3 semester hours)

For an outline of this course, see Mathematics 5 (Multivariable Calculus II--alternate version) in Commentary on A General Curriculum in Mathematics for Colleges, page 77.

2. Intermediate Ordinary Differential Equations. (3 semester hours)

For an outline of this course, see Recommendations on the Undergraduate Mathematics Program for Engineers and Physicists, page 643.

3. Mechanics. (6 semester hours)

The course outlined below differs from that given in certain textbooks in that the discussion of mechanics is interruped at various stages in order to deal with topics in numerical analysis; e.g., after the equations of motion are formulated, various methods for numerically integrating initial value problems are discussed and analyzed. It is assumed that the student has had the linear algebra course as well as the computer science course. Homework assignments that involve use of a computer should be made.

a. Kinematics. (8 lessons) Cartesian coordinates in Euclidean 3-space, cartesian tensors, the numerical tensors δ_{ij}, ϵ_{ijk}. Parametric equations of curves. Velocity and acceleration in cartesian coordinates, in general coordinates. Moving general coordinates and the velocity and acceleration in such coordinates. Equations of straight lines in moving general coordinates. Characterization of inertial coordinate frames.

b. Particle mechanics. (10 lessons) Equations of motion. Initial value problems for a system of ordinary differential equations, existence, uniqueness, continuous dependence on parameters and initial values. Numerical methods for integrating initial value problems, their stability.

c. Perturbation theory. (8 lessons) Physical stability. Numerical stability. Linearization of nonlinear problems.

d. Central forces. (10 lessons) Planetary orbits. Energy integrals, angular momentum integrals. Constants of motion and symmetry properties.

e. Variational principles and rigid body motion. (13 lessons) Hamilton's principle, generalized coordinates of Lagrange, canonical equations, contact transformations, partial differential equations of Hamilton and Jacobi. Rigid body motion.

f. Multidimensional variational principles. (8 lessons) Variation of multiple integrals and applications to problems in statics and dynamics of deformable bodies. Vibrating strings and membranes. Rayleigh-Ritz method. Use of polynomials to derive difference equation approximation to the boundary value differential equations that are the Euler equations of a variational principle. Numerical integration of boundary value problems on the line and in the plane.

g. Continuum mechanics. (21 lessons) Stress and strain tensors. Conservation of mass, momentum and energy. Partial differential equations describing the motion of a perfect fluid. One-dimensional isentropic motions (simple and compound waves). Numerical integration of 1-dimensional motions. Existence of shocks. Numerical integration in the presence of shocks.

4. Numerical Analysis. (3 semester hours)

For an outline of this course, see Mathematics 8 (Introduction to Numerical Analysis) in Commentary on A General Curriculum in Mathematics for Colleges, page 83.

5. Probability and Statistics. (6 semester hours)

For an outline of this course, see Recommendations on the Undergraduate Mathematics Program for Engineers and Physicists, page 642.

6. Functions of a Complex Variable. (3 semester hours)

For an outline of this course, see Mathematics 13 (Complex Analysis) in Commentary on A General Curriculum in Mathematics for Colleges, page 97.

7. Introduction to Functional Analysis. (3 semester hours)

For an outline of this course, see Mathematics Q (Functional Analysis) in A Beginning Graduate Program in Mathematics for Prospective Teachers of Undergraduates, page 125.

8. Electromagnetics. (3 semester hours)

This course combines the essentials of classical electromagnetic theory with the foundations of applications to plasma media. It should take advantage of the preparation in mechanics, especially continuum mechanics, as well as of the fundamentals of electricity and magnetism in the introductory physics course.

a. Electrostatics. (4 lessons) Vacuum field and potential theorems. Dielectrics. Boundary conditions. Energy relations and forces.

b. Magnetic fields. (5 lessons) Current and moving charges. The vector potential. Magnetic media. Energy relations and forces.

c. Maxwell's equations. (6 lessons) The law of induction. Maxwell's equations, extended to moving media. Energy, force, and momentum relations in the electromagnetic field.

d. Wave propagation. (8 lessons) The scalar wave equation. Plane, cylindrical, spherical waves. Homogeneous isotropic media. Dispersion. Nonhomogeneous isotropic media. Rays; the geometrical optics approximation. Wave packets, including nonhomogeneous media and absorption.

e. Electromagnetic waves. (7 lessons) Free space and homogeneous isotropic media. Homogeneous plasmas. Inhomogeneous media. Anisotropic media, including plasma with magnetic field.

f. Radiation. (9 lessons) Simple radiating systems. Radiation by moving charges. Radiation in ionized gases. Synchrotron radiation.

9. Thermodynamics and Statistical Mechanics. (3 semester hours)

There is a current trend to combine the macroscopic and the microscopic aspects of thermal physics from the beginning, instead of giving a careful treatment of classical thermodynamics, with applications, as in Thermal Physics by Philip M. Morse (second

edition; Menlo Park, California, W. A. Benjamin, Inc., 1969). The suggested outline follows the latter plan, as probably more appropriate as a background for varied applications.

Thermodynamics.

a. State variables and equations of state. (3 lessons) Temperature, pressure, heat, and energy. Extensive and intensive variables. Pairs of mechanical variables. The perfect gas and other equations of state.

b. The first law of thermodynamics. (4 lessons) Work, internal energy, heat. Heat capacities. Isothermal and adiabatic processes.

c. The second law of thermodynamics. (6 lessons) Heat cycles. Reversible and irreversible processes. Entropy. Applications to simple thermodynamic systems.

d. The thermodynamic potentials. (3 lessons) Internal energy, enthalpy, Gibbs and Helmholtz potentials. Examples and procedures for calculation.

e. Phrase equilibria. (3 lessons) Melting, evaporation, triple point, and critical point.

f. Chemical applications. (2 lessons) Reaction heats, electrochemical processes.

Statistical Mechanics (Equilibrium).

a. Statistical methods. (3 lessons) Random walk; probability distributions; mean values; binomial, Poisson, and Gaussian distributions.

b. Statistical description of systems of particles. (3 lessons) Ensembles, ergodic hypothesis, postulates, limiting behavior for large N, fluctuations.

c. Quantum statistics. (5 lessons) Maxwell-Boltzmann, Bose-Einstein, and Fermi-Dirac distributions with applications (solids, gases, electron gas, blackbody radiation, etc.).

Microscopic Description of Nonequilibrium.

a. Elementary kinetic theory. (2 lessons)

b. Transport theory. (2 lessons) Based on Boltzmann's equation, in simplified form.

c. Brownian motion. (3 lessons) Possibly including the Fokker-Planck equation. More on random variables. Markov processes, fluctuations, irreversible processes.

10. Partial Differential Equations. (6 semester hours)

This course ordinarily occurs in the fifth year of the sequence, although with certain options (e.g., Orbit Mechanics) it should be taken in the fourth year. The material should strike a reasonable balance between the classical analytical theory of partial differential equations and modern computational aspects of the subject. For that reason, existence theorems and the like should be of the constructive type whenever possible. Further, application to problems in classical and modern physics should constantly be borne in mind. Physical models should be used both to predict results concerning the behavior of solutions to partial differential equations and to interpret phenomena revealed analytically or computationally.

a. Introduction. (6 lessons) Derivation of some equations; discussion of mathematical models, continuous dependence theorems, and relation to physics.

b. Classification and characteristics. (9 lessons) Cauchy problem for first-order equations, formulation and statement of Cauchy-Kowalewski theorem.

c. Hyperbolic equations. (12 lessons) Existence and continuous dependence for second-order equations. Riemann method. Three-dimensional wave equation. Retarded potentials. Numerical methods--finite difference schemes and stability.

d. Elliptic equations. (21 lessons) Potential theory in three dimensions with smooth boundaries. Eigenvalue problems--estimates. Numerical methods.

e. Parabolic equations. (12 lessons) Thermal potential theory. Convergence to steady state and relation to potential problems. Numerical methods and connection, in steady state, to numerical methods for elliptic problems.

f. Integral representation of solutions. (12 lessons) Green's functions. Integral equations.

g. Equations of hydrodynamics. (6 lessons) Shock phenomena, weak solutions. Numerical methods.

11. Optimization. (3 semester hours)

For an outline of this course, see Recommendations on the Undergraduate Mathematics Program for Engineers and Physicists, page 647. A natural successor to this course is the 6-semester-hour course Operations Research in the Operations Research Option, where linear programming techniques are developed in depth and additional topics in dynamic programming, inventory and scheduling problems, Monte Carlo simulation techniques, and queueing theory are introduced.

OPERATIONS RESEARCH IN SYSTEMS ENGINEERING OPTION

Year	Core Courses			Option Courses		
III	1.	Functions of Several Variables	(3)			
	2.	Intermediate Ordinary Differential Equations	(3)			
	3.	Mechanics	(6)			
	4.	Numerical Analysis	(3)			
	5.	Probability and Statistics	(6)			
IV	6.	Complex Variables	(3)	OR1.	Reliability	(6)
	7.	Functional Analysis	(3)			
	8.	Electromagnetics	(3)			
	9.	Thermodynamics and Statistical Mechanics	(3)			
	11.	Optimization	(3)			
V.	10.	Partial Differential Equations	(6)	OR2.	Operations Research	(6)
				OR3.	Systems Simulation	(3)

OR1. Quantitative Methods in Reliability Engineering. (6 semester hours)

The growing complexity of systems over the last two decades has inspired the development of a body of quantitative methods for developing, improving, and measuring system reliability. Since reliability is such a critical factor in the successful completion of every highly technical program, it is of value to the engineer to learn quantitative reliability theory, as presented in the following course. The mathematical and statistical methods of the course are not simply routine applications of well-known theory, but in many cases represent new developments motivated by reliability problems.

The 6-semester-hour course in Probability and Statistics of the core is an essential prerequisite for this course.

First Semester: Probabilistic Models in Reliability.

a. Failure distributions in reliability theory. (8 lessons) Typical failure laws. The exponential as the failure law of complex equipment. Monotone failure rates. Bounds for distributions with monotone failure rate. General failure rates.

b. Prediction of system reliability from a knowledge of component reliabilities. (3 lessons) Analytical methods for computing reliability exactly. Bounds on system reliability based on paths or cuts. Monte Carlo methods. Qualitative relationships for multi-component structures.

c. Redundancy optimization. (5 lessons) Optimal allocation of redundancy subject to constraints. Optimal redundancy assuming two kinds of failure.

d. Operating characteristics of maintenance policies. (6 lessons) Renewal theory. Replacement based on age. Comparison of age and block replacement policies. Random replacement. Repair of a single unit.

e. Optimum maintenance policies. (4 lessons) Replacement policies. Inspection policies.

f. Stochastic models for complex systems. (8 lessons) Semi-Markov processes. Repairman problems. Marginal checking. Optimal maintenance policies under Markovian deterioration.

Second Semester: Statistical Reliability Theory.

a. Estimating reliability parameters assuming form of distribution known. (8 lessons) Maximum likelihood estimation in the

case of normal, exponential, gamma, Weibull, and binomial distributions. Confidence and tolerance limits in these cases. Minimum variance unbiased estimation in these cases.

b. Estimating reliability parameters under physically plausible assumptions. (9 lessons) Errors resulting from incorrect assumption as to form of failure distribution. Maximum likelihood estimation assuming a monotone failure rate. Maximum likelihood estimation assuming a decreasing and then increasing failure rate. Conservative confidence and tolerance limits.

c. Estimating reliability growth. (7 lessons) Form of growth assumed known. Only monotonicity of reliability assumed. Conservative confidence limits.

d. Confidence limits on system reliability using observations on individual components. (6 lessons) Success or failure observations. Life length observations. Asymptotic methods.

e. Hypothesis testing. (9 lessons) Acceptance sampling, fixed sample size, truncated and censored sampling, sequential sampling. Accelerated life testing. Testing for monotone failure rate.

OR2. Operations Research. (6 semester hours)

The accent is on the mathematical aspects of the subject, rather than the management or industrial engineering aspects. It is assumed that time and facilities are available for a computation laboratory in connection with both semester courses. A prerequisite of an introductory computer programming course is desirable.

The first semester develops linear programming in depth, building on the preparation given in a previous one-semester course in Optimization.

First Semester: Advanced Linear Programming.

a. Review of the simplex algorithm. (4 lessons) Variations of the simplex algorithm. Degeneracy, perturbation. Revised simplex method.

b. Games and linear programs. (4 lessons) Matrix games. Equivalence of matrix games and linear programs.

c. The transportation problems. (4 lessons) Elementary

transportation theory. The transshipment problem.

d. Networks and the transshipment problem. (4 lessons) Graphs and trees. Interpreting the simplex method on the network. The shortest route problem.

e. Variables with upper bounds. (3 lessons) The general case. The rounded variable transportation problem.

f. Programs with variable coefficients. (3 lessons) Wolfe's generalized program. Special cases.

g. Decomposition principle for linear programs. (9 lessons) The general principle. Decomposing multistage programs.

h. Convex programming. (4 lessons) General theory. Separable convex objectives. Quadratic programming.

i. Discrete variable problems. (4 lessons) Survey of methods. Gomory's method of integer forms.

Second Semester: Dynamic Programming and Stochastic Models.

a. Dynamic programming. (10 lessons) Principle of optimality. Multistage allocation problems. Arrow-Harris-Marschak inventory model.

b. Dynamic programming and Markov processes. (5 lessons) Discrete dynamic programming. Optimal policies with discounted returns.

c. Monte Carlo techniques. (10 lessons) Production of random variables by computer. Simulating stochastic systems on the computer.

d. Mathematical theory of queues. (14 lessons) Single server; Poisson input; exponential service. Many servers; Poisson input; exponential service. The busy period. Stochastic inventory models.

OR3. Systems Simulation. (3 semester hours)

This course examines those symbol manipulation applications of the computer that involve the numerical and logical representation of some existing or proposed system, for the purpose of experimenting with the model and of comparing methods of operating the system. The primary purpose of the computer is thus not a calculating adjunct to experimentation but is the experimental medium itself. A course in probability and statistics is a prerequisite.

a. Programming languages. (11 lessons) Special languages designed for use in simulation, such as SIMSCRIPT and GPSS. Additional study of the languages will arise in their use throughout the rest of the course.

b. Technical problems of simulation. (14 lessons) Synchronization of events, file maintenance, random number generation, random deviate sampling.

c. Statistical problems peculiar to simulation. (7 lessons) Sample size estimation, variance reducing techniques, problems of drawing inference from a continuous stochastic process.

d. Applications. (7 lessons) Queueing models; storage, traffic, and feedback systems; design of facilities and operating disciplines.

ORBIT MECHANICS OPTION

Year	Core Courses			Option Courses		
III	1.	Functions of Several Variables	(3)			
	2.	Intermediate Ordinary Differential Equations	(3)			
	3.	Mechanics	(6)			
	4.	Numerical Analysis	(3)			
	5.	Probability	(3)			
IV	6.	Complex Variables	(3)	OM1.	Advanced Numerical Analysis	(3)
	7.	Functional Analysis	(3)			
	8.	Electromagnetics	(3)			
	9.	Thermodynamics and Statistical Mechanics	(3)			
	10.	Partial Differential Equations	(6)			
V	11.	Optimization	(3)	OM2.	Advanced Programming	(3)
				OM3.	Celestial Mechanics	(3)
				OM4.	Orbit Theory	(3)
				CT2.	Control	(3)
				CT5.	Data Smoothing and Prediction	(3)

OM1. Advanced Numerical Analysis. (3 semester hours)

This course, with its emphasis on topics in partial differential equations and elementary functional analysis, demands a reasonable amount of mathematical maturity. It should be taken after the first semester of Partial Differential Equations.

a. Matrix inversion and matrix eigenvalues. (10 lessons) Review and extension of iterative methods. Jacobi, Householder, and other methods of finding eigenvalues. Ill-conditioning and error analysis.

b. Ordinary differential equations, boundary value problems, eigenvalue problems. (11 lessons) Finite difference methods, extremal principles.

c. Partial differential equations of second order. (18 lessons) Topics selected from the following: Classification, analytical solutions of well-posed problems for single equations; maximum principles for elliptic and parabolic equations, L_2--or energy--estimates as well as pointwise estimates of solutions; hyperbolic equations, domain of dependence; Fourier analysis and stability for constant coefficient equations, eigenvalues for elliptic equations, iterative methods for difference equations arising from partial differential equations.

OM2. Advanced Programming. (3 semester hours)

This course deals with various types of computer programming and serves to introduce students to the concepts involved in current work in this area. An introductory course in computer science is a prerequisite, and it is assumed that the students have considerable facility in programming with FORTRAN or ALGOL.

a. Survey. (9 lessons) Assembly systems, methods of storage allocation when using these, pseudo-orders, macros, modify and load techniques, monitor and executive systems.

b. Structure of languages. (15 lessons) Study of a particular language such as ALGOL, its ambiguities, its method of dealing with recursions and procedures. List-processing languages, compiler-writing languages.

c. Theory of compilers. (15 lessons) Nature of syntax-

directed compilers, compilers for dealing with problem-oriented languages, compilers for dealing with compiler syntax languages. Discussion of the evolution of a translator from a simple language whose translator is given in machine language.

OM3. Celestial Mechanics. (3 semester hours)

This course in celestial mechanics concerns itself with the mathematical structures underlying the physical theory, and with deriving from them methods which are commonly used to attack the fundamental problems of interest to space research and technology.

a. Introduction. (6 lessons) Description of dynamical systems by means of Lagrangian functions. Lagrangian equations of motion. Ignorable coordinates, energy equation, and other elementary instances of first integrals. Liouville systems in general. Classical examples: harmonic oscillator, simple pendulum, spherical pendulum, central forces, a charged particle in an electromagnetic field, solid body. The principle of dynamical analogy.

b. Phase space. (6 lessons) Legendre duality (with a suggestion about how it is applied to derive state functions in classical thermodynamics). Transition from Lagrangian functions and Lagrangian equations to Hamiltonian functions and canonical equations. Canonical mapping: its definition, its multiplier, and its residual functions. Completely canonical mappings; canonical extensions of coordinate transformations. Generation of canonical mappings by numerical functions. Invariance with respect to the group of canonical mappings: canonical equations, Poisson brackets, Lagrange parentheses.

c. Canonical constants of a dynamical system. (6 lessons) Definition of a set of canonical constants of integration. Variation of canonical constants. The Hamilton-Jacobi equation as an algorithm for constructing sets of canonical constants. The action and angle variables. Separable Hamiltonians; Staeckel systems and Liouville systems. Applications: the problem of two bodies, the problem of two fixed centers. Normal modes of vibrations and vibrations of molecules.

d. Integrals of a dynamical system. (9 lessons) Poisson's theorem about the bracket of two integrals and its dual application to Lagrange parentheses. Integrals in involution; Liouville's theorem. Jacobi-last multiplier. Application to the motion of a solid body. Whittaker's adelphic integral. Application to the investigation of a dynamical system around the equilibrium. Isoenergetic reduction. Application to the regularization and the binary collisions in the problem of two bodies and in the restricted problem of three bodies.

e. Perturbation theory. (12 lessons) Poincaré's method of the small parameter. Birkhoff's method of iterative canonical mappings. Application to the motion of a satellite of an oblate planet.

OM4. Orbit Theory. (3 semester hours)

The purpose of this course is to offer illustrations of mathematical principles and to compel the student to master them securely by careful numerical examples. The material should be arranged to provide a nearly continuous flow of computational work for the laboratory sessions no matter at what level the course is set.

The instructor and his students should be directed to develop the topics all the way down to an efficient and reliable program in the FORTRAN or ALGOL language on an electronic computer. As most of the textbooks on the subject matter cater to computers who use logarithms or hand-operated desk-model calculating machines, special care should be taken to rearrange classical algorithms and formulas for use on an electronic computer.

Topical problems should be selected from at least these four main research areas:

a. Orbit determination. (10 lessons) The obvious reference here is P. Herget's The Computation of Orbits, published privately by the author at Cincinnati Observatory, 1948. This booklet has been updated by the author in his lectures on "Practical Astronomy" and on "Orbit Determination" given at the summer course in Space Mathematics, Cornell University, 1963.

b. Orbit analysis. (12 lessons) The question here is to gain physical information from the comparison between the orbit as

it has been observed and the orbit as it has been computed from a particular mathematical model.

c. Orbit design. (7 lessons) How to produce orbits that satisfy *a priori* conditions (e.g., given initial conditions, mission requirements, optimum characteristics, etc.).

The course should limit itself to well-tried problems and should aim at producing examples where good-quality results can be reached without too much effort. This can be achieved in the restricted problem of three bodies.

After an introduction to that problem, the instructor should review two or three methods for integrating numerically the equations of motion, either in cartesian coordinates or in regularized coordinates. Then should come a development on variational equations. Thereafter the theory of characteristic exponents should be applied to the analysis of a family of periodic orbits.

d. Analytical theories. (10 lessons) How to expand on literal theory by enabling an electronic computer to handle symbol manipulations in a given algebra.

This new field promises to provide mathematicians with powerful tools to develop literal theories in an extensive set of physical problems.

CONTROL THEORY OPTION

Year	Core Courses			Option Courses		
III	1.	Functions of Several Variables	(3)	CT1.	Electric Circuits	(3)
	2.	Intermediate Ordinary Differential Equations	(3)			
	3.	Mechanics	(6)			
	4.	Numerical Analysis	(3)			
	5.	Probability and Statistics	(6)			
IV	6.	Complex Variables	(3)	CT2.	Control	(3)
	7.	Functional Analysis	(3)	CT3.	Laboratory	(3)
	8.	Electromagnetics	(3)	CT4.	Linear Systems	(3)
	9.	Thermodynamics and Statistical Mechanics	(3)	CT5.	Data Smoothing and Prediction	(3)
V	10.	Partial Differential Equations	(6)	CT6.	Advanced Control	(3)
	11.	Optimization	(3)	CT7.	Information Theory	(3)
				CT8.	Advanced Communications	(3)

CT1. Electric Circuits. (3 semester hours)

This is a basic course in circuit theory. The purpose is to teach in a precise language the fundamental facts of circuit theory while developing skills in writing and solving the circuit equations and keeping close contact with physical circuits (filters, amplifiers, digital circuits).

a. Lumped circuits. (3 lessons) Lumped circuit approximation. Kirchhoff laws, relation to Maxwell's equations. Circuit elements; including nonlinear and time-varying elements.

b. Simple circuits. (13 lessons) First- and second-order circuits: zero-input response and zero-state response; response to step, impulse, and sinusoid. Linearity and time-invariance: convolution. Impedance, phasors, frequency response, and resonance.

c. Coupling elements. (2 lessons) Coupled inductors, transformers, and dependent sources. Dependent sources as parts of models for electronic devices.

d. Power and energy. (2 lessons) Energy stored and power dissipated in elements; relation with real and imaginary part of impedance.

e. General methods of analysis. (6 lessons) Graph theory: trees, links, cut-sets, loops. Loop and cut-set analysis, mixed method. Duality. Computer programs for analysis of circuits.

f. Linear time-invariant circuits. (6 lessons) Reduction of systems of equations. Network functions: poles, zeros, gain and phase.

g. Network theorems. (3 lessons) Superposition, Thévenin, Norton, reciprocity. Careful discussion of range of applicability: comments on nonlinear and time-varying circuits.

h. Two-port description. (4 lessons) Two-port description of electronic devices: relation to their graphical characteristics. Linear time-invariant networks as two-ports. Interconnection of two-ports.

CT2. Control. (3 semester hours)

In this course the student learns some basic facts about control systems, their analytical description, and techniques of design. This course is mostly concerned with single-variable control.

a. Description of feedback systems and components. (5 lessons) Advantages and disadvantages of feedback, importance of measuring device, noise problems. Basic components, electrical, hydraulic, pneumatic. Requirements and specifications of control systems. Examples.

b. Linear time-invariant control systems. (20 lessons) Analysis, illustrated by several extensive examples, based on differential equations and integral equations (convolution). Stability, root locus, Nyquist criterion. Design of compensating networks to obtain stability and meet the specifications.

c. Sampled systems. (6 lessons) Examples of systems where the feedback data are naturally sampled periodically. Analysis of sampled systems. Stability, root locus, Jury's criterion. Design of compensating networks.

d. Nonlinear systems. (8 lessons) Local stability near equilibrium. Example of limit cycles. Stability: approximate methods describing functions. Lyapunov's second method. Application to design.

CT3. Laboratory. (3 semester hours)

The purpose of the laboratory is to insure that students connect the classroom concepts and results to physical reality and appreciate the power and limitations of experimental work. Typically, one part of the laboratory could be devoted to circuits work: behavior of linear circuits (including resonance), effects of nonlinear elements on waveform and power spectrum, some pulse circuits. The second part of the laboratory would cover control: study of a typical control system, experiments with various compensations; stability; experiments with and simulation of a nonlinear control system.

CT4. Linear Systems. (3 semester hours)

Purpose: to provide a solid foundation of concepts, facts, and techniques to be used in later courses in control, communication, and circuits.

a. Systems. (6 lessons) State as a parametrization of input-output pairs and as part of the system description. Operator point of view. State equivalence. Linear systems. Linear systems obtained by linearization of ordinary differential equation about a nominal trajectory. Examples throughout.

b. Linear systems of the form $\begin{aligned} \dot{x} &= Ax + By \\ \dot{y} &= Cx + Dy \end{aligned}$ and their discrete time analogs. (6 lessons) Time-invariant case: explicit solution by function of a matrix and Laplace and z-transforms. For simple linear operators, diagonalization, mode interpretation (including numerical techniques). Jordan form, analog computer interpretation. Time-varying case: properties of the state transition matrix. Periodic systems: Floquet theory, kinematic equivalence.

c. Impulse response and transfer functions. (12 lessons) Free use of Fourier and Laplace transforms. Superposition integral. Asymptotics of impulse response and transfer function. Minimum phase. Uncertainty principle. Group delay. Signal flow graphs.

d. Stability. (9 lessons) Characterization of stability for linear time-invariant, periodic and time-varying systems (zero-input stability: Liénard Chipart, Nyquist; bounded input implies bounded output; implications of an impulse response which is in L^1), Lyapunov method.

e. Input-output description and state equations. (6 lessons) Controllability, observability, and normality. Characterization in time-invariant and time-varying cases. Output controllability. Controllability and observability of an interconnection of systems.

CT5. Data Smoothing and Prediction. (3 semester hours)

a. Representation of functions by Fourier series and integrals. The Fourier transform in L^1 and L^2. (8 lessons)

b. Random processes: definition, examples, representations; autocorrelation, power spectrum; estimation of spectral densities. (14 lessons)

c. Linear mean-square estimation, filtering and prediction. The Wiener-Hopf equation; solution by the Wiener filter and Kalman-Bucy filter. (11 lessons)

d. Detection and parameter estimation. Application to digital communications system and radar. (6 lessons)

CT6. Advanced Control. (3 semester hours)

This course treats advanced topics in control so that students can readily read the current literature. Multiple-input multiple-output systems are included.

a. Nonlinear control. (10 lessons) Describing function. Subharmonics. Stability: Sandberg circle criterion, Popov-type criteria. Lyapunov method used as a design tool. Lyapunov method for systems with inputs. Bounds on output.

b. Adaptive control. (8 lessons) Examples of adaptive control: identification techniques and parameter adjustment. Stochastic approximation.

c. Optimum control. (21 lessons) Formulation of the problem. Examples: systems described by ordinary differential equations and difference equations. Maximum principle for differential systems. Numerical methods. Relation of maximum principle with steepest descent.

CT7. Information Theory. (3 semester hours)

a. The concept of the source and of an information measure. Desirable properties of information measure, examples of simple sources. (3 lessons)

b. Codes, their efficiency and redundancy. The efficient

encoding of discrete independent sources. (4 lessons)

c. General discrete sources, Shannon's encoding theorem, the nature of written and spoken English. (4 lessons)

d. The concept of a channel, channel capacity, symmetry of a channel. (3 lessons)

e. The fundamental theorem of information theory, error-detecting and error-correcting codes, the geometric interpretation of coding problems. (17 lessons)

f. Generalization to continuous channels, channel capacity of continuous channels. (8 lessons)

CT8. Advanced Communications. (3 semester hours)

a. Review. (4 lessons) Signal and noise representations, the purpose of modulation.

b. Amplitude modulation. (7 lessons) The generation and detection of AM waves, power spectrum, single side-band and vestigial side-band transmission, effects of distortion and noise.

c. Frequency and angle modulation. (12 lessons) Generation, detection, power spectrum, effects of distortion and noise.

d. Pulse modulation. (10 lessons) Pulse amplitude modulation, pulse position modulation, pulse duration modulation. A brief introduction to pulse code modulation.

e. Design. (6 lessons) The design of optimum receivers in the presence of additive noise and fading.

RECOMMENDATIONS FOR THE UNDERGRADUATE MATHEMATICS PROGRAM FOR STUDENTS IN THE LIFE SCIENCES

An Interim Report of

The Panel on Mathematics for the Life Sciences

September 1970

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1. Introduction

The Panel on Mathematics for the Biological, Management, and Social Sciences was primarily concerned with the mathematics curriculum for prospective graduate students in those fields. Their recommendations, which were published in the 1964 CUPM report Tentative Recommendations for the Undergraduate Mathematics Program of Students in the Biological, Management, and Social Sciences (BMSS), were meant to serve as a basis for discussion and experimentation. From these discussions it became apparent that (1) some of the recommendations would be very difficult for mathematics departments to implement, (2) a different program was needed for the terminal bachelor's degree, (3) the single program presented would not seem to be ideally suited to the diverse fields included in the BMSS disciplines, especially at the advanced level. In response to these findings, CUPM decided to concentrate on individual disciplines and, as a first step, appointed the present Panel on Mathematics for the Life Sciences, charged with making recommendations for the mathematical training of all undergraduate life science students, not only pregraduate students. (Here, life sciences are taken to mean agriculture and renewable resources, all branches of biology, and medicine.)

The Panel on Mathematics for the Life Sciences has undertaken, through conferences and extensive consultation with leaders in the biological field, to learn from them what mathematics they consider to be necessary for their students. In particular, the Panel held several meetings with representatives of the Commission on Undergraduate Education in the Biological Sciences (CUEBS). After the biologists had specified the mathematics needed by students of biology, the Panel proceeded to describe mathematics courses that contain this mathematics. This report is the outcome of these consultations and studies. Finally, the Panel held a special conference in which a preliminary draft of this report was submitted to a group of biologists for comment and criticism. The discussions and conferences with the biologists have emphasized a serious fourth problem: (4) heavy requirements in chemistry, physics, and biology make it difficult for a major in the life sciences to add mathematics courses to his program.

In preparing this report, the Panel considered all four of the problems mentioned above, and it presents herewith its recommendations for a basic mathematics core for life science undergraduate majors (see Part I) and for certain more specialized studies (see Part II).

Part I describes a basic core of mathematics for all undergraduate majors in the life sciences. In Section 2 we describe the level of mathematical preparation on which this core is based. In Section 3 the recommendations for the mathematical core are stated and justified, and in Section 4 we treat some of the principles and details of implementation of this core.

The Panel feels strongly that every life science major should gain substantial experience with computers (digital, analogue, and hybrid). We feel that the time is ripe now for a detailed treatment of the role of the computer in the undergraduate program, especially as it relates to the life science student. This accounts for the detail found in Section 5.

Part II outlines certain more specialized studies which this Panel believes will be important for some students in mathematics as well as for students in the life sciences. Section 6 describes a program for undergraduate preparation for the study of biomathematics at the graduate level. A description of an upper-division course focusing on the building of mathematical models in the life sciences and some suggestions for its implementation appear in Section 7.

Certain of the courses in A General Curriculum in Mathematics for Colleges are cited frequently; their descriptions appear elsewhere in this COMPENDIUM.

I. MATHEMATICS FOR UNDERGRADUATE BIOLOGY MAJORS

2. Background of the Students

In recent years much effort has been expended to improve mathematics education in the elementary and secondary schools. Several programs of improvement in secondary schools have already had considerable effect and we hope that they will have a great deal more. In particular, we hope that mathematics courses in the secondary school will contain a judicious mixture of motivation, theory, and applications. For the purposes of our discussion it is assumed that the student is acquainted with both the algebraic and geometric aspects of elementary functions (see the description of Mathematics 0 in Commentary on A General Curriculum in Mathematics for Colleges, page 75); moreover, we assume that the student has been exposed to the idea of a set, mathematical induction, binomial coefficients, and the summation notation. Thus, our discussion applies to students in the life sciences who are prepared to begin their collegiate mathematics with a calculus course, although departments of mathematics may have to offer precalculus courses in order to prepare some students adequately for this program.

Historically, mathematics has been closely allied to the physical sciences, especially to physics. In secondary schools and in elementary undergraduate courses, applications of mathematics have traditionally been limited to the physical sciences. Therefore, it is not uncommon for students whose interests lie in other fields to enroll in a bare minimum of mathematics courses. If students are to possess the prerequisites stated above, proper counseling both in

high school and in college is imperative. Students must be made aware of the doors that are closed to them in fields of the life sciences, as well as in the physical sciences and engineering, when they terminate their study of mathematics prematurely. We hope that this message will be transmitted to guidance and counseling personnel, and we urge all concerned to give attention to ways by which counseling of potential life science students can be improved in their locality.

3. The Basic Core: Recommendation and Justification

The Panel on Mathematics for the Life Sciences has considered the problem of recommending a basic core of mathematics courses for students in the life sciences. The prospective life science major, whatever his specialty or career goal, now needs more mathematics than was recognized to be the case a few years ago. As a result of its study, the Panel concludes that the mathematical core for the undergraduate life science major should include one year of calculus, some linear algebra, and some probability and statistics.

More specifically, the Panel believes that this core can be provided by the following courses: Mathematics 1 (Calculus I), Mathematics 2 (Calculus II), Mathematics 3 (Elementary Linear Algebra), and Mathematics 2P (Probability). Outlines for all of these courses can be found in Commentary on A General Curriculum in Mathematics for Colleges. In addition, we recommend that each student gain some experience in the use of an automatic computer in the first two years of study. This might come in the form of a sequence of laboratory exercises (see Section 5) in which algebraic language problems are developed and run. Institutions which do not have computation centers may be able to provide service via remote terminals or through the courtesy of nearby organizations. This permits the use of computing algorithms, lecture demonstrations, or problem assignments in biology and mathematics courses at appropriate times.

The recommendations are consistent with the findings of the two life science commissions sponsored by the National Science Foundation. In Publication No. 18 of the Commission on Undergraduate Education in the Biological Sciences, "Content of Core Curricula in Biology" (June, 1967), pp. 30-31, we find: "Fourth, we recommend that careful attention be given to relating biology courses to the background of the student in mathematics, physics, and chemistry...in mathematics, at least through the level now generally taught as calculus, ...some background in physical and organic chemistry." This recommendation clearly indicates that a full-year sequence of calculus (including multivariable calculus) should be taken by a biology major. In this same publication the curricula for biology majors at Purdue University, Stanford University, North Carolina State University, and Dartmouth College are presented. At three of these institutions, one year of calculus is required in addition to some probability and linear algebra. In the remaining institution,

additional calculus is required instead of "finite mathematics" (here taken to mean basic linear algebra with applications--such as Markov chains--and combinatorial probability).

In 1967 the Commission on Education in Agriculture and Natural Resources (CEANAR) charged a committee "to recommend mathematics requirements to be met ten to fifteen years hence in undergraduate curricula for Agriculture and Natural Resources." In its report* this committee chose to state its recommended requirements almost entirely in terms of courses described in the CUPM report A General Curriculum in Mathematics for Colleges. Mathematics 1, 2P, and some computer instruction are recommended for majors in all areas covered by CEANAR. Moreover, for students majoring in technology programs, Mathematics 2 and Mathematics 7 (Probability and Statistics) are recommended; to this students majoring in science programs should add Mathematics 3 and Mathematics 4 (a third course in calculus).

There are other good reasons for recommending this core of four mathematics courses: 1, 2, 3, and 2P. First of all, these are standard mathematics courses whose broad availability should facilitate implementation. Secondly, this curriculum is flexible enough to accommodate a student who may decide to change his major. For example, if in the first year or two he enters a discipline that involves more mathematics, he will not have lost any time. Thirdly, compressing this material into a shorter three-course sequence is unwise from a pedagogical point of view. Very few students are capable of gaining even a minimal mastery of calculus in a one-semester course, and at least one semester is needed to cover a significant amount of linear algebra or of probability. Moreover, if a student eventually decides to take more advanced mathematics and still continue in the branch of life sciences he originally chose, he will have the appropriate prerequisites. In this connection we discuss in Section 7 the role of a course in the applications of mathematics to the life sciences for the research-bound student. Since such a course involves relatively advanced mathematics, it will carry certain further mathematical prerequisites beyond the core itself.

Preparation for research in certain areas of biology will demand competence in mathematics equivalent to the Master's degree level. Some biologists have even asserted that a student who has a Bachelor's degree with a major in mathematics and appropriate courses in chemistry and physics would be welcomed as a graduate student in biology, even though he had had no courses in biology. Further development of this line of thought is found below in Section 6 on biomathematics.

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* See "Undergraduate Education in the Physical Sciences and Mathematics for Students in Agriculture and Natural Sciences," pp. 32-35. This report is available from the Division of Biology and Agriculture, the National Research Council, 2101 Constitution Avenue, Washington, D. C. 20418.

Although the Panel feels that an offering to life science students of fewer than four semesters of mathematics course work will not meet the objectives laid out by the life scientists whom we have consulted, we must recognize that this amount of mathematics is more than will be accepted by some of them as a requirement for all undergraduate life science majors. We have been urged to consider what can be done with three courses. Any three-course program will lose some of the desirable features described in the last paragraphs. We present several options of three courses and point out some of the advantages and shortcomings of each:

(1) Mathematics 1, 2, 3;
(2) Mathematics 1, 3, and 2P;
(3) Mathematics 1, some appropriate interweaving of 2 and 3, and 2P;
(4) Mathematics 1, 2, and one semester of finite mathematics;
(5) An integrated three-semester sequence, specifically designed for life science students and built around finite mathematics, calculus through multivariable calculus, probability, and statistical inference.

It may be well to observe that each of the above options could lead to completion of the core in graduate school, if this were desired. This could be done with standard courses in the case of options 1 or 2 or with one or more special courses in the case of other options.

Some features of the options are highlighted in the following

	Number of special mathematics courses required	Number of core subjects omitted entirely	Full year of calculus included
Core	0	0	Yes
Option 1	0	1	Yes
Option 2	0	1	No
Option 3	1	0	No
Option 4	1	0	Yes
Option 5	3	0	No

The first column measures to some extent the extra load placed on the mathematics department by each option. The offering of each special course involves planning and coordination activities and in a small institution may have a high cost per student because of limited enrollment.

Any option involving special courses inevitably raises the possibility of requiring additional course work (or equivalent thereof) to provide the proper prerequisites for further study in mathematics. The severity of this effect can be assessed only in the context of a

given institution, a given spectrum of courses, and some designated group of students at a given skill level.

Column 3 relates to the remark that offering less than a full year of experience in calculus seems to be insufficient.

The Panel feels that option 1 is the least undesirable, since 1) probability can be added by many students as an elective, 2) this sequence is easier for most mathematics departments to staff than one involving probability, and 3) many schools now offer linear algebra as an integral part of the calculus sequence. Option 2 is considerably less desirable than option 1, since it omits multivariable calculus, a topic that the Panel feels is vital to modern biology (see the previous reference to CUEBS Publication No. 18).

Options 3, 4, and 5 share the disadvantage of having no efficient continuing mathematics course which can be used to complete the core material.

Options 3 and 4 may be desirable for larger institutions in which the mathematics and biology departments can work out arrangements for an additional elective special course which would complete the core. If many departments were to adopt one of these options, graduate departments might choose to <u>require</u> that the core be completed in graduate school. Care must be exercised in implementing option 3 to include topics in calculus that are needed in physics and chemistry prerequisites to biology courses.

The essential feature of option 5 is to construct a three-semester sequence, illustrated by life science examples and containing essential material from the calculus, while interweaving some probability, statistics, and linear algebra in an integrated format. However, in situations where there are substantial numbers of students who have not had appropriate mathematics courses by the time they begin graduate work in biology, the departments of biology and mathematics may wish to collaborate in designing special programs. It is more important for a graduate biology student to understand the basic concepts of the mathematics he uses than to develop the computational skills needed by a physical scientist or engineer. By taking advantage of the maturity, strong motivation, and established field of interest of these students, satisfactory programs (such as option 5 above) stressing the understanding of these mathematical concepts could be designed in such a way as to require less time than the standard undergraduate courses.

Such cooperation involving another department with the mathematics department has proved successful in the past. This was particularly true in some areas of the social sciences, where the demand for such programs eventually diminished when the great majority of entering graduate students came with a sufficient mathematical background. The research needs of some biology departments have motivated them to hire biomathematicians, who could participate in teaching the graduate programs envisioned here.

4. The Basic Core: Implementation

The Panel recommends that all life science majors be required to complete two semesters of calculus and one semester each of linear algebra and probability (including some statistics). These courses, Mathematics 1 and 2, Mathematics 3, and Mathematics 2P, are discussed in detail in Commentary on A General Curriculum in Mathematics for Colleges, page 33. In many undergraduate curricula these courses must serve many needs: prospective biology majors find themselves in the same classes with students from a wide variety of disciplines (such as engineering, economics, business administration, one of the physical sciences, or even mathematics). When this is the case, it is unlikely that special emphasis on biological applications will be featured in any part of this four-course program. A few institutions, however, can afford to present all, or some part, of this core program exclusively for students whose main interests lie in the life sciences. We do not address ourselves to the task of making detailed recommendations to this group since we feel that an institution offering a special mathematics core for life scientists will wish to take advantage of local features and design a hand-tailored program. Between these extremes, we find institutions able to give varying amounts of special attention to life science orientation in the mathematics core. Our suggestions below are directed primarily to this group. We expect that there will be considerable latitude in the extent and manner that these recommendations are utilized.

The life science major should be given more consideration than has been the custom in the past, even by the first group of institutions that cannot afford to provide special courses or sections of courses. Traditionally, the applications given in calculus, for example, are almost exclusively chosen from the physical sciences. With the rapid growth of the life sciences, it is only reasonable that increased emphasis be given to illustrative examples from this field, even in a calculus course in which the interests of most of the students lie elsewhere.

We now proceed to our comments on the modifications necessary to make the core courses more suitable to the needs of the life science students.

[Editor's note: In the case of Mathematics 1 and 2, the Panel's suggestions relative to the original course outlines in the 1965 General Curriculum in Mathematics for Colleges have been incorporated into the revised outlines in Commentary on A General Curriculum in Mathematics for Colleges (1972). Thus, we refer the reader to the new outlines.]

Mathematics 1. Calculus I.

See Commentary on A General Curriculum in Mathematics for Colleges, page 44.

Mathematics 2. Calculus II.

See Commentary on A General Curriculum in Mathematics for Colleges, page 51.

Mathematics 2P. Probability.

Relative to this course, an outline of which is given in Commentary on A General Curriculum in Mathematics for Colleges, page 76 , we make the following comments:

a. With respect to (1) of the GCMC description, the introduction of the probability axioms should be properly motivated by the frequency interpretation (see Hodges, J. L. and Lehmann, E. L. Basic Concepts of Probability and Statistics. San Francisco, California, Holden-Day, Inc., 1964) in order to connect these concepts with the empirical traditions of the life sciences.

b. With respect to sections (2) and (3), some time could be saved by merging the sections so that the Poisson and normal distributions would be introduced as limits of the binomial distribution. For the normal approximation, we feel that the most efficient presentation--in consideration of both time spent and student understanding--would be to discover numerically that a sequence of binomial cumulative distribution functions, after the usual normalization, tends to the normal distribution (see Mosteller, F. R., et al. Probability with Statistical Applications, 2nd ed. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1970).

c. We feel that two-state Markov chains, as a generalization of sequences of Bernoulli trials, should be included in the course. The presentation of these could begin in the discussion of conditional probability in section (1). A solution for the limiting distribution of the process and a numerical demonstration of convergence to this limit should then follow the other limit theorem discussions, taking up one or two lessons.

d. If Mathematics 3 is included as a prerequisite for Mathematics 2P, topics such as the Markov process in (c) can be presented more efficiently and in greater depth in matrix form.

A modified course description of 2P appropriate for students in the life sciences can then be given as follows:

1. Probability as a mathematical system. (11 lessons) Variability of experimental results, sample spaces, events as subsets, probability axioms and immediate consequences, finite sample spaces and equiprobable measure as a special case, random variables (discrete and continuous), conditional probability and stochastic independence, Bayes' formula. Sequences of independent Bernoulli trials, two-state Markov chains.

2. Probability distributions. (15 lessons) Characterization of probability distributions by density and distribution functions, illustrated by the binomial and uniform distributions. Expected values, mean and variance. Chebychev inequality, Poisson distribution introduced as approximation to the binomial, normal approximation to the binomial, Central Limit Theorem, stationary distribution of a simple Markov chain, Law of Large Numbers, discussion of special distributions motivated by relevant problems in the life sciences.

3. Statistical inference. (13 lessons) Concept of random sample, point and interval estimates, hypothesis-testing, power of a test, regression, examples of nonparametric methods, illustrations of correct and of incorrect statistical inference.

Mathematics 3. Elementary Linear Algebra.

[Editor's note: Here the Panel on Mathematics for the Life Sciences referred the reader to Mathematics 3 (Linear Algebra) in the 1965 General Curriculum in Mathematics for Colleges and made several suggestions for modifying this course in order that it be more appropriate for students in the life sciences. Some of the Panel's suggestions were incorporated into the revised version of Mathematics 3 (Elementary Linear Algebra) which appears in Commentary on A General Curriculum in Mathematics for Colleges, page 55. Another appropriate course, featuring many of the Panel's suggested modifications, is Mathematics L (Linear Algebra) of A Transfer Curriculum in Mathematics for Two-Year Colleges, page 231.

5. Recommendations for Computing

Automatic Computing

We recommend that every undergraduate in the life sciences have some contact with an automatic digital computer, and that this contact begin as early as possible in his program of study. Among the many bases for this recommendation are: that many mathematical models in the life sciences, as witnessed by the current technical literature, are procedural in nature and are best studied with the computer; that many analytic techniques of experimental biology are of practical value only when applied with an automatic computer; and that the automatic computer could play an important role in undergraduate biology lectures and laboratory if the students were prepared to make use of it.

This recommendation is stated separately from our recommendation of a CORE mathematics program for the life sciences student because we feel that experience in automatic computing will become part of a general liberal arts requirement rather than part of a major in either biology or mathematics. In many colleges a first course in computing is not the responsibility of the mathematics

department but of a computer science department or a department in which computer applications are already numerous. As applications in subject-area courses increase, the need for an introduction to computing separate from the courses in biology, chemistry, mathematics, and physics may disappear.

CUPM has established a panel to consider instruction in computer science and the use of computers for instruction in mathematics. Our recommendations may be used to select options from their recommendations when they become available and to set an amount of experience appropriate for the biology major. [See Recommendations for an Undergraduate Program in Computational Mathematics (1971) and Recommendations on Undergraduate Mathematics Courses Involving Computing (1972).]

It should be noted that our basic recommendation in automatic computing is minimal. For example, the recommended experience does not include the introduction to analog or to hybrid analog-digital computing, and it includes only the briefest view of the complex problems of numerical analysis. We hope that some analog experience could be gained in advanced biology laboratory work. We also urge that the student be cautioned against the misuse of computing techniques, to avoid any tendency toward confusing the mastery of a programming language with an adequate knowledge of mathematics.

We suggest two alternatives for a one-semester course by which this computing experience can be gained. These are described in detail in Section 9. [See also the course C1 in Recommendations for an Undergraduate Program in Computational Mathematics, page 563.] The first alternative is an informal program of weekly lectures and discussions of one hour extending through the freshman year, supplemented by a large number of assigned programming exercises to be developed and run at the student's convenience. It would amount to about one half a semester course for which credit might or might not be given. The second is a formal 3-semester-hour course with five or six assigned programming exercises, to be taken normally in the sophomore year. These suggestions will be stated more completely, but first it seems proper to point out the advantages and problems of the two approaches.

Basic computer programming skills have commonly been acquired through programs of self-instruction. Many computer scientists feel that it is best to provide the student with a computing facility, some reference manuals as to its use, an introductory lesson or two, and then to stay out of his way as he practices by developing programs which are of particular interest to him or are relevant to his other studies. They would not give formal academic credit for this work. We would temper this plan by continuing the lessons or discussion periods beyond the most basic introduction and assigning some specific programs to insure that the student is exposed to various classical and valuable computing techniques. Even with this modification, the plan has the obvious advantage of not making significant demands on either faculty or student time, an important factor

considering the already heavy required curriculum in the life sciences. It reasonably could, in fact, be carried out without academic credit. Unfortunately, the student freedom which permits this implies a lack of control over facility usage, so this alternative can prove expensive in machine charges.

The second alternative, a formal semester course introducing the student to computing, has the disadvantage of adding to an already exacting schedule. A most troublesome additional problem is that we do not feel that the introductory course in many computer science programs is appropriate to students in the life sciences. In that they are planned to set the foundation for further work in mathematics or computer science, they often do not cover computing applications adequately. The course we propose may, therefore, add an additional load on the mathematics faculty, particularly distasteful because of its partial redundancy. A formal course has two powerful advantages, however. The students will be brought to a higher level of competency and machine charges per student can be kept relatively low. This applied computing course could be relevant to fields beyond the life sciences, of course, and could be planned to serve all undergraduates not intending to specialize in computer science.

Continuing the Computing Experience

Given the contact with computers which we recommend, the student will be able to use the computer to extend his studies both in mathematics and in the life sciences. Experience has shown that he will, in fact, do so. It is important, therefore, that facilities be available to support this use. While accurate estimates of potential use are impossible, many students will continue computing at about the rate begun in the introductory course if given a chance.

Of importance in taking proper advantage of the student's computing experience is the use of computing exercises and demonstrations whenever relevant in the regular biology curriculum. We point out that the relevance is striking in many areas. For example, a course in population genetics could use a computing facility as a regular laboratory instrument, and some topics such as genetic drift are difficult to present without the computer. Too often, the student will recognize this value before the instructor. It is essential that every effort be made to introduce the potential of applied computing, as well as all other mathematical techniques, to the life science faculty. Among the possible means for this faculty education are: the involvement of the life science faculty in the program of computer instruction, the preparation and distribution of materials and understandable manuals on local computing facilities, the preparation and distribution of computer demonstration and laboratory materials for specific courses, and, most important, a demonstration of interest by the mathematics or computing faculty in biological research along with patient collaborative effort with life scientists.

6. Undergraduate Preparation for Biomathematics

The present state of biomathematics is such that one cannot expect to study this subject as an undergraduate. The best that can be expected of an undergraduate curriculum is to provide the student with a strong background in mathematics, physics, chemistry, and biology as preparation for graduate study. Indeed, because of its dependence upon the other sciences, biology may be emphasized the least in the undergraduate program and then, presumably, the most in the graduate program. In most colleges the undergraduate who is enrolled in such a program will be regarded as a major in mathematics.

Before going into details concerning the mathematics component, we consider some general principles on which a biomathematics program should be based.

(1) About one third of the student's undergraduate curriculum will be devoted to the mathematical sciences, including statistics and computing. A second third will be devoted to physics, chemistry, and biology, and the remainder to the humanities and social sciences to fulfill degree requirements. Since the student will normally be a major in mathematics, it is important that departments of mathematics allow their majors to choose electives freely in the biological sciences.

(2) Many institutions give several different versions of basic courses in the sciences. The crucial difference is usually the extent to which mathematics is used. It is vital, therefore, that the student plan his program so as to take the most sophisticated version of each course that is available. This injunction applies especially to courses in physics and in physical chemistry.

(3) Very few universities have a department of biomathematics. Most graduate students who study this subject will be enrolled in some life science department. It is essential, therefore, that the undergraduate program of such a student include enough courses in biology for him to gain admission to a graduate program in a life science area. This need not imply that the undergraduate program must contain very many courses in biology. Many leading life science departments will admit a person with as strong a background in mathematics and chemistry as is contemplated here if he has had as few as four semester courses of undergraduate biology and some may even require no undergraduate biology.

(4) It is neither practical nor desirable for a student to make an irrevocable commitment to a particular specialty early in his college career. As was stated in the preceding paragraph, a student who elects the program that is being presented here should be qualified for admission to a graduate life science program. He may then choose to specialize in some area of biology other than

biomathematics. On the other hand, he may decide to do graduate work in mathematics only. By adding a substantial course in abstract algebra to the program described below, he should become eligible for admission to most graduate departments of mathematics.

(5) Of the natural sciences, chemistry will receive the greatest emphasis. Courses in organic chemistry and the strongest possible course in physical chemistry will certainly be included in the program; biochemistry may also be included, although some schools prefer to introduce this topic at the graduate level.

With these considerations in mind, we now turn to the mathematics in this program. The computer experience and the core of four mathematics courses discussed in Sections 4 and 5, as well as in the GCMC report, form the foundation of this preparation. To this we add semester courses in Calculus, Advanced Multivariable Calculus, Statistics and Probability, and a two-semester sequence of Real Variable Theory, as described in Mathematics 4, 5, 7, 11, and 12 in Commentary on A General Curriculum in Mathematics for Colleges, page 33 The GCMC Mathematics 10, preferably in the version described in Section 7 below, and a Numerical Analysis course (see Mathematics 8, page 83) should be included.

A biomathematician will need to know more mathematics than is presented in this program. For example, he will have only a touch of differential equations in Mathematics 2 and 4, and will ordinarily need considerably more probability and statistics than is covered in Mathematics 2P and 7. Thus, his graduate program will include additional work in mathematics, although it will consist predominantly of biology. A biomathematics student (as well as other biology graduate students) may wish to follow a plan that is currently used in many other graduate fields: electing one mathematics course each term until the Master's degree requirements in mathematics are met. A few biomathematicians may wish to include course preparation for the Ph.D. degree both in mathematics and in the life sciences.

7. A Course in Applications of Mathematics in the Life Sciences

A course in applied mathematics (Mathematics 10) is briefly described in Commentary on a General Curriculum in Mathematics for Colleges. The essential feature of this course is "model building and analysis" coupled with appropriate interpretation and theoretical prediction. The philosophy of this approach to applied mathematics is well stated on page 92, and we recommend the reader's careful attention to that material in order to establish the necessary point of view for consideration of a course entitled Introduction to Applied Mathematics: Life Sciences Option. [See also the 1972 report Applied Mathematics in the Undergraduate Curriculum, page 705.

Three versions of a model-building course are developed in detail in Applied Mathematics in the Undergraduate Curriculum, page 705. We consider here another version designed for students with a particular interest in the life sciences.

Applications of mathematics in the life sciences may be classified basically into two broad categories, deterministic and stochastic. Moreover, a third category should also be added, that of mixed models, wherein the particular phenomenon under consideration may be modeled in either deterministic or stochastic fashion.

Specific prerequisites for a life sciences version of Mathematics 10 will vary according to which topics are studied, but in any case they include the basic core, supplemented suitably--usually with additional work in calculus and differential equations (Mathematics 4 and 5) and perhaps with additional work in probability and statistics (Mathematics 7).

One feature of life science models is that the mathematics used tends to be either almost trivial or relatively advanced; good "junior-level" models seem hard to find. Thus, for the present, successful offering of a life science version of Mathematics 10 would seem to call for an instructor who is well qualified both in mathematics and in the life sciences. Moreover, this instructor should be broadly interested and knowledgeable in applied mathematics and, in particular, in model building. Earlier CUPM reports have recommended that, in the absence of such a member of the mathematics faculty, Mathematics 10 should not be offered. It has been found in a number of institutions, however, that a viable alternative may be obtained through a joint effort of an interested member of the mathematics faculty and specialists in various other disciplines. Both mathematics and biology can thus be adequately represented and the essential feature of strong motivation is present. An undergraduate seminar led jointly by such a faculty team, with models being proposed by the members of the class, has been found to work well in practice. A format suitable for this purpose has been described by S. A. Altman ("A Graduate Seminar on Mathematics in Biology." CUEBS News, Vol. V, No. 1, October, 1968, pp. 9-10).

An institution with a strong, modern biological sciences department should be able to offer a course such as that suggested above. This is especially true if the members of the life sciences faculty are interested in bringing in mathematical ideas and there is also present at the institution a mathematics cadre interested in the applied mathematical sciences. Several members of the Panel have had some experience in offering courses based on model building in both physical and life sciences. Such an approach revolves around an artful use of the case study method, with the class thereafter pursuing the mathematical structure, detective story fashion, wherever it may lead. Usually the mathematical structure itself is developed en route only to the extent that is demanded by the model, although appropriate avenues are of course indicated to the students for following up any particular portions of the mathematics that may especially interest them.

We conclude this section with a few general comments concerning mathematical models in the life sciences.

Model construction consists, for the mathematician at least, of laying down an appropriate axiom system, either as a formal set of axioms or by means of a system of defining equations. Equations of motion in physiology and biophysics, linear algebra formulations of protein sequences or of population state vectors, dynamical systems describing population interactions, combinatorial models of genetic phenomena or of macromolecule configurations are all instances of such axiom systems. Once an appropriate mathematical structure (i.e., a set or sets with operations) has been specified, the further analysis proceeds within the mathematical structure, emerging at certain strategic times with interpretations or theoretical predictions drawn from the mathematical deductions themselves. To the extent that these conclusions are in accord with those aspects of the actual phenomenon that are regarded as significant in that context, so, too, may the original mathematical model be regarded as a good one. One of the virtues of such a procedure, as noted in the GCMC report, is that "the attempt to build a satisfactory mathematical model (often) forces the right question about the original situation to come to the surface." It is clear, therefore, that the modeling process is often one of successive approximations, hopefully convergent to a sound theory at some stage. An essential part of the instructor's responsibility would seem to be conveying to the life sciences student the realization that once an appropriate mathematical structure has been determined via axiomatization, he can work strictly within this mathematical structure, to come back in the end with certain interpretations and theoretical predictions relevant to the particular life science phenomenon under consideration. All too often there seems to be what amounts to almost a mental block in many biologists' thinking that precludes their leaving the realm of empirical laws and statistical description (mathematics as curve fitting) to work within the mathematical structure itself. The great significance of this latter mode of procedure for the astonishing growth of the physical sciences during the past half century has been well described by Mostow, Sampson, and Meyer (Fundamental Structures of Algebra. New York, McGraw-Hill Book Company, 1963. Preface) in the following terms:

> "The great evolution of the physical, engineering and social sciences during the past half century has cast mathematics in a role quite different from its familiar one of a powerful but essentially passive instrument for computing answers. In fact that view of mathematics was never a correct one... . Its inadequacy is becoming increasingly apparent with the growing recognition that mathematics is at the very heart of many modern scientific theories--not merely as a calculating device, but much more fundamentally as the sole language in which the theories can be expressed. Thus mathematics plays an organic and creative part in science, as a limitless source of concepts which provide fruitful new ways of representing natural phenomena."

The objective of the proposed course in applications of mathematics in the life sciences is to develop in students the capability to utilize these powerful mathematical methods in the fashion indicated above.

III. APPENDICES

8. Course Outlines for Mathematics 0, 1, 2, 3, 2P

See Commentary on A General Curriculum in Mathematics for Colleges, page 33.

9. Course Outlines for an Introduction to Computing

Following are outlines of programs for the two alternatives suggested in Section 5. [See also the course C1 in Recommendations for an Undergraduate Program in Computational Mathematics, page 563.]

Introduction to Computing: Alternative 1

The course is comprised of weekly or biweekly one-hour lecture and discussion meetings and ten or more student programming exercises. It is set primarily for freshmen, although, if done with informality, it could involve the entire life science community, including the faculty. The prime goal is the development of basic applied programming skills in an algebraic language. There is little concern for the logical organization of the machine or detailed representation of information in the computer.

Materials

1. An introduction to a common algebraic language, such as FORTRAN, PL/1, ALGOL, or the conversational languages BASIC, CAL, JOSS.
2. A reference for elementary numerical methods.
3. A reference for statistical methods.
4. A reference for simulation and other general problem-solving techniques.

Facilities

Computing facilities will be required to handle the submittal

of approximately 60 batch process jobs of very short duration per student over the period of the course. If a time-shared facility is available, about 20 to 25 console hours will be required equivalently. The conversational use of a time-shared facility is to be preferred from the point of view of efficient use of student time.

Faculty

In order to assure relevance of the exercises to the life sciences, it would be desirable for an instructor from the life sciences faculty to handle the lectures and discussions for the life science students in this course, initially, with the advice and assistance of a member of the mathematics or computer science faculty. This would also serve as a logical entree to the education of the life science faculty to the potential of automatic computers and related models for their fields. We feel that many biologists will accept the challenge posed in this context, when assured adequate guidance.

There are a number of possible logistic problems related to running the programming exercises that will usually make teaching assistants at a very junior level valuable to this course.

Content

Topics	Suggested Problems
(Approximate number of lecture hours in parentheses)	(These may be presented in the context of a biological problem)
1. Algorithms, flowchart representations. (1)	
2. First principles of an algebraic language. (6)	2a. Mean and standard deviation of a sample.
BASIC, FORTRAN, CAL, PL/I, or ALGOL. Organized so that students may begin programming as soon as possible.	2b. Selection sort.
	2c. Table look-up.
	2d. Linear interpolation in a table.

3. Very simple introduction to numerical calculus. (This can be carried out before the students have had any appreciable instruction in calculus.) (4)	3a. Area under a curve by Simpson's rule.
	3b. Euler's method (point-slope).
Cautionary discussion of error in calculation, with examples.	3c. Root finding by method of false position.
4. Pseudo-random numbers. Simulation. (2)	4. Simulation of the rolling of a die.
5. Introduction to the literature of computer programs. (2)	5. Use of a standard data analysis package such as the BIMD statistical programs.
	Additional program or programs on topics of special interest to the student.

Introduction to Computing: Alternative 2

This course is a one-semester 3-credit-hour introduction to applied computing. While the concern for representation of algorithms and data overlaps that of a first course in computer science, our suggestion differs in that the accent is always on application, on problem solving with a digital computer. There are three or four suggested programming exercises in an algebraic language and one or two in special-purpose languages suited for biological problems. The course is set primarily for sophomores.

Materials

1. An introductory text on computing.
2. References for elementary numerical methods and statistical methods.

Faculty

The course must be handled by a specialist in computing, as opposed to Alternative 1, although the elective programming exercise could be directed by a teaching assistant from the life sciences.

Facilities

Computing facilities will be required to handle the submittal of approximately 30 batch process jobs of short duration per student.

About 15 console hours on a time-shared remote access computer would be required equivalently.

Content

Topics	Suggested Problems
(Approximate number of lecture hours in parentheses)	Problems listed under alternative 1 and others, such as
1. The concept of an algorithm: discussion of its connotations. (2)	
2. Representation of algorithms: natural language, flowchart, algebraic language. (2)	
3. Principles of an algebraic language: FORTRAN, PL/I, ALGOL, or a conversational language: BASIC, CAL, etc., as available. Illustrations from simple numerical and statistical methods. (10)	
4. The evaluation of algorithms, logical organization of a computing machine. (5)	4. Test for well formation of string of parentheses.
5. A sampling of computer applications and methods. Simple symbol manipulation, list structures, simulation examples, pseudo-random number generation, a simulation language such as SIMSCRIPT, and specific applications from the life sciences. (12)	5a. Generation of Markov chain from transition matrix (presented in behavioral terms) in algebraic language, or a flow simulation in SIMSCRIPT. 5b. Elective problem on a topic from the life sciences such as an epidemic simulation, or the analysis of data from an actual experiment. 5c. Numerical integration, or linear regression with printer plot of graphical output.
6. Discussion or demonstration of special computing equipment involving graphical displays or real-time control of experiments. (3)	

APPLIED MATHEMATICS

IN THE

UNDERGRADUATE CURRICULUM

A Report of

The Panel on Applied Mathematics

January 1972

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I. INTRODUCTION AND STATEMENT OF RECOMMENDATIONS

Traditionally, attempts to solve problems in the physical sciences have stimulated and, in turn, extensively utilized basic developments in mathematics. This essential interaction between mathematics and the sciences is experiencing new vigor and growth. Recently, mathematical methods have been introduced into the social and life sciences, and even into some areas of the humanities. This has led to the development of new mathematical ideas and to new ways of using mathematics. The Committee on the Undergraduate Program in Mathematics (CUPM) appointed a Panel on Applied Mathematics to consider the implications for the undergraduate curriculum of this new growth of the uses of mathematics.

Instead of training students to handle all of the steps involved in solving a realistic problem, typical courses in applied mathematics generally confine themselves to a treatment of various mathematical techniques; in particular, mathematical model building is neglected. While courses in mathematical techniques are necessary, they do not provide a sufficiently broad training for students interested in applied mathematics.

The Panel therefore makes the following recommendations:

1. Every mathematics department should offer one or two courses in applied mathematics which seriously and comprehensively treat realistic problems and which emphasize model building.

2. Mathematics courses in the first two years of college should contain many realistic applications.

3. Every student taking a substantial number of courses in mathematics should include at least one course in applied mathematics.

4. A concentration in applied mathematics should be made available if the resources of the college permit.

The Panel is aware that the fourth recommendation is the most difficult to implement, especially in smaller departments. However, we feel strongly that most college departments can begin to implement the first three recommendations without undue difficulty or delay. For instance, having one instructor offer a course emphasizing model building could be an initial step toward implementing the first recommendation. Although the course may not have all of the desired characteristics the first time it is taught, the instructor's experience, along with ideas from this report, should enable him to come closer to meeting the objectives described here when he teaches the course again. Instructors in calculus, for example, can help to implement the second recommendation by introducing in their courses some applications different from the usual ones. In any case, the first of these recommendations can be effected by instituting one or

two new courses at the upper-division level, and the second by incorporating applications in the lower-division courses.

II. DISCUSSION

Pure mathematics has undergone tremendous development during the past 25 years. Consequently, the recent generation of mathematicians is concerned primarily with pure mathematics, not only in research but also in educational activities. This is evidenced by the abstractness of some high school mathematics courses and the early introduction of axiomatic courses in colleges.

While the Panel applauds the advances in pure mathematics, it feels that it is unfortunate that education in applied mathematics has not received the same attention as that in pure mathematics. As a result, many other departments offer courses having substantial mathematical content, and mathematics faculties have tended to be unaware of the mathematization of many areas. It is encouraging, however, that there seems to be a recognition of this tendency and that a sympathetic interest in applications of mathematics is spreading. There is much more emphasis now than there was ten years ago on areas which directly attack problems of contemporary society such as ecological studies, city planning, water and atmosphere restoration, etc. This interest manifests a return to an attitude held in earlier times when mathematics was viewed as closely related to other areas such as the physical sciences and engineering. The unique way in which mathematics can contribute to an understanding of important problems in modern society is acknowledged, and many mathematicians have been attracted to the new ideas involved in recent applications because they are eager to have their teaching and research contribute to solutions of problems which are practical and contemporary.

These recent applications have contributed to changes in applied mathematics, both in its nature and in its methods. Applied mathematics may once have been identified exclusively with areas of analysis which had particular bearing on physics and engineering. But because mathematics is used in the social, life, and managerial sciences, and even in the humanities, applied mathematics must now include topics such as linear programming, graph theory, optimization theory, combinatorics, game theory, and linear algebra, in addition to those which have been traditionally associated with it. Similarly, methods of applied mathematics may have been thought of as involving complicated calculations with numbers or analytic expressions. While techniques for calculation are important, they are only part of the professional resources of an applied mathematician. Theory construction and model building are now essential for him. In studying the role of applied mathematics in the undergraduate curriculum, the Panel has taken into account these new topics and methods.

Having considered all of these points, we conclude that undergraduate instruction in applied mathematics must have a strong component specifically devoted to model building, and that undergraduates generally should be more aware of the many uses of mathematics in other areas.

III. NEW COURSES IN APPLIED MATHEMATICS

In our considerations we have been guided by the steps a working applied mathematician follows in studying a given situation. This process has been described in many ways by various authors. We use a description which is reminiscent of the one given by Murray Klamkin in the American Mathematical Monthly, 78 (1971) pp. 55-56 (ascribed to Henry O. Pollak):

1. Recognition of the nonmathematical problem.

2. Formulation of the mathematical model.

3. Solution of the mathematical problem.

4. Relevant computations.

5. Explanation of results in the context of the original problem.

Courses in mathematical topics give training in the solutions of mathematical problems (step 3), and courses in computer science and numerical analysis explain computational and approximative techniques (step 4), but very few courses adequately treat the processes involved in recognition, formulation, and explanation (steps 1, 2, and 5). While the student must, of course, have sufficient mathematical and computational techniques at his command to solve the mathematical problems he confronts and to obtain the numerical results which are needed, we are convinced that the training of a student of applied mathematics must be more comprehensive. He must be thoroughly grounded in the techniques of mathematical model building, and he must have ample practice in interpreting the results of his mathematical solution in the original setting.

The first recommendation of the Panel is that each department should offer a course or two in applied mathematics which treat some realistic situations completely, beginning with a careful analysis of the nonmathematical origin of the problem; giving extremely careful consideration to formulation of a mathematical model, solution of the mathematical problem, and relevant computations; and presenting thoughtful interpretations of the theoretical results to the original problem. In other words, there should be a few courses

which give the students the experience of grappling with an entire problem from beginning to end.

To aid colleges in implementing the first recommendation, the Panel has constructed outlines of courses which emphasize model building. These courses are not intended to replace courses stressing mathematical techniques which are offered for students majoring in other areas, nor should they replace those standard mathematical offerings in which applications play a useful motivational role. Service courses are valuable and should continue to be offered by the mathematics department; indeed, they should be designed in active collaboration with members of concerned departments. Courses in mathematical topics which have their origins in applications are also important. However, the courses we recommend here provide a complementary training by giving students active experience in mathematical model building.

The Panel has given at the end of this report three course outlines which illustrate how a course stressing model building can be designed. These outlines are centered around the topics of optimization, graph theory and combinatorics, and fluid mechanics. The optimization course is intended as an example of a sophomore-junior course, the course on graph theory and combinatorics is appropriate at the junior level, and a course along the lines of the fluid mechanics option can be taught at the senior level. The optimization course and the course in graph theory and combinatorics can be offered at various levels by changing the level of rigor, varying the pace, concentrating longer on problems from a specific area, etc. These particular topics were chosen as unifying themes because of the experiences, interests, and competencies of individual Panel members, and because the courses on optimization and on graph theory and combinatorics illustrate the use of topics not traditionally viewed as being part of applied mathematics. In choosing these topics, the Panel does not mean to exclude other topics which might be used as the unifying element of an applied mathematics course. On the contrary, we hope that these outlines will stimulate instructors to construct similar courses around other topics. In fact, within reasonable limits, the particular topics chosen are not nearly so crucial as the emphasis on the model building process.

IV. GUIDELINES FOR TEACHING THE NEW COURSES

In planning or in teaching courses which emphasize model building, the instructor should keep in mind certain points which are essential for proper implementation of our recommendations.

First, the role of model construction must be made clear and amply illustrated throughout the course. The student must have as much experience as possible in constructing models. Real-life situations are often so complex that it is impossible to formulate a satisfactory model immediately; quite often it is necessary to construct a succession of models in an effort to find a satisfactory one. The student should have experience with this process. Furthermore, he should be aware that there may be several approaches which lead to essentially different mathematical models for the same problem. Therefore, a critical evaluation of the steps in constructing a model is essential in order that the student know what kind of information he can expect or cannot expect from a model and that he be able to choose the model which is most effective for his purpose.

Constructing a model for a given situation requires originality and a thorough understanding of the original nonmathematical situation. To appreciate what is involved, students must be active in formulating models. This aspect of the training is so important that the instructor should be willing to sacrifice some topics to insure the student's thorough grounding in model building. If the instructor conducts his class in the traditional lecture fashion, then he should prepare homework projects which require his students to formulate and to refine models for various situations. However, the Panel explicitly calls attention to the possibility of conducting these courses as seminars in which students and faculty members work cooperatively. Such a seminar could be organized around various problems, or it could develop a model for a complicated system which can be subdivided into smaller units. A benefit of the latter format is the experience of teamwork. Another possibility is for students to choose projects which they pursue independently. These projects could range from original investigations to reports based on the literature. In this case, students should periodically report their progress to the other participants in the seminar.

It is important to realize that model building has many forms. The activity which is most usually associated with the term <u>modeling</u> and which is actually always present in some form consists of formulating in explicit terms the dependence of the phenomenon under investigation upon the relevant factors. A classic example is the construction of a model for the motion of a vibrating string leading to a linear partial differential equation. In this case the factors which are to be neglected as well as those which have considerable effect on the motion can be identified, and the sort of physical assumptions which simplify the model are relatively clear. With appropriate assumptions, an analysis of the physical laws governing the motion of a particle lead to a mathematical model for the motion of the string consisting of a partial differential equation and suitable boundary conditions. The solution of this mathematical problem aids in the description of the motion of the string. The degree to which the solution of the mathematical problem contributes to an understanding of the physical one depends upon the degree to which the assumptions fit the real situation.

The model for motion of a vibrating string is a deterministic one; that is, it is based on the assumption that the physical laws and the initial conditions determine the response of the system exactly. Such models are not always appropriate, and there are instances in which uncertainty in the real situation should be reflected in the model, as, for example, in stochastic models. As an illustration, consider the construction of a model for the spread of a disease. The number of people who become ill during an epidemic depends on a number of factors associated with the disease--its virulence and period of contagion for example--and also on the random contacts between infected and susceptible individuals. In some instances the results obtained by ignoring the probabilistic features of the situation may be adequate, while in others inclusion of the probabilistic features may be required in order to obtain a satisfactory fit between the predictions of the model and the results of observations.

Alternatively, it may be that any model which accounts for what appear to be the essential features and which is formulated for mathematical analysis will lead to mathematical problems which are either totally intractable or beyond the scope of investigation. In such cases a computer simulation may be useful. Simulations may be performed on both deterministic and stochastic models, and they may provide much of the same type of information that is obtained from a mathematical analysis when such analysis is feasible.

The point is that there are many kinds of models, and the student of applied mathematics should be aware of them. Consequently, the topics for investigation must be chosen carefully so that different types of models will be illustrated.

Second, the problems chosen for investigation must be realistic. In this report, when we use the term "realistic" referring to problems or situations, we have in mind those which arise directly from nature or from social behavior and which have some current significance. We label as "artificial" those problems which seem to be designed purely to illustrate some mathematical point. While some artificial problems have undeniable pedagogical value, relying almost exclusively on such problems will not instill the attitude of mind which should characterize the modern applied mathematician. In a contrived situation it is difficult to create and maintain interest in the multitude of concerns which arise in problems occurring in the real world. Since it is the Panel's intention that the student recognize the complexities of the real world and that he come to terms with these complexities in his model building process, the student must face real problems. In the course outlines we have given references to assist those who wish to acquaint themselves with significant problems in other fields.

Third, the original nonmathematical situation should not be forgotten once a mathematical formulation has been achieved. The results of the mathematical study need to be interpreted in the original setting. Stopping short of this gives the impression that

manipulation of symbols or that techniques of computation or approximation are the important points, whereas they are only intermediate steps, although absolutely essential, in studying a realistic nonmathematical situation. For this reason we urge that in these courses the situations should not only be realistic but that they should be treated as completely as possible.

Fourth, the mathematical topics treated should be worthwhile and have applicability beyond the specific problem being discussed. They should be appropriate to the level at which the course is offered; problems and examples should be chosen to illustrate more than just elegant or ingeneous applications of mathematically trivial ideas. It is impossible for a single course to contain all of the mathematical techniques which all students may need; nevertheless, it is possible to select as illustrative techniques those which will be valuable to a large portion of the students.

Finally, an instructor of applied mathematics should not view his work as being confined to one academic department or, for that matter, restricted to his college or university. Applied mathematics affords unique opportunities for cooperative projects with other members of the college community and with people outside the college whose professional work is related to mathematics. We encourage instructors to invite active participation by students and faculty members from other departments in planning and conducting courses or seminars. In some instances it may be valuable to include nonacademic professional people having interests and competencies related to the area being studied; their experience and point of view may add a new dimension to the investigations. It is our view that instructors in applied mathematics are in a particularly good position to initiate cooperative ventures of this type.

V. USE OF COMPUTERS IN APPLIED MATHEMATICS

Mathematics education has been influenced in several ways by the recent trend toward the widespread use of computers. This is particularly true of instruction involving applications. The role of computing in the mathematics curriculum is being studied in detail by the CUPM Panel on the Impact of Computing on Mathematics Courses [see Recommendations on Undergraduate Mathematics Courses Involving Computing, page 571], and comments on computing as a part of a concentration in applied mathematics can be found in Section VIII of this report. The purpose of this section is to draw attention to the ways in which machine experience can reinforce ideas and techniques which the student is learning and thereby contribute to the teaching of applications.

The use of computers makes it possible to consider situations having a much greater complexity than would be possible if the associated numerical work were to be carried out by hand or with the assistance of a desk calculator. This is true not only in courses specifically oriented toward applications but also in the standard undergraduate courses. As an illustration of the sort of activity which illustrates the process of applied mathematics and which becomes feasible through the use of computers, consider the example of determining as a function of time the position and velocity of a rocket traveling to the moon. The depth of the study obviously depends heavily on the audience, but certain versions are appropriate for students in courses in elementary calculus or ordinary differential equations. A sample discussion in the spirit of this report would include the following features.

1. Newton's laws of motion and gravitation and a mathematical model for the system. A careful discussion of the idealizations and approximations made in constructing the model.

2. Derivation of the differential equations governing the motion of a rocket in one dimension between the earth and moon.

3. Discussion of the qualitative features of the solution.

4. Selection of a numerical method.

5. Preparation and testing of a computer program for the integration of a system of first-order ordinary differential equations.

6. Use of the program to obtain quantitative information on the motion of the rocket. Determine the escape velocity of the earth-moon system and compare it with that of the earth alone.

7. Comparison of results predicted by the 1-dimensional model with observed phenomena and a discussion of the inadequacies of such a model.

8. Derivation of the differential equations describing the motion of a rocket in two dimensions.

9. Numerical solution of these equations in two dimensions [repeat steps 3, 4, and 5 in this case]. Use a plotter to graph the trajectories as functions of initial velocity and firing angle.

10. Comparison of these results with observations. Discussion of discrepancies.

In addition to its use in the activities described above, the computer can also be used to obtain the best values of parameters occurring in the model and to test the validity of the model. The latter usually involves comparing predictions based on the model

with experimental data by using statistical techniques. Finally, both analog and digital computers are useful tools for simulation when the situation cannot be modeled in a form susceptible to mathematical analysis.

VI. RECOMMENDATIONS CONCERNING MATHEMATICS COURSES IN THE FIRST TWO YEARS

The Panel believes that many students lose their enthusiasm for mathematics even as a tool because their mathematics courses seem unrelated to their own discipline. A large segment of students in lower-division mathematics courses is primarily interested in fields outside mathematics. These students want to use the ideas and techniques of mathematics in their fields of interest; they are not interested in majoring or minoring in mathematics. We feel that the best way to demonstrate the power and utility of mathematical ideas to these students and thereby to sustain their interest is to introduce applications to other fields in the early mathematics courses. Therefore, the second recommendation of the Panel is that a greater number of realistic applications from a greater variety of fields be introduced into the mathematics courses of the first two years.

The suggestions made earlier about the choice of problems and examples apply here too. Instructors should strive to avoid artificial or contrived examples and applications. It is especially important to formulate the problem clearly and to mention explicitly the assumptions, approximations, and idealizations used to obtain a reasonable mathematical model. If simplifications are needed to make the mathematical problem workable, then they should be clearly stated and discussed. In other words, the applications should be significant and their treatment should be as complete and intellectually honest as the level of the course will allow.

The applications should be chosen from various fields in order to illustrate the use of a mathematical model or idea in different settings. If the course and the background of the students permit, some problems should be treated which require one to construct a succession of mathematical models in an effort to conform better to experimental data. Numerical methods might be included.

As we have already mentioned, some students who are not mathematics majors lose enthusiasm for mathematics because their courses do not contain applications. However, the Panel is also concerned that the mathematics major have an appreciation for the importance of mathematics in other areas. Even if he becomes a research mathematician, he is very likely to teach some undergraduate mathematics courses. His effectiveness in these courses can be greatly increased by a grasp of the relations among different branches of mathematics

and the relations between mathematics and other disciplines. Therefore, we feel that he should see many significant applications in his elementary mathematics courses.

Unfortunately, very little literature on applications of elementary mathematics exists at the present time. One source is the Proceedings of the Summer Conference for College Teachers on Applied Mathematics held at the University of Missouri--Rolla with the support of the National Science Foundation, published by CUPM. These proceedings contain applications of elementary calculus, linear algebra, elementary differential equations, and probability and statistics.

Textbooks for most undergraduate mathematics courses vary considerably in their emphasis on applications, and instructors should consult various books so that they can provide their classes with a variety of interesting applications. For example, in differential equations there are many modern texts which contain discussions of genuine applications. Two books which contain a variety of applications not duplicated in many other places are:

Bellman, R. and Cooke, K. L. Modern Elementary Differential Equations, 2nd ed. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1971.

Carrier, G. F. and Pearson, C. E. Ordinary Differential Equations. Waltham, Massachusetts, Blaisdell Publishing Company, 1968.

Also, modern texts in general physics and mechanics usually have examples suitable for discussion in a course on differential equations.

Another standard undergraduate course--linear algebra--has many applications to both physical problems and linear programming. In addition to the references listed in connection with the optimization outline given later in this report, the following text deserves mention:

Noble, B. Applied Linear Algebra. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1969.

The following book is a collection of realistic problems suitable for undergraduate mathematics courses. The problems are catelogued according to the mathematical tools used in their solution. Every teacher of freshman and sophomore mathematics should be aware of this source of applications.

Noble, B. Applications of Undergraduate Mathematics in Engineering. New York, The Macmillan Company, 1967.

As an example of the way in which a specific subject matter area may be used to provide applications for elementary courses,

consider the biological sciences. Population growth, for example, can serve as a motivation for the introduction of elementary differential equations. Also, population growth problems can be considered from a probabilistic point of view; indeed, many problems in the biological and social sciences admit both deterministic and stochastic models, so it may be wise to introduce probability along with calculus in order to be able to study both kinds of models. Books are now available which take this approach; for example,

Chover, J. The Green Book of Calculus. Menlo Park, California, W. A. Benjamin, Inc., 1971.

Stein, Sherman K. Calculus in the First Three Dimensions. New York, McGraw-Hill Book Company, 1967.

The instructor who wishes to include applications to the biological sciences will find the following references useful. Although some of this material can be treated with little modification in lower-division classes, these sources are more suitable for the instructor than for the student.

Gerstenhaber, M. et al. AMS Lectures on Mathematics in the Life Sciences: Some Mathematical Problems in Biology, vol. I. Providence, Rhode Island, American Mathematical Society, 1968. (See particularly the first paper.)

Pielou, E. C. An Introduction to Mathematical Ecology. New York, John Wiley and Sons, Inc., 1969.

Smith, J. M. Mathematical Ideas in Biology. New York, Cambridge University Press, 1968.

VII. COMMENTS ON SECONDARY SCHOOL TEACHER TRAINING

Our third recommendation is that every student whose degree program includes a substantial number of courses in mathematics should take at least one course in applied mathematics. This recommendation clearly should apply to mathematics majors, but the Panel wishes to emphasize that every prospective secondary school teacher of mathematics should also have at least one course in applied mathematics. The role of applied mathematics in the training of teachers of secondary school mathematics has been underscored by the American Association for the Advancement of Science* and by other CUPM panels. [See Recommendations on Course Content for the Training of Teachers

* Guidelines and Standards for the Education of Secondary School Teachers of Science and Mathematics. Washington, D. C., American Association for the Advancement of Science, 1971.

of Mathematics, page 158.] The AAAS recommendations state that "an undergraduate program for secondary school mathematics teachers should ... provide substantial experiences with mathematical model building so that future teachers will be able to recognize and construct models illustrating applications of mathematics." The CUPM Panel on Teacher Training recommends that prospective teachers should complete a major in mathematics and that the courses in the program should include not only a mixture of motivation, theory, and application but also an introduction to model building. Indeed, that Panel recommends that a course in applied mathematics is particularly desirable as an upper-division option for the mathematics major.

The Panel on Applied Mathematics strongly supports these recommendations and emphasizes the following reasons for a secondary school teacher of mathematics to have a knowledge of applications:

1. Appropriate applications provide excellent motivational material.

2. The teacher should be aware that most of the mathematics encountered in the secondary school has its origins in problems in the real world, and he should know what these origins are.

3. The teacher should be aware of the applications of mathematics in the social and life sciences as well as in the physical sciences. Since mathematical notions are occurring with increasing frequency in elementary texts in the social and life sciences, and since it is unlikely that most teachers of these subjects have adequate mathematical training to appreciate this material, the mathematics teacher may well be called upon to serve as a resource person for other teachers.

4. Carefully selected applications may aid significantly in developing the student's ability to recognize familiar processes which occur in complex situations.

Further discussion of these and other ideas can be found in references [E] and [P] at the end of this section.

We make the following recommendations:

1. In those courses of the basic curriculum which are taken by substantial numbers of prospective secondary school teachers (viz., Mathematics 1, 2, 3, 4 and 2P of Commentary on A General Curriculum in Mathematics for Colleges (CGCMC)), applications of the subject to problems arising outside mathematics should receive more attention than is generally given now.

2. Each prospective teacher should be strongly encouraged to take one of the courses proposed in Section III of this report or a course in applied mathematics designed especially for secondary school teachers. Sample materials appropriate for an applications-oriented course for teachers include [B], [Po], and [S].

References

[B] Bell, M. S., ed. Some Uses of Mathematics: A Source for Teachers and Students of School Mathematics. (Studies in Mathematics, vol. XVI, SMSG) Pasadena, California, A. C. Vroman, Inc., 1967.

[E] Engle, A. "The relevance of modern fields of applied mathematics for mathematics education." Educational Studies in Mathematics, 2 (1969), pp. 257-269.

[P] Pollak, H. O. "How can we teach applications of mathematics?" Educational Studies in Mathematics, 2 (1969), pp. 393-404.

[Po] Pólya, G. Mathematical Methods in Science. (Studies in Mathematics, vol. XI, SMSG) Pasadena, California, A. C. Vroman, Inc., 1963.

[S] Schiffer, M. M. Applied Mathematics in the High School. (Studies in Mathematics, vol. X, SMSG) Pasadena, California, A. C. Vroman, Inc., 1968.

VIII. RECOMMENDATIONS CONCERNING A CONCENTRATION IN APPLIED MATHEMATICS

The fourth recommendation of the Panel is that an undergraduate concentration in applied mathematics should be offered if the resources of the college permit. In many institutions there are students who desire such a program. These students should take some courses in model building such as those described in Section III, and they should be trained in mathematical topics useful in applications. We are concerned both that the training of the students properly reflect the changes taking place in applied mathematics and that a department of mathematics be able to begin implementation of our recommendations immediately with a relatively small change in course offerings. For these reasons our recommendations center around courses of the type we have already described and courses in various mathematical techniques which are common in many colleges.

A student interested in a concentration in applied mathematics should take three courses in calculus (Mathematics 1, 2, 4 of CGCMC) and a course in linear algebra (Mathematics 3 of CGCMC). (For those who notice the omission of differential equations, we point out that Mathematics 2 of CGCMC contains an introduction to differential equations.) To insure training and practice in modeling, he should take at least one and preferably two of the new courses described in this report. A student who has a particular area to which he wishes to

apply his mathematics should select courses in mathematical topics which are useful in that area as well as courses in the field of application which utilize significant mathematics. The topics suggested below can be organized into courses in various ways. However, we do recommend that applications be introduced in these courses, and we feel that the comments made in Section VI on applications in the freshman and sophomore courses are particularly appropriate here.

A student who is interested in applications to the physical sciences or in some areas of life sciences (e.g., ecology) should take a physical science version of an applied mathematics course such as the one in fluid mechanics outlined in this report. His further mathematics courses should include as many of the following topics as possible: probability theory; elementary partial differential equations (some of this is already contained in the fluid mechanics course); topics in ordinary differential equations such as asymptotic solutions, stability, and periodic solutions; boundary value problems (including Fourier series); computer-oriented topics from numerical analysis such as those which emphasize numerical solutions of ordinary differential equations, numerical linear algebra, solution of nonlinear equations, or numerical quadrature.

A student interested in applications to business and social sciences should take courses such as the optimization course and the graph theory course outlined in this report. His further mathematics course work should include as many topics as possible from the following: probability theory and applications as described in the report of the CUPM Panel on Statistics, Preparation for Graduate Work in Statistics; statistics as described in the same document; computational linear algebra.

Furthermore, because much work in applied mathematics involves computations, approximations, and estimates, it is clear that students concentrating in applied mathematics should have training in the use of computers. Beyond increasing computational power, a knowledge of the uses of computers can provide a new perspective for formulating and analyzing problems of applied mathematics. Consequently, the Panel strongly recommends that the following phases of computer experience be included in the program of every student of applied mathematics:

1. Computer programming. The student should have sufficient familiarity with a programming language to be able to use computer facilities in ways that are appropriate for his mathematical course work.

2. Computational mathematics. The approximations, estimations, algorithms, and programming necessary to derive numerical solutions of mathematical questions should be presented.

3. Training and experience in the use of a computer at the various stages of solving a problem in applied mathematics. The

student should have experience in using the computer to organize large quantities of numerical data and to simulate complicated processes.

IX. COURSE OUTLINES

To exemplify the kinds of courses recommended in Section III, the Panel has constructed three course outlines. These courses do not deal merely with mathematical topics; they are courses in which the momentum comes from real situations. In particular, stress should be placed on model building and on interpretation of mathematical results in the original nonmathematical situation.

These outlines are not offered as perfect models of the kinds of courses we recommend. Rather, they represent our present best efforts to construct courses with these new emphases. We hope that they will produce a thoughtful response in the form of even better outlines for applied mathematics courses.

It is essential that these outlines be read with the recommendations of Section IV in mind. Also, the reader should have in hand one or two of the primary references in order to find examples of the kind of treatment we are suggesting.

In reading these outlines, in teaching these courses, or in constructing other courses along the lines of our recommendations, instructors should strive to stay well between the extremes of: (a) a course about mathematical methods whose reference to science consists mainly of assigning appropriate names to problems already completely formulated mathematically, and (b) a kind of survey of mathematical models in which only trivial mathematical development of the models is carried out.

The course in optimization was planned as a one-quarter course, with additional material in the sections marked * bringing the total to a one-semester course. The courses in graph theory and combinatorics and in fluid mechanics were designed as one-semester courses.

The number of lectures specified indicates the relative emphasis we have in mind for the various topics and serves as an actual time estimate for a well-prepared class. The Panel appreciates the fact that some instructors will find these time estimates somewhat unsuitable (for instance, they do not take into account the pursuit of finer points or the review of prerequisite material) and will find it necessary to make modifications in the courses for their classes. The Panel was tempted to construct less ambitious outlines but decided against this, because it felt that a prospective

instructor would be helped by having more examples of the treatment we recommend rather than fewer. Nonetheless, a valuable course can be constructed by choosing a few of the topics listed and treating them carefully and thoroughly. Furthermore, if the students become actively engaged in the model building activity, then the time estimates given are not appropriate. In any case, we encourage instructors to engage in open-ended discussions with class participation in the modeling aspect of the course and, if necessary, to restrict the subject matter content of the course in order to accommodate this.

IX.1. OPTIMIZATION OPTION

This course was designed to provide an introduction to the applications of mathematics in the social and management sciences. The goals of this course, as stated in Sections III and IV, are a study of the role of mathematics as a modeling tool and a study of some mathematical notions of proven usefulness in problems arising in the social and management sciences. The mathematical content consists of programming and game theory. This selection is a considered choice, although it is recognized that several other alternatives could serve as well.

The proposed course can be taught at several levels to fit the competencies and interests of the class. In particular, one version might be appropriate for freshmen whereas another might be appropriate for upper-class students in the management and social sciences. The course outlined here is intended for an average junior-level class. The students should have completed the equivalent of two semesters of calculus and should have some familiarity with elementary probability theory. Linear algebra is not included as a prerequisite, as the necessary background is developed in the course. No specific knowledge of any other discipline is assumed.

A bibliography and an appendix, important adjuncts to the course coutline, are found after the outline. References to the bibliography are enclosed in square brackets [ ], and references to the appendix are enclosed in braces { }. The bibliography contains a selection of books and other references which have proved useful in courses of this sort. Certain references have been designated as primary references, and comments have been provided which indicate those features of particular interest for an instructor. Most of the citations in the course outline are to the primary references. The instructor should have at least one of the primary references at hand while reading the outline. The appendix contains examples of the types of problems which can be studied using the ideas and methods of this course.

COURSE OUTLINE

1. Mathematical foundations of model building (4 lectures)

The real world and abstractions to mathematical systems; axiom systems as used in model building.

The ideas of a mathematical model and model building are introduced by using several examples which can be developed quickly and which illustrate applications in several different fields. Typical examples might be drawn from business (programming models for resource allocation), ecology (linear programming models of pollution control), psychology (2- or 3-state Markov chain models for learning), and sociology (game theory models for conflict). Assumptions made in the construction of these models should be carefully identified. The status of empirical "laws" should be discussed: law of gravity, law of reflection, law of supply and demand. It should be pointed out that all model building requires some essentially arbitrary decisions on the part of the person who is constructing the model. For example, whether to select a deterministic or a stochastic model is ultimately a decision of the investigator. In most instances there is no single best model. A model which was constructed to account for observed phenomena of one type may not give predictions which agree with other observations. The role of approximation and idealization in model building is fundamental. Approximations which are made and justified for real-world reasons should be distinguished from those whose basis is mathematical. Students need practice in making connections between assumptions about the real world and axioms in a mathematical system. Some of the examples should bring out the fact that an important (and frequently difficult) part of model building is asking the right question and viewing the real world problem from the right perspective. Some attention should be given to the practical problems of critically evaluating models and estimating parameters.

Most of the references contain some comments on model building. The initial chapters of primary references [D], [KS], and [Sa] have more comprehensive discussions. The books [ABC] and [LR] discuss modeling from the point of view of the social scientists.

2. Linear programming models (18 lectures)

a) Construction of linear programming models. (1 lecture) A detailed discussion of a real-world situation which can be reasonably modeled in terms of a linear program.

Examples similar to {1} or {2} might be used. Assumptions which lead to the axioms of linearity should be explicitly noted and adequately justified. It may be that the linear model is meant to serve only as a first approximation to a more complicated situation. Also, a linear model is frequently realistic only for restricted values of some variables. Such questions need to be considered. It is desirable to introduce both deterministic and stochastic models and later to compare two models of the same situation. The history of the development of linear programming during and after World War II is interesting. The book [D] is a useful reference for this material.

b) The basic problem. (6 lectures) The algebra and geometry of systems of linear inequalities in R^n. Matrix and vector notation and elementary linear algebra. Systems of linear equations and their application to systems of linear inequalities (e.g., if A is an $m \times n$ matrix and $\underline{b} \in R^m$ ($\underline{b} \neq 0$), then there exists $\underline{x} \in R^n$ satisfying $A\underline{x} = \underline{b}$ or there exists $\underline{y} \in R^m$ satisfying $A^t\underline{y} = 0$, $\underline{b}\cdot\underline{y} \neq 0$).

The notion of duality and the fundamental theorem should be introduced and illustrated. Consider complementary slackness and its economic interpretation. Selections from the primary references [D], [Ga], [SpT], and [W] provide appropriate sources.

* Proof of the fundamental duality theorem.

c) Algorithms: the simplex method. (6 lectures)

Much of the usefulness of linear programming models rests on the fact that the resulting mathematical problems can be efficiently solved. Accordingly, it is important to give some attention to computation, although only a bare introduction is proposed here. The method can be introduced as a sequence of replacement operations similar to a method for solving systems of linear equations. Algebraic and economic considerations can be used to describe and motivate the method. The concept of degeneracy arises naturally, but a

complete discussion of this idea is beyond the scope of the course. Larger and more realistic problems should be solved, and students with computer competence should be encouraged to use it. The references are the same as those cited in b).

* Further remarks on degeneracy.

* A proof of the convergence of the simplex algorithm.

d) Refined models: linear programming and uncertainty. (5 lectures)

These models should be introduced by discussing the inadequacy of deterministic models for certain problems. One example is the allocation of aircraft to routes (this is discussed in Chapter 28 of [D]). There is no single formulation for stochastic models, as for deterministic ones, and there is little general theory. However, this is an important modeling technique which serves to demonstrate how models can be refined to take account of additional information. Examples can be given which show that one is not usually justified in simply substituting expected values for coefficients which are actually random variables. The basic problem is to formulate the stochastic model in such a way that relevant information can be obtained by studying an ordinary deterministic model. Chance constrained programming provides an interesting special case. Primary references [D] and [W] contain this material.

* Multistage models and dynamic programming.

* Geometry of the simplex method.

* Linear models of exchange and production.

3. Game-theoretic models (10 lectures)

a) Games and decision-making with uncertainty models for systems involving opposing interests. (3 lectures) The role of games as a modeling technique in the social sciences. The basic assumption of rational behavior and its validity.

Introduce utility theory, in both its qualitative and quantitative aspects. Consider individual decision-making under uncertainty and compare this to games. Discuss examples and, in particular, the relevance of a mathematical theory of games for the real world. The basic reference [LR] is useful here. Both [LR] and [BN] discuss

game theory from the social scientist's point of view.

b) Games with two sets of opposing interests. (3 lectures) Two-person zero-sum matrix games and the connection between such games and linear programming.

Although such games are of limited use in applications, they provide a convenient vehicle for introducing basic notions of strategy and payoff. The fundamental (minimax) theorem of two-person zero-sum games. In primary references [D] and [Ga] this material is closely connected with linear programming. The discussion in [LR] is more comprehensive, and the notions of extensive and normal forms for games are introduced.

c) Nonzero-sum games. (3 lectures)

Games of the "prisoner's dilemma" type are of particular interest to the social scientist and can be used to illustrate the difficulties which arise in more complex models. The theory for such games is not nearly so well developed as for the games of b), but the study should bring out many of the questions that arise in mathematical work in the social sciences. Primary reference [LR] contains some of this material; more detailed expositions can be found in [BN] and, among the additional references, in [R].

d) n-person games. (1 lecture)

There is a qualitative difference between two-person situations and those involving three or more independent interests. Thus, there are new difficulties which arise in modeling three-interest conflict situations. The notion of a "solution" to such games requires careful analysis. The role of bargaining and coalitions is important in such models. See primary reference [LR].

* Games of timing. Reference [Dr] is especially complete on this topic.

* Two-person cooperative games.

References

A bibliography consisting of several hundred items on the topics listed in the course outline could easily be compiled. Thus, with some exceptions, the list of references is restricted to those sources specifically cited in the course outline. Several of the books listed here contain extensive bibliographies. The books given

extended annotation are, with one exception, examples of writing which reflect the spirit of the course. The exception [0] is a mathematics textbook which presents some of these notions from a purely mathematical point of view. Critical reviews are indicated according to the following scheme: AMM, American Mathematical Monthly; MR, Mathematical Reviews; OR, Operations Research; and SR, SIAM Review. Also, each reference has been broadly classified according to whether it is primarily concerned with the mathematical content (M) or applications (A), and whether it is most useful for the student (S) or instructor (I). Several of the other references have been given a one-line annotation where useful.

Primary References

[D] Dantzig, George B. Linear Programming and Extensions. Princeton, New Jersey, Princton University Press, 1963, 625 p. AMM 72, p. 332; MR 34 #1073; OR 14, p. 734. (M and A, S) A textbook on mathematical programming written by one of the founders of the field. It includes chapters on the history of the subject and on model formulation. Chapter 3 contains five detailed examples. Standard topics in linear programming, extensions to integer, stochastic, and nonlinear programming, and many applications. Connection between programming and matrix games is included. Basic linear algebra is covered rapidly, and some probability is needed for the chapters on stochastic programming and games. No other prerequisites. Last two chapters contain detailed examples of formulation and study of models for nutrition and transportation. Extensive bibliography, many examples, and exercises.

[Dr] Dresher, Melvin. Games of Strategy: Theory and Applications. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1961, 184 p. AMM 69, p. 243; MR 22 #13310; OR 10, p. 272. (M and A, S) The basic theory of two-person zero-sum games is developed in the first three chapters independent of its connection with linear programming. The remaining chapters include methods of solution, extensions to games with an infinite number of strategies, and games of timing. Basic calculus and probability are required. Several thoroughly discussed examples and applications, particularly to military problems. No exercises.

[Ga] Gale, David. The Theory of Linear Economic Models. New York, McGraw-Hill Book Company, 1960, 330 p. MR 22 #6599. (M and A, S) Theoretically oriented presentation of linear programming and game theory with emphasis on the use of these concepts in developing linear economic models. An advanced undergraduate textbook with no specific mathematics prerequisites, but almost everything is proved and several proofs require considerable sophistication. Chapter 2 provides a self-contained study of linear algebra including

linear inequalities. The approach is both algebraic and geometrical throughout. Validity of the models is not discussed. Exercises are a definite asset; they vary from routine to nontrivial extensions of the theory.

Kaufmann, A. and Faure, R. Introduction to Operations Research. New York, Academic Press, Inc., 1968. (A, S) Collection of 18 chapters, each a completely worked-out independent example, generally written in anecdotal form. Few specific mathematics prerequisites; calculus and finite mathematics certainly sufficient. Basic ideas thoroughly explained but involved mathematical arguments are avoided. Applications are mostly to business situations and problems have an aura of reality.

[KS] Kemeny, J. C. and Snell, J. L. Mathematical Models in the Social Sciences. Waltham, Massachusetts, Blaisdell Publishing Company, 1962, 145 p. AMM 71, p. 576; MR 25 #3797. (A, S) Collection of eight independent examples of the construction and study of mathematical models drawn from several scientific disciplines. Stated mathematics prerequisites are one year of calculus and a good course in finite mathematics, but most students will require more background. No specific social science knowledge is assumed. There is an introductory chapter on the methodology of mathematical model building. Exercises and projects at the end of each chapter.

[LR] Luce, R. D. and Raiffa, H. Games and Decisions. New York, John Wiley and Sons, Inc., 1957, 509 p. MR 19, p. 373. (A, S) This is more a book about the concepts and results of game theory than a mathematics textbook; there are almost no proofs. Modest prerequisites: some knowledge of finite mathematics plus a bent for mathematical thinking. Thoroughly motivated discussions of the heuristic considerations which precede the mathematical formulation of the problems. These discussions are colored by a social science point of view. The introductory chapters consider the role of game theory in the social sciences and give a relatively complete discussion of utility theory including an axiomatic treatment. Extensive bibliography. No exercises.

[O] Owen, Guillermo. Game Theory. Philadelphia, Pennsylvania, W. B. Saunders Company, 1968, 228 p. MR 36 # 7420. (M, I) Mathematics textbook on two-person (chapters 1-5) and n-person (chapters 6-10) game theory, including those aspects of linear programming which are important for the study. Assumes basic calculus and probability. Convexity used but developed in an appendix. Chapters on infinite games and utility theory. Written in Definition-Theorem-Proof style. Many exercises of varying difficulty. Few applications.

[Sa] Saaty, Thomas L. Mathematical Methods of Operations Research. New York, McGraw-Hill Book Company, 1959, 421 p. AMM 68, p. 188; MR 21 #1223. (M and A, I) A textbook on operations research consisting of three major units. Part I contains chapters on the scientific method, mathematical existence and proofs, and some methods of model formation. The first chapter is particularly relevant for this course. Part II includes classical optimization techniques as well as linear programming and game theory. Part III is devoted to probability theory and its applications, particularly to queueing. There are many examples with convenient references to the literature, and a large bibliography accompanies each chapter. Assumes basic calculus and matrix theory. Some sections require multidimensional calculus. No exercises.

Simonnard, Michele. Linear Programming (translated by William S. Jewell). Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1966, 430 p. MR 34 #1079, original French edition MR 25 #1952; SR 9, p. 608. (M, I) A textbook on linear programming covering the general theory (chapters 1-7), integer programming (chapters 8-9), and the transportation problem (chapters 11-15). The connection of linear programming with game theory and extensions to stochastic and dynamic programming are omitted. The book is oriented in a practical direction, and emphasis is on effective methods. It contains references to the literature, and there is an extensive bibliography but no exercises. Appendices on linear algebra, convex polyhedra, and graphs. No specific prerequisites.

[SpT] Spivey, W. A. and Thrall, R. M. Linear Optimization. New York, Holt, Rinehart and Winston, Inc., 1970, 530 p. (M and A, S) A mathematics textbook on linear programming with emphasis on the development of the simplex algorithm. The approach is a spiral one, and most topics are developed at several levels of difficulty. Chapter 2 discusses modeling and presents several examples. There is a chapter on game theory. The necessary background material on foundations, sets, functions, and linear algebra is given in appendices. Many exercises. Suitable as a text for students with limited backgrounds.

[W] Wagner, H. M. Principles of Management Science with Applications to Executive Decisions. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1970, 562 p. (A, S) A textbook of mathematical model building and optimization in a business setting. Prerequisites are a year of calculus and finite mathematics. No knowledge of business administration or economics is assumed. Emphasis on linear, dynamic, and stochastic programming with chapters on waiting-line models and computer simulation. Broad selection of exercises ranging from computational to "Form a mathematical

model for ... ." Some proofs, but many results are provided only heuristic justification.

Additional References

Ackoff, Russell L., with the collaboration of S. K. Gupta and J. S. Minas. Scientific Method: Optimizing Applied Research Decisions. New York, John Wiley and Sons, Inc., 1962. AMM 72, p. 216; OR 11, p. 157. The philosophy and formulation of mathematical models, including utility theory.

[ABC] Atkinson, R. C.; Bower, G. H.; Crothers, E. J. An Introduction to Mathematical Learning Theory. New York, John Wiley and Sons, Inc., 1965.

Bellman, Richard, ed. Mathematical Optimization Techniques. Berkeley, California, University of California Press, 1963.

Blalock, H. M. Theory Construction, from Verbal to Mathematical Formulations. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1969. AMM 77, p. 216.

[BN] Buchler, I. R. and Nutini, H. G., eds. Game Theory in the Behavioral Sciences. Pittsburgh, Pennsylvania, University of Pittsburgh Press, 1969.

Charnes, Abraham and Cooper, W. W. Management Models and Industrial Applications of Linear Programming, vol. I, II. New York, John Wiley and Sons, Inc., 1961. MR 28 #1003a,b.

Dantzig, George B. and Veinott, A. F., eds. Mathematics in the Decision Sciences, Parts 1, 2. (Lectures in Applied Mathematics, vol. XI, XII) Providence, Rhode Island, American Mathematical Society, 1968.

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Gale, D.; Kuhn, H. W.; Tucker, A. W. "Linear programming and the theory of games." Cowles Commission for Research in Economics, Monograph #13. New York, John Wiley and Sons, Inc., 1951.

Goldman, A. J. and Tucker, A. W. "Theory of linear programming." Annals of Mathematics Studies No. 38. Princeton, New Jersey, Princeton University Press, 1956, pp. 53-98.

Hadley, G. Linear Programming. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1962. AMM 71, p. 815; SR 6, p. 191.

Rapoport, Anatol. Fights, Games, and Debates. Ann Arbor, Michigan, University of Michigan Press, 1960.

Rapoport, Anatol. Two-Person Game Theory: The Essential Ideas. Ann Arbor, Michigan, University of Michigan Press, 1966. MR 35 #1356. Nonmathematical exposition of basic ideas of two-person game theory.

[R] Rapoport, Anatol and Chammah, A. M. Prisoner's Dilemma; a Study in Conflict and Cooperation. Ann Arbor, Michigan, University of Michigan Press, 1965.

Shubik, Martin, ed. Game Theory and Related Approaches to Social Behavior. New York, John Wiley and Sons, Inc., 1964. OR 12, p. 637. Collection of articles on the applications of game theory in the social sciences.

Spivey, W. A. Linear Programming: An Introduction. New York, The Macmillan Company, 1963.

Vajda, S. Mathematical Programming. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1961.

Ventzel, E. S. Lectures on Game Theory. New York, Gordon and Breach, Science Publishers, Inc., 1961.

Wilder, R. L. Introduction to the Foundations of Mathematics, 2nd ed. New York, John Wiley and Sons, Inc., 1965.

Williams, J. D. The Compleat Strategyst, rev. ed. New York, McGraw-Hill Book Company, 1966. Intended for readers with no background in mathematics, this book develops the basic ideas of game theory through a variety of clever examples.

Appendix

The problems given here are indicative of the sorts of questions that can be studied using the techniques and ideas of this course. Problems similar to these should be approached in the spirit of Section 1 of the outline, where the question is phrased in real-world terms and a mathematical model is constructed. In such a discussion, close attention should be paid to assumptions, both explicit and tacit. The student should be made aware of the strengths and shortcomings of the resulting models.

1. Linear programming

Here are two linear optimization problems, one concerning diet and another concerning transportation, posed in a business context. The first is given in considerable detail, while the second is merely sketched. Possible extensions are indicated.

1.1 This problem, the determination of an adequate diet of minimum cost, was one of the first studied using a linear programming model. Detailed comments on the formulation of a mathematical model may be found in [D] and in the original paper of G. J. Stigler ("The cost of subsistence," J. of Farm Econ., 27 (1945), pp. 303-314). The following is a linear programming model.

Consider n different types of foods (apples, cheese, onions, peanut butter, etc.) and m nutrients (proteins, iron, vitamin A, ascorbic acid, etc.). In the original problem of Stigler, $n = 77$ and $m = 9$. Suppose that one can determine the daily allowance of each nutrient required by an individual and the nutrient values of the foods per dollar of expenditure. (These assumptions are at best approximations and should be presented as such.) Let

a_{ij} = amount of nutrient i obtained from an expenditure of one dollar on food j,

b_i = daily requirement of nutrient i,

x_j = number of dollars spent on food j.

With these definitions the condition that the diet provide at least the daily requirement of each nutrient becomes

$$\sum_{j=1}^{n} a_{ij}x_j \geqq b_i, \quad i = 1, 2, \ldots, m.$$

The problem of finding an adequate diet of least cost is then the problem of minimizing $\sum_{j=1}^{n} x_j$ subject to the above inequalities.

1.2 Suppose an oil company has m producing wells, n refineries, and pipelines connecting certain pairs of wells and refineries. Given the output at each well, the demand at each refinery, and the cost of transporting one barrel of oil through each pipeline, determine how the production of the wells should be allocated among refineries in order to minimize transportation costs.

1.3 In 1.2 consider the case that only allocations in whole barrels are permitted. Also consider the case where supply, demand,

or other parameters are not known exactly, but instead some random behavior for each is assumed.

2. Game theory

Here are two examples involving decision making under uncertainty. The first example can be completely analyzed in terms of the elementary theory; the second cannot, but it illustrates a game that occurs frequently in the social sciences.

2.1 Two political parties compete for public favor by stating their views in n different media, labeled $1, \ldots, n$. Each party has finite resources and must distribute its expenditures among the various media without knowing the intentions of the opposing party. The payoff (a numerical measurement of the gain of one party or, equivalently, the loss of the other) resulting from use of medium i is given by a function $p(i,x,y)$ depending only on the medium and the resources x and y committed to that medium by the opposing parties. The payoff for the entire game is the sum of the payoffs in individual media. Given a knowledge of the resources and payoffs, how should each political party allocate its expenditures?

The following is a very simple model of a social situation involving conflicting interests. Models of this sort and their refinements are currently being studied by mathematically oriented social scientists. Although these models are only rough approximations to very complex situations, the results obtained from them are far from completely understood from a psychological and sociological point of view.

2.2 In an isolated and self-contained environment two retail stores compete for the local soft drink market. Each retailer handles only one brand of soda pop, different from the brand handled by the other retailer, and the two brands are identical in quality. In ordinary circumstances each retailer pays 70¢ for a carton of pop which he sells for $1. However, the soft drink distributors realize that from time to time price competition will develop, and they agree to sell their products to the retailer at 60¢ per carton provided that it is offered at retail for 80¢ per carton. Every Saturday each retailer must decide independently what his price for soda pop will be for the following week. Each has available the following information concerning demands: At the usual price they will each sell 1,000

cartons per week. If one retailer discounts while the other does not, then the discount store will sell 2,000 cartons while the store maintaining the usual price will sell only 200 cartons. If both stores sell at the discount price, then the total demand will be for 2,300 cartons and each will sell half that amount. Supposing that this decision must be made each week, how should the store managers proceed?

IX.2 GRAPH THEORY AND COMBINATORICS OPTION

This is an outline for a one-semester course designed to acquaint students with some fundamental concepts, results, and applications of graph theory and combinatorial mathematics. Only high school mathematics is required, but the student needs to be thoroughly familiar with this material. It should be kept in mind that this course represents just one of a number of (essentially equivalent) possible courses and is intended to offer the student not only specific facts and applications but also a feeling for the underlying philosophy of combinatorial mathematics.

A bibliography and an appendix follow the course outline. References to the bibliography are in brackets [ ], and references to the appendix are in braces { }. The bibliography contains references to books and other sources, together with comments about the primary references. The appendix contains examples of problems which can be treated using the ideas and methods of this course.

COURSE OUTLINE

1. Mathematical foundations of model building (4 lectures)

Real models, mathematical models, axiom systems as used in model building. (For discussion, see Section 1 of the course outline for the Optimization Option.)

2. Graph theory (18-20 lectures)

a) Basic concepts: relations, isomorphism, adjacency matrix, connectedness, trees, directed graphs, Euler and Hamiltonian circuits. (3 lectures)

In this section the student is introduced to a number of elementary (but fundamental) ideas of graph theory. He should be given

as soon as possible the opportunity to formulate and discuss various models of real situations in these terms. [BS] is an especially good source of appropriate, relatively simple examples.

This material is available from numerous sources. The presentation in [L] is suitable here; more technical treatments are given in [Harl] and [01], while that of [02] is probably too elementary. Other sources are [Bel] and {1, 2}.

b) Circuits, cutsets, spanning trees, incidence matrices, vector spaces associated with a graph, independent circuits and cutsets, orthogonality of circuit and cutset subspaces. (5-7 lectures)

The linear algebra required for this section is minimal and, if necessary, could be developed in several hours. The concepts covered here lead directly to one of the more important applications of graph theory, namely, electrical network analysis. This material is covered rather briefly in [L] with no applications, very compactly in [Bec], more completely in [BS], and comprehensively in [SR] (on which an entire course could easily be based). [SR] is also an excellent source of applications of these topics.

c) Flows in networks, max-flow min-cut theorem, Ford-Fulkerson algorithm, integrity theorem, applications (e.g., linear programming, König-Egerváry theorem, multicommodity flows, marriage theorem). (4 lectures)

An appropriate discussion of this material occurs in [L] and in selected passages of [BS]. An exhaustive treatment occurs in [FF], which is also a good source of examples and applications ({3} is typical).

This section allows for a wide selection of applications for which these techniques are appropriate. Examples of multicommodity flow problems might be given here in order to illustrate the difficulties often encountered in more complex models.

d) Planarity, Kuratowski's theorem, duality, chromatic graphs, matching theory. (6 lectures)

The concepts presented in this section allow the student to become familiar with some slightly more advanced material in graph theory. These can be used to model more complex situations, e.g., {4} and {5} (cf. [Si], [Ben]).

This material may be found in nearly all standard graph theory texts (e.g., [O1], [Bel], [Harl], or briefly in [L]). Typical applications occur in [BS]. Example {4} gives a nice application of some of these subjects (cf. [Si]). These topics are perhaps not so fundamental as the preceding and may be omitted if time pressure is a problem.

3. Combinatorial mathematics (19-22 lectures)

a) Basic tools: permutations, combinations, generating functions, partitions, binomial coefficients, recurrence relations, difference equations, inclusion-exclusion. (10-12 lectures)

The concepts introduced in this section are fundamental and should be part of every applied mathematician's stock in trade. Typical applications of this material are literally too numerous to be singled out. See, e.g., [F], [Rio], [L], [Bec], [Sa], [Kn], [Pe].

Two standard sources are the initial chapters of [F] and [Rio], but these might tax some students a bit. [L] is easier to read but says less. Crisp discussions of most of the material are given in [Ry].

b) Somewhat more advanced material. Systems of distinct representatives, Möbius inversion, theorems of Ramsey type, block designs, Hadamard matrices. (3-4 lectures)

It is important for the student to see models which use somewhat more sophisticated concepts from combinatorial mathematics. Good examples of this are the studies of the dimer problem and the Ising model presented in [Pe] and the analysis of telephone switching networks in [Ben]. The topics listed in this section serve to introduce the student to more advanced ideas. (Of course, other similar topics listed in the available references may be substituted at the discretion of the instructor.) These subjects are covered adequately (although perhaps somewhat disjointly) in [Hal]. The treatment in [Ry] would be suitable for the better students. The relevant sections of [Hal] are suitable if more emphasis on block designs is desired. Historically, block designs arose primarily in the design of statistical experiments. Recently, these concepts have been useful in a variety of fields, e.g., coding theory [Berl], spectroscopy [SFP], and data compression. (Also see {5}.)

c) Pólya counting theory: equivalence classes, (permutation) groups, cycle structure, Burnsides' theorem, Pólya's theorem, generalizations. (6 lectures)

Historically, this subject arose from Pólya's work on enumerating chemical isomers ([Po]; see also {6}). Typical applications include enumeration of Boolean functions [Sle] and enumeration of random walks on lattices [Pe]. Other examples are also available in [L], [Be2], [Rio].

[L] is appropriate here if only minimal depth is required. [Bec, Ch.5] gives a more detailed picture. The presentation of [Rio] has a reputation of being somewhat hard to read. Pólya counting theory offers students an opportunity to apply some elementary concepts from group theory to their models. Of course, several additional lectures may be needed to prepare students who have had no exposure to the concept of an equivalence relation or a group. Numerous examples and applications of this material are available, e.g., [Sle], [L], [Be2], [Rio].

It should be kept in mind that the particular choice of models and results presented is not critical. The underlying object here is to develop in the student a feeling for the formulation and analysis of various models using the ideas of combinatorial mathematics.

Many of the topics covered involve techniques for which efficient algorithms are known (e.g., network flows, matching, connectivity, and planarity). It would be quite appropriate for students to implement these algorithms on computers if facilities are available. This very effectively illustrates the savings in time and money achieved by using an efficient algorithm rather than, for example, an enumerative search.

References

In the list of references below, there is no attempt to be exhaustive. Each primary reference is accompanied by a short description and a suggestion whether it is of interest mainly to the instructor (I) or to a student in the kth year of college (S-k). References to Mathematical Reviews (MR) are given.

Primary References

[Be1] Berge, Claude. The Theory of Graphs and its Applications. New York, John Wiley and Sons, Inc., 1962, 247 p. MR21 #1608. (I; S-3,4) This is one of the original standard texts on graph theory. In addition to the standard topics, e.g., chromatic numbers, connectivity, planarity, Hamilton paths, and transport networks, this volume contains several nice chapters dealing with games on graphs and Sprague-Grundy functions. There are no exercises and a moderate selection of examples.

[BS] Busacker, Robert G. and Saaty, Thomas L. Finite Graphs and Networks: An Introduction with Applications. New York, McGraw-Hill Book Company, 1965, 294 p. MR 35 #79. (S-2) This is a nice introduction to the basic topics of graph theory, slanted somewhat toward applications to network theory. A major feature of this book is the 140-page section on applications. They are varied and interesting and include applications to economics and operations research (linear programming and PERT), combinatorial problems, games, communication networks, statistical mechanics, chemistry, genetics, human sciences, group theory, and a number of other subjects. Exercises are included.

[FF] Ford, L. R., Jr. and Fulkerson, D. R. Flows in Networks. Princeton, New Jersey, Princeton University Press, 1962, 194 p. MR 28 #2916. (I) This book, written by two of the principal developers of the field, contains the most complete treatment of network flows. A sampling of the contents includes the max-flow min-cut theorems, the König-Egerváry theorem, sets of distinct representatives, linear programming and duality, Dilworth's theorem, minimal cost flow problems, and 0-1 matrices. There are some examples but no exercises. It is probably more useful as a reference than as a text.

[Ha1] Hall, Marshall, Jr. Combinatorial Theory. Waltham, Massachusetts, Blaisdell Publishing Company, Inc., 1967, 310 p. MR 37 #80. (S-3,4) In addition to most of the standard topics, some less common subjects such as Möbius inversion and finite geometries are touched upon. By far the chief emphasis of the book is on block designs, a topic on which the author is well qualified to write. Some exercises and a few examples are contained in the book.

[L] Liu, C. L. Introduction to Combinatorial Mathematics. New York, McGraw-Hill Book Company, 1968, 393 p. MR 38 #3154. (S-1,2) This is a very well-rounded presentation of most of the basic concepts of both graph theory and combinatorial mathematics mentioned in the course outline. Numerous exercises and examples are contained in the book, although it is not particularly strong in mentioning

applications to other fields. However, this defect is offset by an extensive bibliography. The book is on the whole quite readable and could easily serve as a textbook for a freshman-sophomore course.

[Ol] Ore, Oystein. Theory of Graphs. Providence, Rhode Island, American Mathematical Society, 1962, 270 p. MR 27 #740. (I; S-4+) This is currently the most mathematical treatment of graph theory available. The subject material ranges widely and includes, e.g., product graphs, Euler paths in infinite graphs, homomorphic images of graphs, the axiom of choice, partial orders, and groups and their graphs. Unfortunately, the density of definitions is rather high (especially at the beginning), and this may discourage many readers. The patient reader will be well-rewarded for his perseverance, however. While numerous examples are included, the primary orientation of the book is toward basic concepts rather than applications of graph theory.

[Rio] Riordan, John. An Introduction to Combinatorial Analysis. New York, John Wiley and Sons, Inc., 1958, 244 p. MR 20 #3077. (I; S-3,4) This is the standard modern text on combinatorial analysis. It contains complete discussions of permutations and combinations, generating functions, inclusion and exclusion, occupancy problems, permutations with restricted positions, and, above all, enumeration (including Pólya theory). There are many exercises and examples at a variety of levels with which the reader may test his skill. An extensive list of references is also included. This is suitable as a text for upper-division and good lower-division students or as a reference for an instructor.

[Ry] Ryser, H. J. Combinatorial Mathematics, MAA Carus Monograph 14. New York, John Wiley and Sons, Inc., 1963, 154 p. MR 27 #51. (I; S) This short book, already considered by many to be a classic in its field, provides the reader with an accurate (although necessarily abbreviated) introduction to some fundamental ideas in combinatorics. The subjects include permutations and combinations, inclusion and exclusion, recurrences, Ramsey's theorem and applications, systems of distinct representatives, 0-1 matrices, orthogonal Latin squares and the Bruck-Ryser theorem, block designs, and perfect difference sets. Each chapter concludes with a number of references. There are few examples and no exercises. The initial sections of the book could be read by freshmen; the latter material would be more suitable for a good junior or senior.

Additional References

Avondo-Bodino, G. Economic Applications of the Theory of Graphs. New York, Gordon and Breach, Science Publishers, Inc., 1962.

[Bec] Beckenbach, E. F., ed. Applied Combinatorial Mathematics. New York, John Wiley and Sons, Inc., 1964. (I; S-3,4) A formidable collection of chapters written by a number of experts in combinatorial fields. Especially good is the section by deBruijn on counting theory.

[Ben] Beneš, V. E. Mathematical Theory of Connecting Networks and Telephone Traffic. New York, Academic Press, Inc., 1965.

[Be2] Berge, Claude. Principles of Combinatorics. New York, Academic Press, Inc., 1971.

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[Ber1] Berlekamp, E. R. Algebraic Coding Theory. New York, McGraw-Hill Book Company, 1968.

Combinatorial Structures and Their Applications. Proceedings of Calgary International Conference on Combinatorial Structures and Their Applications. New York, Gordon and Breach, Science Publishers, Inc., 1970.

[ER] Erdös, P. and Rényi, A. "On random graphs I." Publicationes Mathematicae, 6 (1959), pp. 290-297.

[F] Feller, W. An Introduction to Probability Theory and Its Applications, vol. 1, 3rd ed. New York, John Wiley and Sons, Inc., 1968.

[GP] Gilbert, E. N. and Pollak, H. O. "Steiner minimal trees." SIAM Journal, 16 (1968), pp. 1-29.

Guillemin, E. A. Introductory Circuit Theory. New York, John Wiley and Sons, Inc., 1953.

[Har1] Harary, F. Graph Theory. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1969. A coherent presentation of the basic results in what could be called "exact" graph theory (as opposed to the asymptotic graph-theoretic results of Erdös and others, cf. [ER]).

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Harris, B. ed. Graph Theory and Its Applications. New York, Academic Press, Inc., 1970.

Harrison, M. A. Introduction to Switching and Automata Theory. New York, McGraw-Hill Book Company, 1965.

Kattsoff, L. O. and Simone, A. J. Finite Mathematics with Applications in the Social and Management Sciences. New York, McGraw-Hill Book Company, 1965.

Kemeny, J. G.; Snell, J. L.; Thompson, G. L. Introduction to Finite Mathematics, 2nd ed. Englewood Cliffs, New Jersey, Prentice-Hall, Inc., 1966.

[KL] Kernighan, B. and Lin, S. "A heuristic algorithm for the traveling salesman problem." Operations Research, to appear.

[Kn] Knuth, D. E. The Art of Computer Programming. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., Vol. I (1968), Vol. II (1969), Vol. III (1971). The first two volumes of this (intended) seven-volume series are literally overflowing with a wealth of results, applications, references, and problems having to do with computational (and combinatorial) algorithms (among other things).

König, D. Theorie der endlichen und unendlichen Graphen. New York, Chelsea Publishing Company, 1950.

[Kr] Kruskal, J. B., Jr. "On the shortest spanning subtree of a graph and the traveling salesman problem." Proceedings of the American Mathematical Society, 7 (1956), pp. 48-50.

[Li] Lin, S. "Computer solutions of the traveling salesman problem." Bell System Technical Journal, 44 (1965), pp. 2245-2269.

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MacMahon, P. A. Combinatorial Analysis, vols. I and II reprinted in one volume (originally published in two volumes in 1915 and 1916). New York, Chelsea Publishing Company, 1960. This is the old classical treatise on combinatorial analysis.

Marcus M. A Survey of Finite Mathematics. Boston, Massachusetts, Houghton Mifflin Company, 1969.

Milne-Thomson, L. M. The Calculus of Finite Differences. New York, St. Martin's Press, Inc., 1951.

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Moon, J. Topics on Tournaments. New York, Holt, Rinehart and Winston, Inc., 1968.

Noble, B., ed. Applications of Undergraduate Mathematics in Engineering. New York, The Macmillan Company, 1967 (especially Ch. 2 and Sect. 4.6).

International Conference on Combinatorial Mathematics, proceedings. New York Academy of Sciences, July 31, 1970.

Recent Trends in Graph Theory. Proceedings of the First New York City Graph Theory Conference. Jamaica, New York, St. John's University, June, 1970.

[O2] Ore, Oystein. Graphs and Their Uses. New York, Random House, Inc., 1963. A good reference for graph theory as applied to electrical network theory.

[Pe] Percus, J. K. Combinatorial Methods. New York University, Courant Institute of Mathematical Sciences, 1969. An excellent reference for the application of combinatorial techniques to some of the classical lattice statistics problems of physics, e.g., the dimer problem and the Ising model.

[Po] Pólya, G. "Sur le nombre des isomètres de certains composes chimique." C. R. Acad. Sciences, 202 (1936), p. 1554.

Richardson, W. H. Finite Mathematics. New York, Harper and Row, Publishers, 1968.

Rota, G.-C. "On the foundations of combinatorial theory, I." Theory of Möbius Functions. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 2 (1964), pp. 340-368.

[Sa] Saaty, T. L. Optimization in Integers and Related Extremal Problems. New York, McGraw-Hill Book Company, 1970. A recent book which contains a wide variety of interesting results and examples from optimization theory, many of which are well in the spirit of this course.

[SR] Seshu, S. and Reed, M. B. Linear Graphs and Electrical Networks. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1961.

[Si] Sinden, F. W. "Topology of thin film RC circuits." Bell Systems Technical Journal, 45 (1966), pp. 1639-1662.

[Sle] Slepian, D. "On the number of symmetry types of Boolean functions of n variables." Can. J. Math., 5 (1953), pp. 185-193.

[SFP] Sloane, N. J. A.; Fine, Terrence; Phillips, P. G. "New methods for grating spectrometers." Optical Spectra, 4 (1970), pp. 50-53.

Whitworth, W. A. Choice and Chance, reprint of the 5th ed. (1901). New York, Hafner Publishing Company, Inc., 1965.

Appendix

1. Organization X has offices located in a number of cities. It wishes to establish a communication network among all its locations so that any two offices may communicate with one another, possibly by going through some of the other locations. Furthermore, it is desired to minimize the total length (cost) of the network. How should the cities be connected? If one is allowed to locate switching junctions arbitrarily rather than just at the office locations, then how can a minimal network be obtained? (See [Kr] and [GP].)

2. A (traveling) salesman has a fixed set of locations (farm houses) that he is required to visit. He leaves from his home office, travels to each location once in some order, and then returns. In what order should he visit the locations in order to minimize his total distance, cost, time (energy)? (See [Li] and [KL].)

3. An oil company has a number of oil wells (sources) and a number of refineries, customers, etc. (sinks), all connected by some intricate network of pipelines. The portions of pipeline between various points of the network have different (known) capacities. How can one route the oil through the system in order to maximize the flow of oil to the sinks? What if the direction of flow in certain pipelines is restricted? What if there are several grades of oil available in varying amounts from the sources and it is desired to maximize the value of the mixture received at the sinks? (See [FF].)

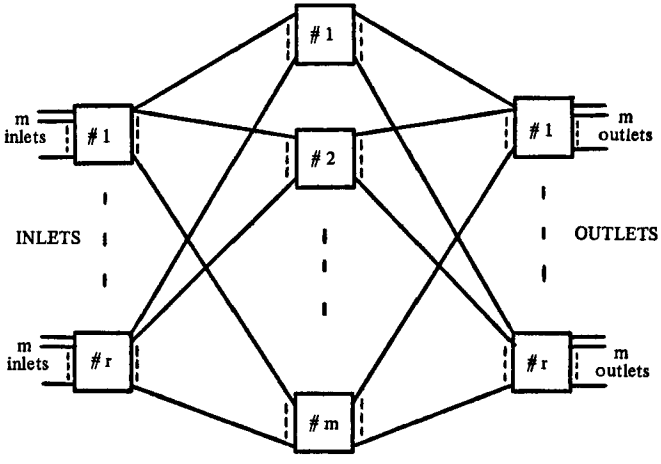
4. Certain integrated circuits can be made by depositing very thin metallic and dielectric films in suitable patterns on an insulating substrate. Ordinarily printed circuits are strictly planar; crossovers are made only by leading one of the conductors entirely out of the plane of the circuit. In the thin film technique, however, conductors can be separated by thin insulating layers within the plane of the circuit, causing a nonzero capacitance between the crossing conductors. Thus, crossovers can be permitted, provided this nonzero capacitance between the crossing conductors is permitted. The general problem is to determine which circuits can be realized by some suitable thin film circuit. This leads to a number of interesting questions in graph theory, one of which is the following: Given a set $S = \{s_1,\ldots,s_n\}$ of arcs or "strings," what are necessary and sufficient conditions on a set P of pairs $\{s_i,s_j\}$ so that there is some configuration of the s_k <u>in the plane</u> for which s_i and s_j intersect if and only if $\{s_i,s_j\}$ belongs to P? (See [Si].)

5. The Hall theorem on systems of distinct representatives occurs in a variety of applications. Several of these are:

a) In a certain company, n employees are available to fill n positions, each employee being qualified to fill one or more of these jobs. When can each employee be assigned to a job for which he is qualified? (See [Bel].)

b. An $m \times n$ chessboard has a certain subset of its squares cut out. When is it possible to place a collection of 2×1 "dominoes" on this board so that each of its squares is covered exactly once? (See [Pe].)

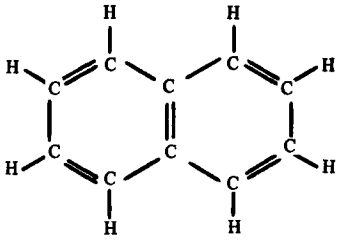
c) A telephone switching network connecting mr inlets with mr outlets is made up of three stages as indicated in the figure. (See [Ben].)



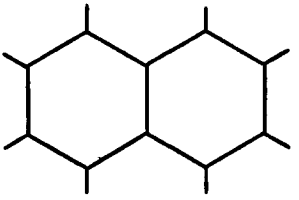
Each square box represents a switching unit for which any of the possible permutations of connecting its local inlets to its local outlets is possible. The problem is to show that this network is <u>rearrangeable</u>, i.e., given any set of calls in progress and any pair of idle terminals, the existing calls can be reassigned new routes (if necessary) so as to make it possible to connect the idle pair. How is the reassignment made so as to change the minimum number of existing calls? (cf. [Ben].)

d) If there are as many r-element subsets of an n-element set as there are k-element subsets, then it is possible to associate with each k-element subset a distinct r-element subset which contains it. How?

6. A naphthalene molecule $C_{10}H_8$ (See figure on next page.)



contains 8 hydrogen atoms which are available for substitution. The symmetry group G of the underlying figure



has order 4 and consists of the identity and three rotations about axes which are horizontal, vertical, or coming out of the page. Think of this group as just permuting the eight hydrogen atoms. The identity fixes them all and has cycle index S_1^8; each of the other three permutations moves them in four pairs of two each and contributes to the cycle index S_2^4. The cycle index of the group (considered as acting only on the hydrogen atoms) is thus

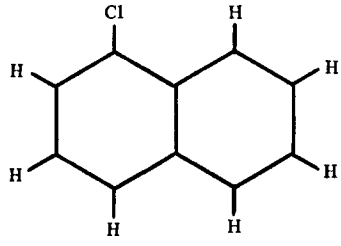
$$P_G(S_1,S_2) = \frac{1}{4}\,(S_1^8 + 3S_2^4).$$

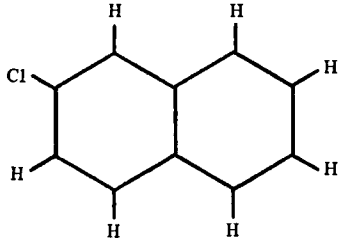
Now suppose we replace k of the hydrogen atoms by chlorine atoms and r of the hydrogen atoms by bromine atoms. How many different molecules can be formed? This is exactly the kind of question that Pólya's theorem answers.

Answer: In $P_G(S_1,S_2)$ replace S_1 by $1 + x + y$ and S_2 by $1 + x^2 + y^2$. Then the coefficient of $x^k y^r$ is the desired number. In fact, after making the substitution, we have

$$\frac{1}{4}\left[(1 + x + y)^8 + 3(1 + x^2 + y^2)^4\right] = 1 + 2x + 2y + 10x^2 + 14xy + 10y^2 + \ldots .$$

Each term in this series can be interpreted: 1 corresponds to the original molecule (no substitution); 2x corresponds to





(substituting one Cl for an H), etc. (For the notation used here, see [L] and [Bec]; problems of this type occur in [Pe].)

IX.3 FLUID MECHANICS OPTION

The course described below is based on a view of applied mathematics as a natural science distinguished from other natural sciences by a mathematical content that is significant in its own right. Fluid mechanics was chosen because it exemplifies applied mathematics in this sense: it is important historically, it encompasses many interesting physical problems, and it can be taught in the spirit of this report. However, to teach such a course at the undergraduate level requires special care in order to avoid the two possible extremes of, on the one hand, pursuing mathematical topics for their own sake and, on the other hand, studying physical models which involve only trivial mathematical ideas.

The approach to the subject proposed here has been selected with the audience and our objectives in view. Although this material can be taught from a more modern perspective, it would then require more sophisticated mathematical techniques and would be feasible only with very well prepared undergraduates. Our approach was selected because we feel that it is accessible to a wide audience and because it effectively attains our goals.

The course is intended for seniors. Prerequisites are elementary courses in calculus, differential equations, linear algebra, and physics. A course in advanced calculus or analysis is desirable. The student should be familiar, for instance, with the mathematical issues involved in the termwise integration of infinite series. This course should be valuable in solidifying and extending the student's grasp of areas of analysis and differential equations. The course outlined here does <u>not</u> assume prior knowledge of complex analysis, partial differential equations, or fluid mechanics.

A potential instructor of this course is faced with issues not present in the preceding outlines. It requires more specialized knowledge and would most easily be offered by someone with a background in applied mathematics. Nevertheless, we feel that the present outline is sufficiently detailed so that it can serve as a guide to instructors and so that it can encourage teachers to experiment with courses in this area. The main point which the instructor must keep in mind is that this is to be a course about applied mathematics using fluid mechanics as its representative element; it is not a course on fluid mechanics alone.

The main needs of the instructor, in addition to mathematics, are a basic knowledge of classical physics, a willingness to read, and perhaps above all an interest in nature. Those who are not specialists in fluid mechanics will find it particularly important to read this outline with one of the references at hand. While there are many books on fluid mechanics, there are very few which emphasize the point of view which the Panel has taken here. A list of books which may be helpful to the teacher is given in section 5 below, with brief comments. References to specific sections of some of these

books are given for each of the topics in the detailed outline in section 4. Unfortunately, there seems to be no book which would be completely satisfactory as a text for this kind of a course; the book by Prandtl, which includes most of the topics in the outline discussed clearly from the physical point of view, is perhaps the most appropriate. But the mathematical side of many of these discussions will require appreciable expansion for the purposes of this course; for digressions of a more purely mathematical nature which will from time to time be appropriate, one can perhaps rely on the general mathematical background of the instructor.

COURSE OUTLINE

This course has two main parts, the first of a fairly general nature concerned with the mathematical formulation of continuum models for fluids and the second dealing with more specific problems illustrative of the more important simplified models. In the outline each part is broken down into several areas, for each of which some remarks in the style of a "catalog description," with some suggestions on the general approach, are given. These remarks are followed by a list of topics for each of several lectures on this area, with attention drawn to specific sections of the references in which a treatment of these topics is given. For definiteness, these specific references have been restricted mainly to the books of Prandtl and Yih. The format of the references is indicated by this example: [P] II:1.1 means section 1.1 in the second chapter of Prandtl's book.

1. Continuum models for fluids

This part of the course concerns primarily the formulation and basic properties of the principal mathematical models used in fluid mechanics. Here one can well emphasize the central role of model building in applied mathematics and the importance of models which are both mathematically self-consistent and capable of being critically compared with the experimental or observational facts which they are supposed to describe. Fluid mechanics is a particularly good example to illustrate that a mathematical model can be very helpful even though it is in a sense definitely incorrect (e.g., the molecular structure of matter is completely missing from continuum models) and that in reality all theoretical science is done in terms of models, none of which should be assigned any absolute validity.

a) The concept of continuous matter as a useful macroscopic model of real matter. (2 lectures) Mass and density. Kinematics: velocity field and the idea of a "fluid particle" as a theoretical concept in the continuum model, not the same thing at all as a mole-

cule. "Eulerian" and "Lagrangian" variables and the mathematical form of the continuum model. The continuity equation.

Mathematical ideas: flow as a continuous mapping, Jacobians in the transformation of multiple integrals, the divergence theorem. One might well emphasize here the reverse of the familiar physical "proof" of the divergence theorem--the mathematical theorem shows that the continuum model is in accord with our intuitive ideas about the continuity of matter.

References. Continuum models for fluids; mass, density, "fluid particle," velocity, acceleration: [Y] I:1,2,3,4 and [P] II:1.1. Continuity equation, divergence theorem, and Jacobians: [Y] I:5,6 and [P] II:1.2. More kinematics: [Y] I:7,8,9.

b) Dynamics. (4 lectures) Introduction to the basic ideas from particle mechanics (momentum, force, kinetic energy) into the continuum model. Pressure and stress. Stress tensor and the momentum equation. Mechanical energy equation. Angular momentum and symmetry of the stress tensor in the absence of body torques and "torque-stresses."

Mathematical ideas: divergence theorem again, with more vector calculus. Tensors as geometric objects. Components of a symmetric second-order tensor form a symmetric matrix, hence have real eigenvalues and an orthonormal basis of eigenvectors (principal stresses).

References. Pressure and stress: [P] I:1,2,3. Concept of a tensor as a geometric object and its representation by components; stress tensor of a continuum: [Y] I:10. Yih's rather classical and formalistic view of tensors might well be given a more geometrical and contemporary flavor. Symmetrical second-order tensors, in particular stress and deformation tensors, relationship between a second-order tensor and the matrix of its components; application of linear algebra; principal stresses and eigenvalues, principal directions, etc.: [Y] I:11,12. Navier-Stokes form of the stress tensor and the corresponding fluid equations of motion: [Y] I:13 and [Y] II:1,2. Conservative body forces and the mechanical energy equation, vorticity equation: [Y] II:3,4.

c) Thermodynamics. (3 lectures) The equation of state. Internal energy, heat, and entropy. Heat conduction and the total

energy equation.

In the absence of sufficient background in physics, this part may have to be limited mainly to equations of state in the simplest cases: incompressible fluids and the isothermal and adiabatic ideal gas. However, thermodynamics, where accessible, provides a good source of exercises in changing variables, Jacobians, etc., and also often illustrates rather well the advantages of a careful mathematical formulation over a loose intuitive description.

At this point various examples of hydrostatics problems can conveniently be introduced. Two important points to be emphasized here (and throughout the course in other contexts) are: i) Hydrostatics is a "simplified model," relevant not only when there is strictly no motion but also a good approximation in appropriate circumstances (vertical accelerations small compared with that of gravity). One can introduce here the idea of simplifying the model on the basis of the smallness of certain dimensionless parameters characteristic of the particular case in hand. ii) By discussing some problems related to familiar situations, one can help the student to form the habit of using mathematics to enhance his perception of nature. For example, the hydrostatics of the isothermal and adiabatic atmospheres can answer questions like: Is it plausible that oxygen should be needed when climbing Mt. Everest? or How much colder is it likely to be on the top of some local peak than it is at ground level?

Mathematical ideas: in addition to Jacobians, etc., some simple ordinary differential equations.

References. Review of thermodynamics: [Y] appendix I, possibly truncated and treated in a mathematically more sophisticated manner. Ideal gases: [Y] appendix I and [P] I:5. Heat or energy equation: [Y] II:8. Applications from hydrostatics: [P] I:6,7,10.

2. The more simplified models

Geometrical or physical parameters needed to specify a problem completely lead to characteristic dimensionless parameters (e.g., Mach number, Reynolds number) whose smallness or largeness in particular cases indicate the usefulness of simplified models (e.g.,

incompressible or inviscid flow). In the discussion of simplified models, emphasis should be shared between their general properties (e.g., Kelvin's circulation theorem) and careful consideration of the extent to which the simplified model is in fact relevant. In particular, the prevalence of nonuniform convergence in going over to the simplified model and the kinds of additional considerations required in the regions of nonuniformity ("boundary layers") should be brought out, at least qualitatively. In assessing relevance, it is probably best to include with the general discussion a number of applications of the basic models to concrete situations. Simple and familiar cases which emphasize the two points mentioned under 1c) in connection with hydrostatics should be considered where possible.

a) Ideal irrotational flow and surface waves on water. (5 lectures)

Here there are a number of opportunities for introducing important mathematical ideas and techniques, for instance: i) some general properties of harmonic functions; ii) solution of boundary value problems for Laplace's equation by superposition of wave solutions (i.e., "separation of variables" or use of Fourier representations); iii) free waves--phase and group velocity; iv) forced waves, e.g., the linear wave-maker problem (radiation condition at infinity, Sturm-Liouville equations, and eigenfunction expansions for boundary value problems).

If the students have not seen a proof of the Fourier series theorem, the instructor might like to insert a lecture on this topic, proving the theorem for piecewise continuously differentiable periodic functions.

References. General properties of the inviscid flow model, dimensionless form of general equations, and inviscid flow as idealization for large Reynolds number: [Y] II:5,6. Kelvin circulation theorem: [Y] III:1,2 and [P] II:2.8. Bernoulli theorems: [Y] III:8,9 and [P] II:2.3,4. Ideal irrotational flow: [Y] IV:1,2,3,4 and [P] II:2.9. Surface waves: [P] II:2.15 and [Y] V:1,2.1,2.2,2.3. Standing waves and group velocity: [Y] V:2.4,2.5,2.6,2.11. The wave-maker problem: [Y] V:2.13.

b) Linear shallow-water theory. (4 lectures)

This provides another simplified model and gives opportunity for further discussion of Sturm-Liouville eigenvalue problems. The relationship with variational techniques can be brought out here in the estimation with trial functions or comparison theorems of the resonant frequencies of soup bowls, swimming pools, harbors, and lakes.

Some properties of the wave equation, for instance the significance of characteristics, can also be included. (Nonlinear shallow-water theory, its analogy with compressible flow and shock waves might be discussed, but probably there will not be sufficient time for this.)

References. Linear shallow-water theory: [L] pp. 169-72, 189. Forced motion and normal modes: [L] pp. 176-79. Tides: [L] pp. 180-82, 198-200. Variable depth or section: [L] pp. 185-86, 191-93.

c) Ideal flow past bodies. (5 lectures)

Flow past circles and spheres gives simple problems in potential theory which can be tied in with Fourier series and spherical harmonics, notably by considering flow past near-circles or near-spheres; ideas of regular perturbation theory enter here as well.

D'Alembert's "paradox" provides a striking example of the failure of a simplified model when interpreted too literally, combined with its rescue and continued usefulness when the main source of the difficulty (flow separation) is identified and appropriately modeled. The elementary theory of airfoils and drag estimates via dynamic pressure arguments could be discussed with questions like: Why do sailplanes have very long slender wings? How big should a parachute be? How much air resistance is a car subject to?

References. Examples: [Y] IV:7.4, [P] II:2.9, and [Y] IV:18. Flow past a near-sphere: [Y] IV:13, possibly generalized and with further discussion of spherical harmonics. (Yih's discussion is perhaps too brief and formalistic, and the fact that surface harmonics are to spheres what sines and cosines are to circles is rather obscured.) Perhaps another mathematical digression could be added here: students are too often so put off by excessive emphasis on associated Legendre functions that they never seem to realize that the rotation group is behind it all. Two-dimensional flows with

circulation: [P] II:10. Blasius and Kutta-Joukowski theorems: [Y] IV:22,22.1. Airfoils: [P] III:16,17.

d) Inviscid flow with vorticity. (3 lectures)

Some interesting phenomena of this sort can be studied without too much complication by considering linearized flow in rotating systems. Unfortunately, a more complete picture of the applications of hydrodynamic theory in meteorology and oceanography probably involves too many other considerations to be feasible in this course.

References. Geostropic flow and the Taylor-Proudman theorem: [G] I and [Y] III:12,12.1,12.2. Effect of the earth's rotation on atmospheric and oceanic flows: [P] V:8. Motion of parallel rectilinear vortex filaments: [L] pp. 154-55.

e) Viscosity. (4 lectures)

References. Couette and Poiseuille flow: [Y] VII:2.1,2.2,2.3, 2.10,2.11 and [P] III:1.9b. Ekman and Stokes-Rayleigh boundary layers: [Y] VII:2.12,3.4; [P] V:9; and [G] 2.3. Secondary flow: [P] III:8, [G] 2.4, and [P] V:9. The boundary layer: [P] III:3 and [Y] VII:5,6.

f) Instabilities. (2 lectures)

(Why does water run out of an inverted glass even though the atmospheric pressure can support the weight of a 30-foot column of water--and why does it not similarly run out of a narrow tube?) Kelvin-Helmholtz instability, although an over-simplified model, can be related to wave generation by wind.

References. Gravitational instability: [Y] IX:2.1,2.3 and [P] V:16. Kelvin-Helmholtz instability: [Y] IX:8.1, [L] p. 232, [Y] IX:6.1.

References

The books referred to in the outline are:

[G] Greenspan, H. P. The Theory of Rotating Fluids. New York, Cambridge University Press, 1968. A rather advanced text or research monograph.

[L] Lamb, H. Hydrodynamics. New York, Dover Publications, Inc., 1945 (reprint). The classic work in the field. Its rather old-fashioned mathematical style and extensive character,

combined with a certain tendency to present important results without adequate identification, make it sometimes rather difficult for the novice. As with some other classics, results given almost parenthetically in Lamb continue occasionally to be rediscovered (and published!).

[P] Prandtl, L. Essentials of Fluid Dynamics. New York, Hafner Publishing Company, 1952. An excellent and interesting book from the physical point of view, with clear discussions of many scientific and engineering applications. Most of the less elementary mathematical aspects, however, have (intentionally) been left aside.

[Y] Yih, Chia-Shun. Fluid Mechanics. New York, McGraw-Hill Book Company, 1969. A good graduate-level textbook, emphasizing the theoretical side. Quite a few exercises.

Some other books which the instructor may find useful to have on hand are listed below.

Additional References

Batchelor, G. K. An Introduction to Fluid Mechanics. New York, Cambridge University Press, 1967.

Landau, L. D. and Lifschitz, E. Fluid Mechanics. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1959.

Rouse, H. Fluid Mechanics for Hydraulic Engineers. New York, McGraw-Hill Book Company, 1938.

Von Arx, W. S. Introduction to Physical Oceanography. Reading, Massachusetts, Addison-Wesley Publishing Company, Inc., 1962.

The books by Batchelor and by Landau and Lifschitz are both good; Landau and Lifschitz is written perhaps more from the physicist's point of view, Batchelor from the applied mathematician's.

Also, a good engineering text such as the book by Rouse and, in connection with 2d), the book by Von Arx, may be found helpful.

There are a number of interesting 8mm. film strips on topics in fluid mechanics, as well as some longer films, prepared by the National Committee for Fluid Mechanics Films and available from Encyclopaedia Brittanica Films. They do not on the whole contribute much on the mathematical side but may well add interest and appreciation for the physics. Some which might be found useful in connection with the course outlined above are:

FM-3: Shear Deformation of Viscous Fluids [continuity equation]

FM-14A and B: Visualization of Vorticity with Vorticity Meter [continuity equation, conservative body forces and the mechanical energy equation, vorticity equation]

FM-13: The Bathtub Vortex [general properties of the inviscid flow model, two-dimensional flows with circulation, effect of the earth's rotation on atmospheric and oceanic flows]

FM-10: Generation of Circulation and Lift for an Airfoil [airfoils]

FM-11: The Magnus Effect [secondary flow]

FM-6: Boundary Layer Formation [the boundary layer, Ekman and Stokes-Rayleigh boundary layers]

FM-31: Instabilities in Circular Couette Flow [instabilities, Couette and Poiseuille flows]