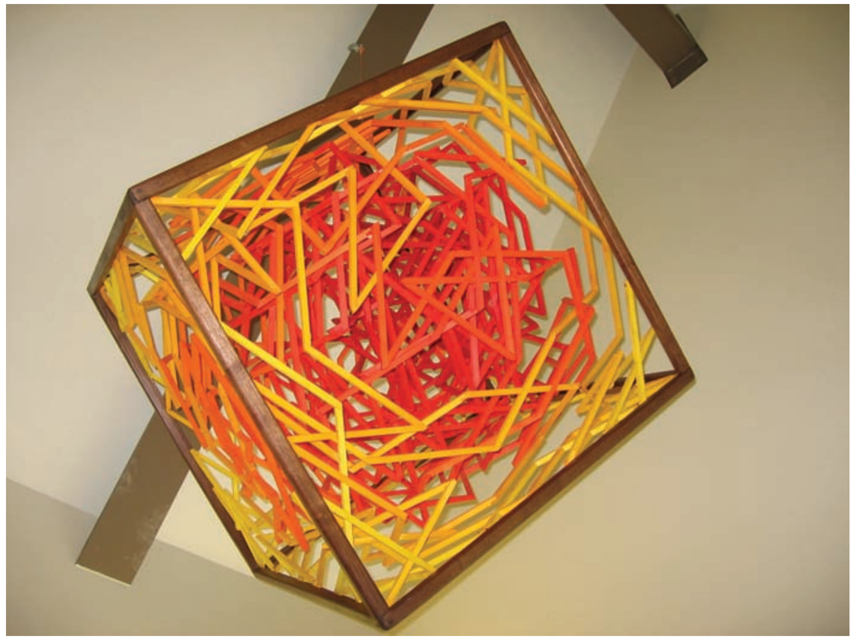
A Knight's Tour de Force

Adam H. Berliner

Built in the Norman Gothic style, the ivy-covered, limestone-sided Regents Hall of Mathematical Sciences stands quietly on the campus of St. Olaf College in Minnesota. The inside of the building, however, is modern, with newly remodeled offices, study spaces, computer labs, and high-tech classrooms. The top floor, with a pitched ceiling and gabled windows overlooking the campus and the surrounding countryside, is a large, open space where students gather to study, socialize, and relax.

Aiming to draw attention to mathematics past and present, Professor Emeritus of Mathematics Loren Larson decided this space called for a special piece of mathematical art: a three-dimensional knight's tour. Now, several months after the grand opening of the building, the finished piece, titled *Synergy*, hangs ever so delicately from the ceiling. The structure is remarkably complex, but the interconnections among individual pieces seem to work together to form a coherent whole. Many who wander by are mystified: what does this colorful display of carefully interwoven wooden sticks represent?



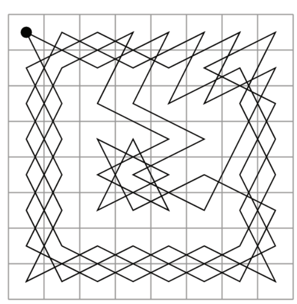
"Synergy" by Loren Larsen, on display at St. Olaf College.

When his student constructed a model of a three-dimensional knight's tour using hail screen and yarn, Larson's idea to construct a larger-scale wooden version was born.

A Brief Tour of Knight's Tours

On an ordinary 8×8 chessboard, a knight moves two squares horizontally or vertically and one square in the perpendicular direction. The number of possible knight moves from a given square—eight at most—depends on the location of the square within the chessboard. *A knight's tour* is a sequence of 64 knight moves that visits each square exactly once. A knight's tour that begins and ends at the same square is called *closed*. Figure 1 shows a closed knight's tour on a standard chessboard; squares are numbered in the order they are visited.

Why would a knight's tour grace the halls of a mathematics building? One of the first papers analyzing knight's tours was by Euler, in 1759. Euler presented several knight's tours, some closed, on the standard 8×8 chessboard. "Although the number of these routes is not infinite, it will always be so great that one could never exhaust it," Euler wrote. Indeed, the problem of counting closed knight's tours was solved only



1	52	13	34	3	54	11	24
14	33	2	53	12	23	56	9
51	64	35	4	55	10	25	22
32	15	42	39	36	5	8	57
63	50	37	6	41	44	21	26
16	31	40	43	38	7	58	45
49	62	29	18	47	60	27	20
30	17	48	61	28	19	46	59

Figure 1. A closed knight's tour on the 8×8 chessboard.

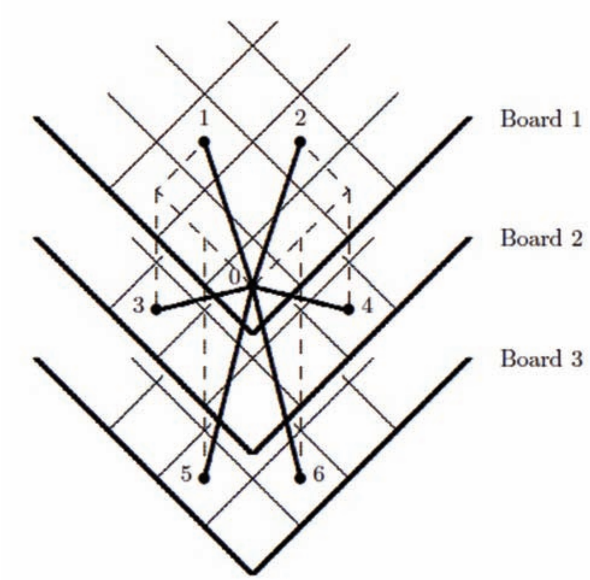


Figure 2. The six possible three-dimensional moves from a corner of an $8 \times 8 \times 8$ board.

in 1997, when Brendan McKay computed the number as 13,267,364,410,532. McKay's result corrected a 1996 paper, which had overstated the total at 33,439,123,484,294.

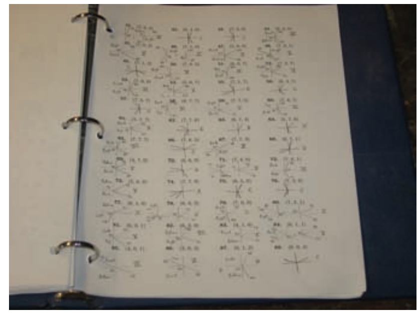
Since Euler, several algorithms for finding closed knight's tours have been proposed, including one described in 1823 by H. C. von Warnsdorff. At each stage, the knight moves to the available square that has the fewest possible *following* moves. Ties are broken by randomly choosing one of the eligible squares. While Warnsdorff's algorithm is not guaranteed to produce a knight's tour, the result is usually close enough so that a slight modification will create a tour.

Warnsdorff's algorithm can also be applied to find knight's tours of nonstandard chessboards, including some that are not square. It is not hard to show that closed knight's tours do not exist for boards whose dimensions are $1 \times n$ and $2 \times n$. As a fun exercise, one can also show that no closed knight's tours exist for boards of dimension 3×6, 3×8, and $4 \times n$. In 1991 Allen Schwenk showed that as long as one of the dimensions of the board is even, every other $m \times n$ chessboard allows a closed knight's tour.

Constructing a Knight-mare

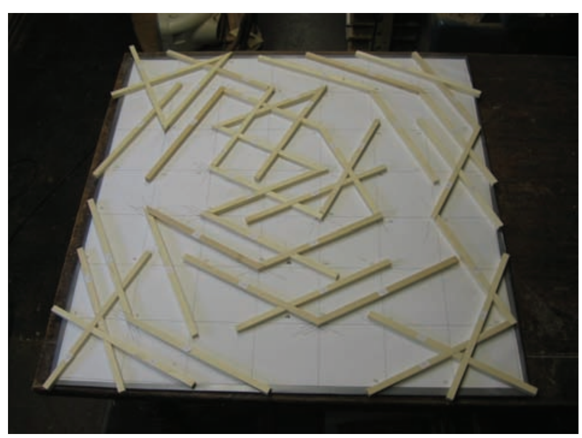
Larson wanted his creation to be big and bold, but intricate and precise: he would construct a giant *three-*dimensional closed knight's tour of an $8 \times 8 \times 8$ chessboard. Three-dimensional knight's moves are much like those in two dimensions—a knight moves two squares in any possible direction and one square in any perpendicular direction. The number of three-dimensional moves a knight can make is higher than the number of two-dimensional moves. For example, if the knight starts in a corner, there are only two possible two-dimensional moves, but six possible three-dimensional moves (see figure 2). The maximum number of three-dimensional moves a knight can make from any given square is 24.

Larson's interest in three-dimensional knight's tours dates back to January 1975, when he offered a course on chess and mathematics. In one student project, Noreen Herzfeld found a closed knight's tour on a three-dimensional $8 \times 8 \times 8$ chessboard by writing a computer program using Warnsdorff's algorithm. When Herzfeld constructed a model of this tour using heavy-duty window screen and yarn, Larson's idea to construct a larger-scale wooden version was born. Herzfeld, now a professor of theology and computer science at St. John's University in Collegeville, Minnesota, has become a prominent author on technology and religion. With Herzfeld nearby as a reminder and retirement successfully under way, Larson could finally implement his plan.



Cataloguing the 512 knight moves.

Larson knew how complex and difficult the venture would be. Constructing a closed, 512-move knight's tour made the scale of the project daunting. Since each knight's move lies in a plane perpendicular to one of the coordinate axes, the construction would involve 24 different grids, 8 for each of the 3 axis directions. Each knight's move would be represented by a wooden stick, and the sticks joined end-to-end, in order. In addition to showing the complexities of a three-dimensional tour, Larson wanted to highlight



The complicated crossings required each stick to be uniquely carved and notched.

Warnsdorff's algorithm by displaying the order in which the 512 moves were made. He therefore decided to dye each stick a slightly different color, progressing gradually from yellow to maroon, with shades of orange, apricot, and crimson in between. Larson used soft pine, a wood that easily takes on bright, vivid colors.

Because the completed piece would hang on a cable, overall stability was a crucial consideration. Because the 512 sticks needed to cross each other repeatedly, Larson had to cut differently shaped notches into almost every stick. To ensure that the pieces fit properly, he used vectors representing the directions of each move and dot products to calculate the appropriate angles. Cutting accurate angles at the end of each stick was too complicated for Larson's machines, so he was forced to finish carving each element by hand. Each of the 512 sticks took about an hour to craft, Larson estimates, and some were trickier than others.

Once the sticks were cut and sanded, the coloring process began. Helped by his grandson Jakob, Larson dyed each stick separately. Then came the arduous task of gluing everything together. "We began at the top, using gravity to work with us rather than against us, by tying the top layer of sticks to a screen with pipe cleaners," Larson recalls. "From there, we worked down through the center, hanging sticks from the top by cords or propping them up from the bottom, and then out towards the faces."

Loren Larsen and his grandson Jakob working on the final stages.

The final steps were to construct a walnut frame and to enlist some friendly St. Olaf professors to help carry the structure up the steep hill from Larson's workshop to its final destination.

Although the finished sculpture is beautiful in its own right, Larson sees the piece as symbolic of the neural connections implicit in mathematical problem solving. The orange and yellow sticks represent understanding of the problem and possible approaches. Gradually the focus narrows toward the central ideas and converges to the heart of the problem, represented by a spherical tangle of red sticks in the middle.

Gazing up at the finished creation, former *Math Horizons* editor Deanna Haunsperger called it a "Knight-mare." Larson estimates the construction process consumed around 1,000 hours of labor. Still, Larson recalls, the project was anything but tedious. Momentary frustrations and technical challenges aside, "the variability in the construction of the pieces kept the project from feeling like production work." Was the project worth all the effort? Larson's response is simple: "As in much of mathematics, the sustaining motivation was artistic—the satisfaction gained from the problem solved, the piece completed." ■

Shading the sticks to reflect the order of the moves.

Further Reading

Euler's early work on knight's tours can found in "Solution d'une curieuse que ne paroit soumise à aucune analyse," which is part of *Mémoires de l'Académie Royale des Sciences et Belles Lettres de Berlin, Année 1759*, **15** (1766). Martin Loebbing and Ingo Wegener's original (and incorrect) estimate of the number of knight's tours on a standard chessboard appeared in a 1996 article in the *Electronic Journal of Combinatorics*, and Brendan McKay published his correction in a technical report on the computer science homepage at the Australian National University. A. J. Schwenk's article "Which Rectangular Chessboards Have a Knight's Tour?" was published in *Mathematics Magazine* **64** (1991).

About the author: *Adam Berliner is an assistant professor of mathematics at St. Olaf College. He studies combinatorics and graph theory, and advises St. Olaf's Problem Solving Club.*

email: `berliner@stolaf.edu`

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