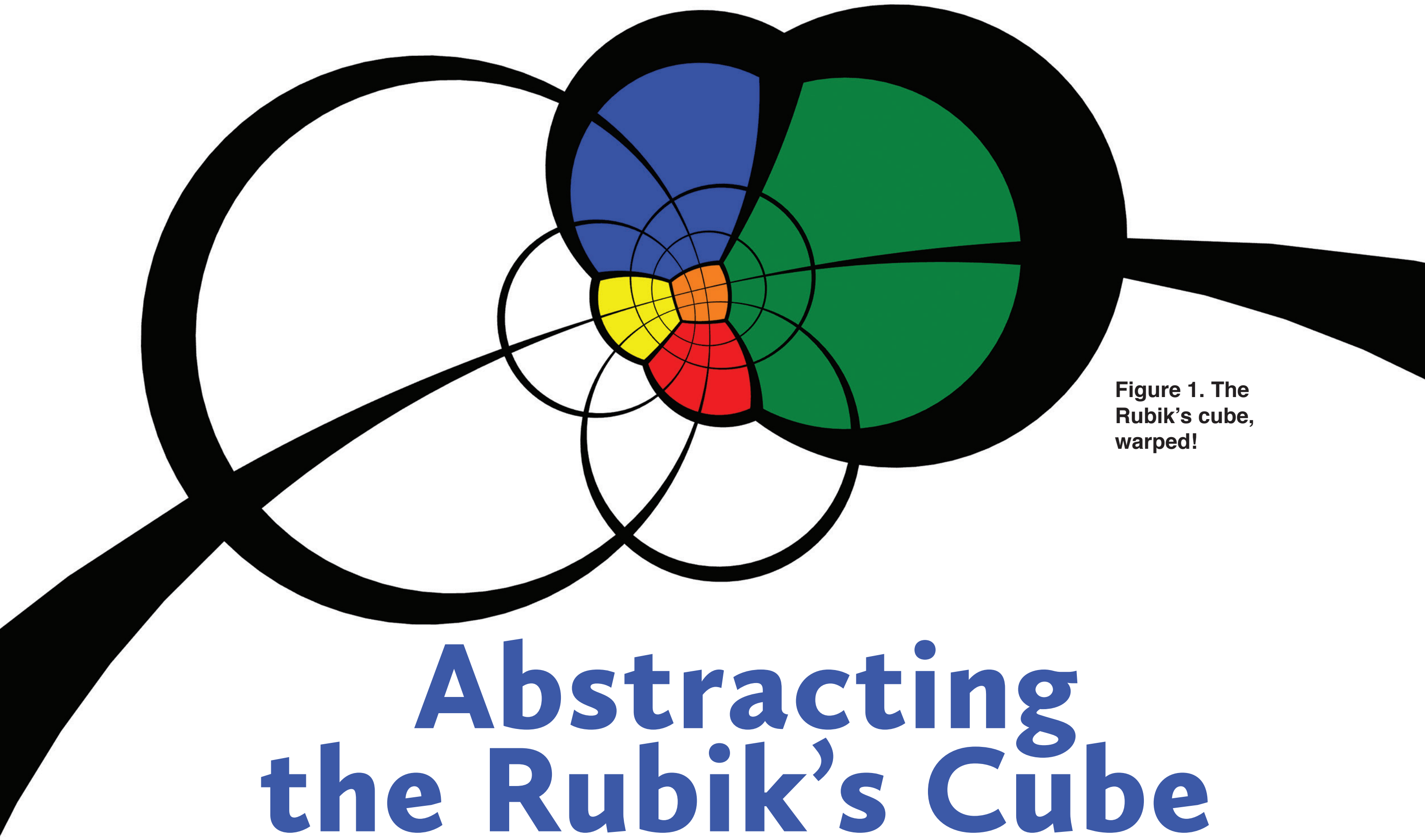

Figure 1. The Rubik's cube, warped!

Abstracting the Rubik's Cube

Roice Nelson

David Hilbert wrote, "The art of doing mathematics consists in finding that special case that contains all the germs of generality."

Over the past few decades, a growing group of puzzle enthusiasts known as *hypercubists* have generalized the Rubik's Cube in ways that traverse a wide expanse of mathematical ground. The explorations have been a microcosm of mathematical progress. Finding and studying these puzzles provides a rich way to approach varied topics in mathematics: geometry (higher dimensional, non-Euclidean, projective), group theory, combinatorics, algorithms, topology, polytopes, tilings, honeycombs, and more.

For this group of people, twisty puzzles are more than just a casual pastime. Elegance is a core principle in their quest.

Hypercubes

We can change many properties of the classic $3 \times 3 \times 3$ Rubik's Cube, such as its shape or twist centers, to make new and interesting puzzles (see figure 2). But the hypercubing group began by changing a more abstract property, namely, the dimension.

Don Hatch and Melinda Green created an exquisite working four-dimensional $3 \times 3 \times 3 \times 3$ (or 3^4) analogue, which they called MagicCube4D. Every property of this puzzle is upped a dimension: Faces, stickers, and twists are three-dimensional rather than two-dimensional. Figure 3 shows the ordinary Rubik's cube and the hyperpuzzle using a central projection that reduces the dimension by one; it is as if we are looking into a box, with the nearest face hidden.

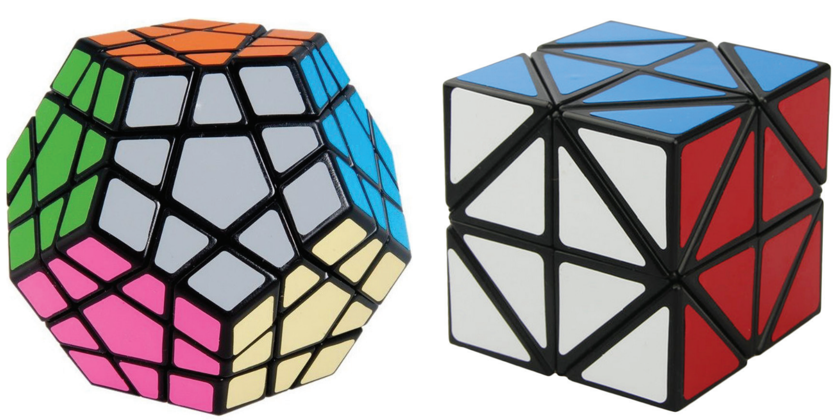

Figure 2. (a) Megaminx uses a dodecahedral shape rather than a cube. (b) The Helicopter Cube twists around edges instead of faces.

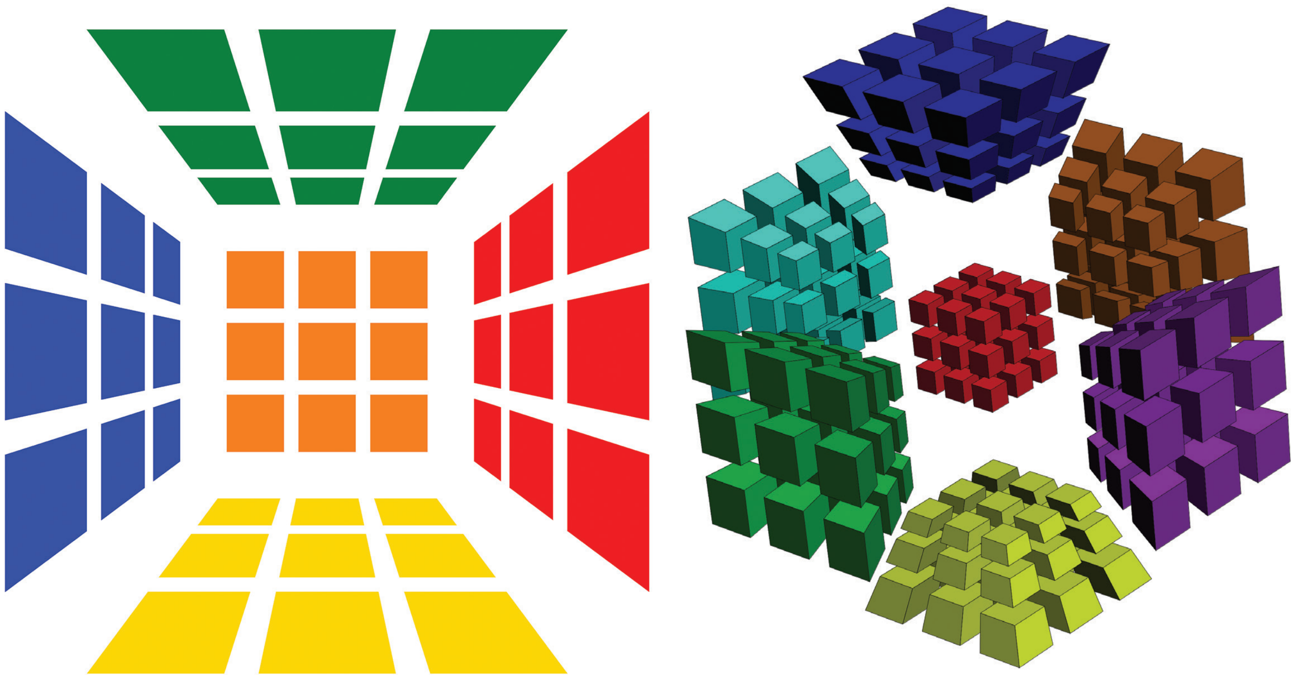

Figure 3. Projection tricks can help us visualize higher-dimensional Rubik's Cubes. (a) The 3^3, projected so a two-dimensional "flatlander" sees five of the six cube faces. (b) The 3^4, projected so a three-dimensional being sees seven of the eight hypercube faces.

The 3^3 Rubik's Cube has $6 \cdot 3^2 = 54$ stickers that can live in a mind-boggling 4.325×10^{19} possible states. The hypercubical 3^4 has $8 \cdot 3^3 = 216$ stickers and the number of possible puzzle positions explodes to an incomprehensible 1.756×10^{120}. Calculating this number is a challenge that will test your group theory mettle!

But as Edwin Abbott wrote in *Flatland,* "In that blessed region of Four Dimensions, shall we linger on the threshold of the Fifth, and not enter therein?"

The group didn't stop at four dimensions. In 2006, a working five-dimensional puzzle materialized with $10 \cdot 3^4 = 810$ hypercubical stickers and 7.017×10^{560} states, pushing the boundaries of visualization. Figure 4 shows a shadow of a shadow of a shadow of the five-dimensional object. Nonetheless, as of mid-2017, around 70 people have solved this puzzle.

In June 2010, Andrey Astrelin stunned the group by using a creative visual approach to represent a seven-dimensional Rubik's Cube. Yes, it has been

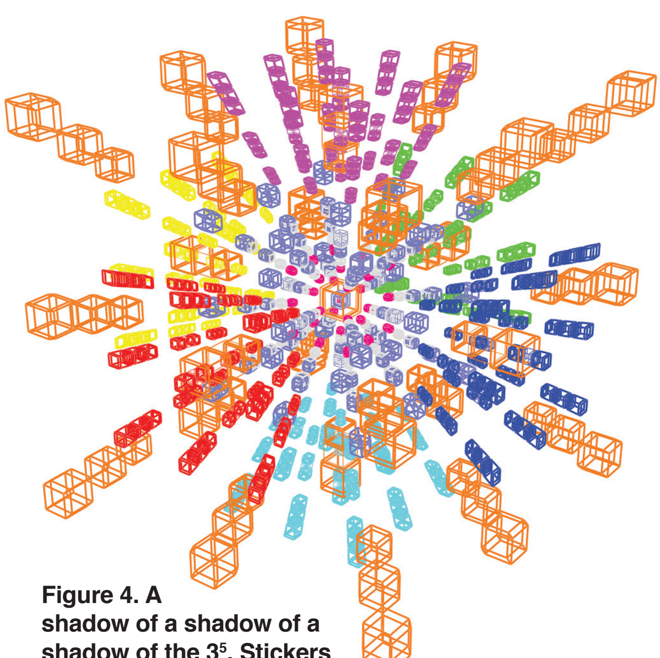

Figure 4. A shadow of a shadow of a shadow of the 3^5. Stickers are little hypercubes.

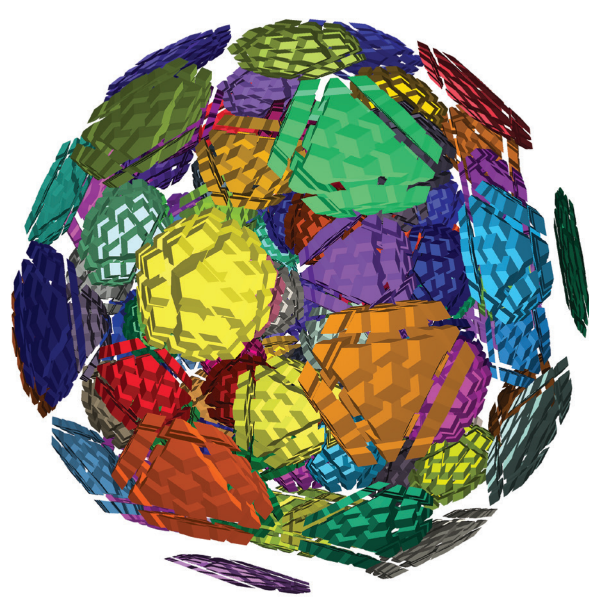

Figure 5. The Magic120Cell, or the 4D Megaminx, has 120 dodecahedral faces. It derives from the 120-cell, one of six Platonic shapes in four dimensions.

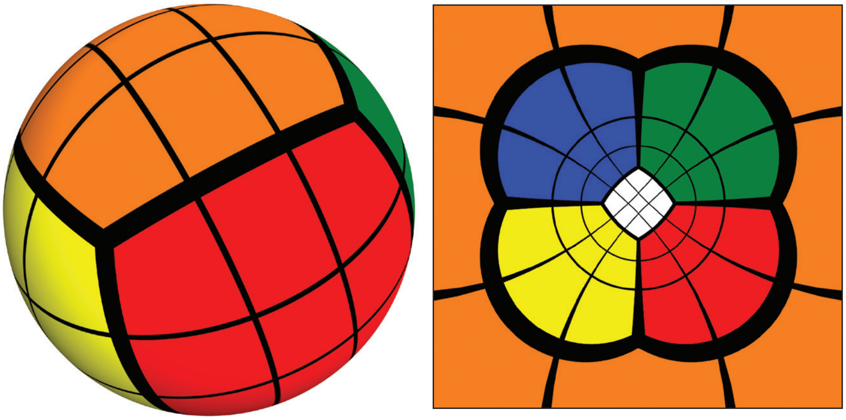

Figure 6. (a) The Rubik's Cube projected radially onto a sphere yields a two-dimensional tiling of the sphere. (b) It is then stereographically projected onto the plane.

solved. Can you calculate the number of stickers on the 3^7?

You may also enjoy trying to work out the properties of a two-dimensional Rubik's Cube. What dimension are the stickers?

Of course, we can play the shape-changing game in higher dimensions too, yielding a panoply of additional puzzles. There are five Platonic solids in three dimensions, but six perfectly regular shapes a dimension up, and you can attempt to solve twisty puzzle versions of all of them! Figure 5 shows one of the most beautiful in its pristine state.

Shapes in arbitrary dimensions are called *polytopes*, or *polychora* in four dimensions. In addition to the regular polychora, there are many uniform polychora, and quite a few have been turned into twisty puzzles. Uniform polychora can break regularity in various ways. They may have multiple kinds of three-dimensional faces, or the faces may be composed of uniform (that is, Archimedean) polyhedra.

Curved Twisty Puzzles

"For God's sake, I beseech you, give it up. Fear it no less than sensual passions because it too may take all your time and deprive you of your health, peace of mind and happiness in life."

No, these were not desperate pleas to a hypercubist about excessive puzzling adventures. Such were the words of Farkas Bolyai to his son János, discouraging him from investigating Euclid's fifth postulate. János continued nonetheless, which led him into the wonderful world of hyperbolic geometry.

We will also not heed the elder Bolyai's advice. Let's use topology to abstract away a different property of Rubik's Cube—its cubeness. To do so, project the cube radially outward onto a sphere (see figure 6a). Notice that all the important combinatorial properties remain. Furthermore, what were planar slices of the Rubik's Cube are now circles on the sphere's surface. A twist simply rotates the portion of the surface inside one of these twisting circles. In short, we are viewing the Rubik's Cube as a tiling of the sphere by squares, sliced up by circles on the surface.

Inspired by this example, we can consider other colored regular tilings, and a huge number of new twisty puzzles become possible, some living in the world of hyperbolic geometry!

For two-dimensional surfaces, there are three geometries with constant curvature: spherical, Euclidean, and hyperbolic. These geometries correspond to whether the interior angles of a triangle sum to greater than, equal to, or less than 180 degrees, respectively. Intuitively, we can think of the surface of a sphere, a flat plane, and a Pringles potato chip as representative surfaces for these geometries.

Each surface of constant curvature can be tiled with regular polygons. The *Schläfli symbol* encodes regular tilings with just two numbers, $\{p,q\}$. This denotes a tiling by p-gons in which q such polygons meet at each vertex. The value $(p-2)(q-2)$ determines the geometry: Euclidean when equal to 4, spherical when less, and hyperbolic when greater.

For example, $\{4,3\}$ denotes a tiling by squares with three arranged around each vertex, that is, the cube. As we saw in figure 6a, this gives a tiling of the sphere, and indeed, $(4-2)(3-2) = 2 < 4$.

Euclidean geometry is the only one of the three geometries that can live on the plane without any distortion. A lovely way to represent the other geometries on the plane is via *conformal*, or angle preserving, maps. The *stereographic projection* is a conformal map for spherical geometry. Figures 1 and 6b show the stereographic projection of the spherical Rubik's Cube onto the plane. For hyperbolic geometry we use the *Poincaré disk*, which squashes the infinite expanse of the hyperbolic plane into a unit disk (see figure 10).

One challenge of turning Euclidean and hyperbolic tilings into twisty puzzles is that unlike spherical tilings, which are finite, tilings of these two geometries go on forever. To overcome this hurdle, we begin with a tiled surface, called the *universal cover*; choose a certain subset of tiles, called the *fundamental domain*; and *identify* its edges to form a *quotient surface*. Intuitively, we glue the edges of this region together to turn the infinite tilings into finite puzzles. Figures 7, 8, and 9 show a few examples.

One of the crown jewels of this abstraction is the *Klein quartic* Rubik's Cube, composed of 24 hepata-

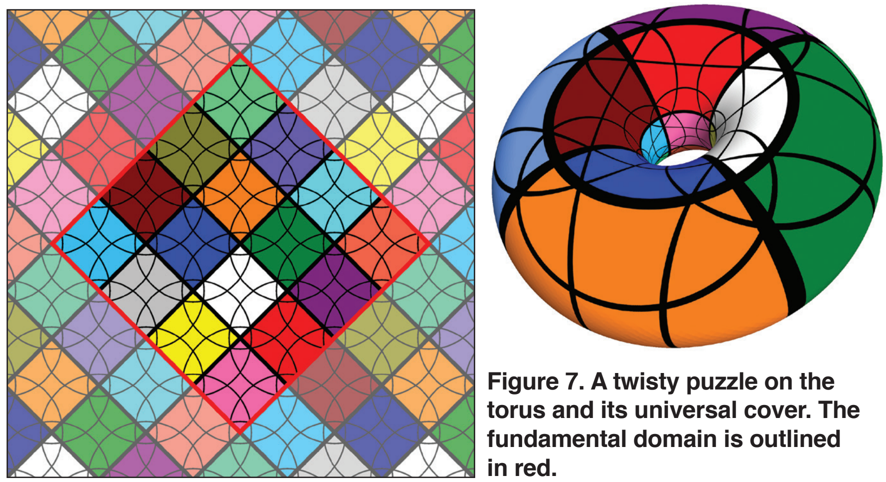

Figure 7. A twisty puzzle on the torus and its universal cover. The fundamental domain is outlined in red.

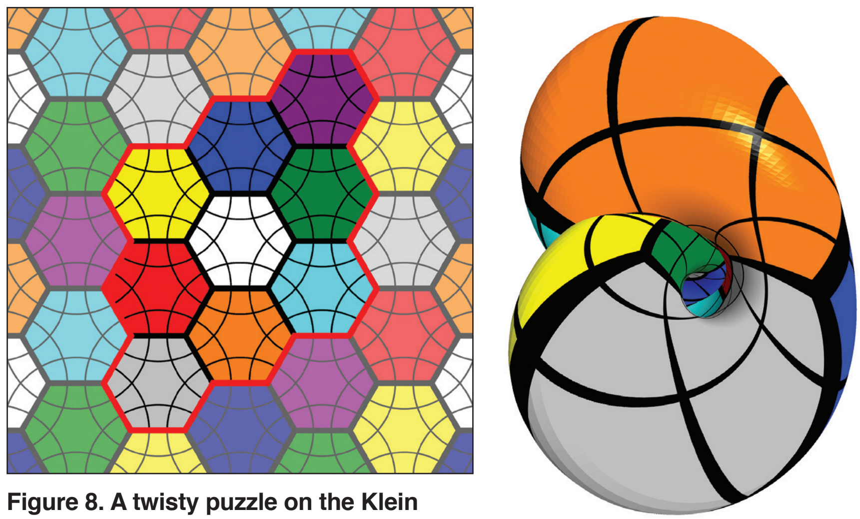

Figure 8. A twisty puzzle on the Klein bottle and its universal cover. The fundamental domain is outlined in red.

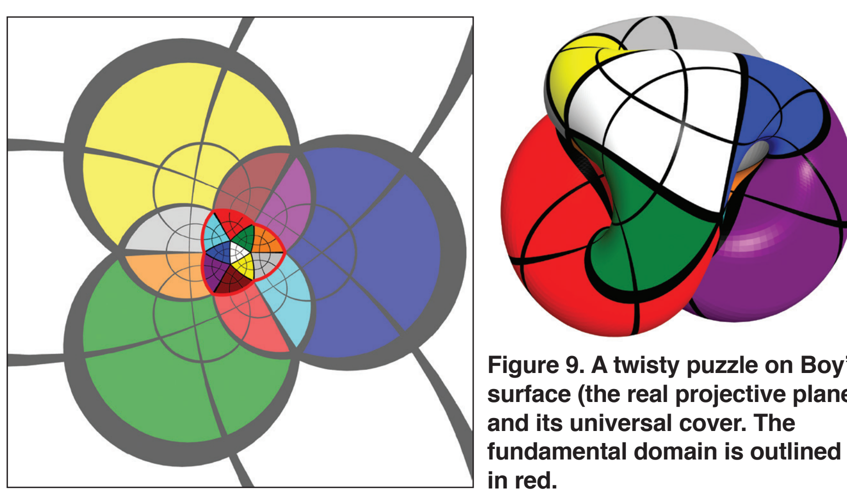

Figure 9. A twisty puzzle on Boy's surface (the real projective plane) and its universal cover. The fundamental domain is outlined in red.

gons, three meeting at each vertex. It has "center," "edge," and "corner" pieces just like the Rubik's Cube. The universal cover is the {7,3} hyperbolic tiling, and the quotient surface is a three-holed torus. This puzzle contains some surprises; if you solve layer by layer, as is common on the Rubik's Cube, you'll be left with two unsolved faces at the end instead of one.

All these puzzles and more are implemented in a program called MagicTile. The puzzle count recently exceeded a thousand, with an infinite number of possibilities remaining.

More Puzzles

There are even more intriguing analogues that we have not yet seen. Let me mention two of my favorites. The first is another astonishing set of puzzles by Andrey Astrelin based on the {6,3,3} *honeycomb* in three-dimensional hyperbolic space, $\mathbb{H}^3$ (see figure 11). The faces are hexagonal {6,3} tilings, with three faces meeting at each edge. Gluing via identifications serves to make the underlying honeycomb finite in two senses: the number of faces and the number of facets per face. If we take a step back and consider where we started, this puzzle has altered the dimension, the geometry, and the shape compared to the original Rubik's Cube!

The second is a puzzle created by Nan Ma based on the 11-cell, an abstract regular polytope composed of 11 hemi-icosahedral cells (see figure 11). This is a higher-dimensional cousin of the Boy's surface puzzle in figure 9. The 11-cell can only live geometrically unwarped in 10 dimensions, but Nan was able to preserve the combinatorics in his depiction.

With so many puzzles having been uncovered, one could be forgiven for suspecting there is not much more to do. On the contrary, there are arguably more avenues to approach new puzzles now than 10 years ago. For example, there are no working puzzles in $\mathbb{H}^3$ composed of finite polyhedra. There are not yet puzzles for uniform tilings of Euclidean or hyperbolic geometry, in two or three dimensions. Uniform tilings are not even completely classified, so further mathematics is required before some puzzles can be realized.

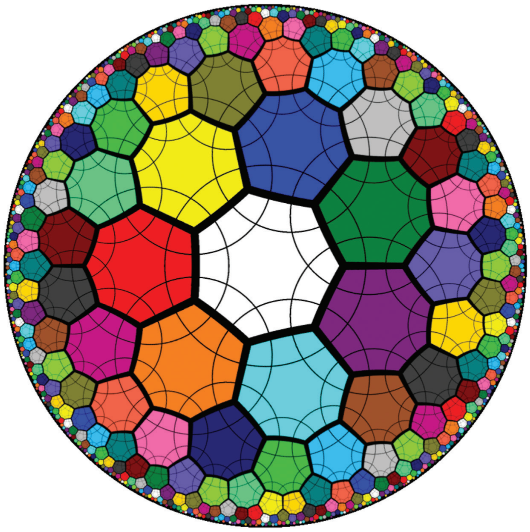

Figure 10. Klein quartic Rubik's Cube on the hyperbolic universal cover. The quotient surface is a three-holed torus.

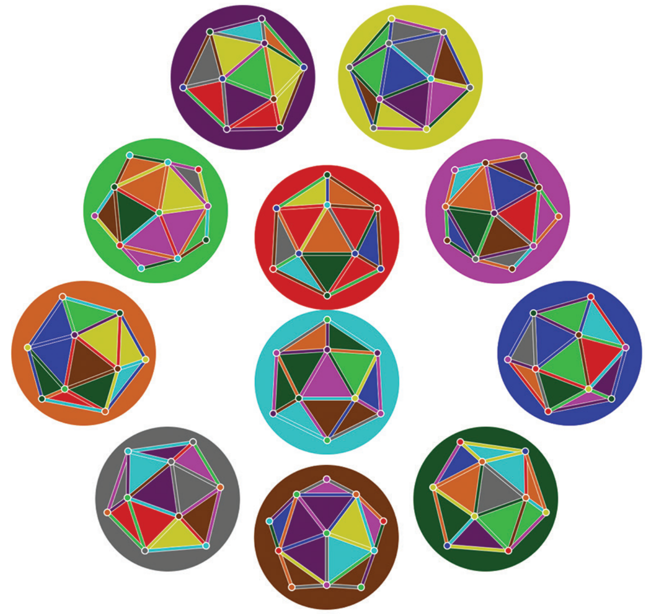

Figure 12. The scrambled Magic 11-Cell.

Melinda Green has been developing a physical puzzle that is combinatorially equivalent to the 2^4. The idea of fractal puzzles has come up, but no one has yet been able to find a good analogue.

In addition to the search for puzzles, countless mathematical questions have been asked or are ripe for investigation. How many permutations do the various puzzles have? What checkerboard patterns are possible? Which n^d puzzles have the same number of stickers as pieces? How many ways can you color the faces of the 120-cell puzzle? What is *God's number* for these higher dimensional Rubik's Cubes; that is, what is the minimum number of moves in which the puzzle can be solved, regardless of starting position? The avenues are limited only by our curiosity.

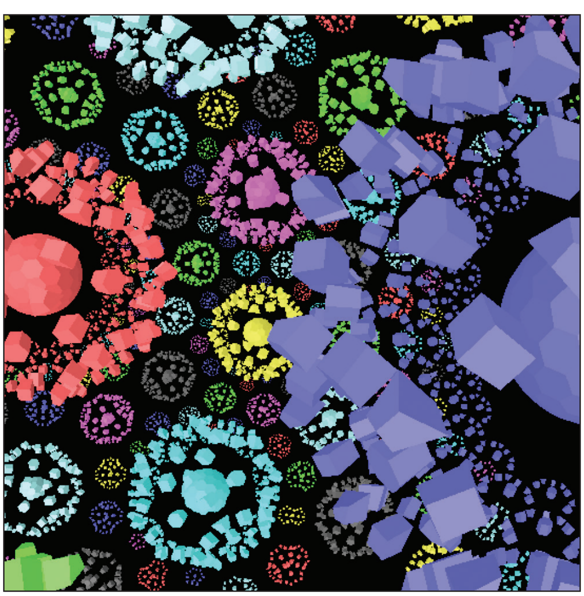

Figure 11. An in-space view of the Magic Hyperbolic Tile {6,3,3} puzzle in three-dimensional hyperbolic space.

As John Archibald Wheeler wrote, "We live on an island surrounded by a sea of ignorance. As our island of knowledge grows, so does the shore of our ignorance." ■

Further Reading

The *MagicCube4D* website (*superliminal.com/cube/cube.htm)* contains links to all the puzzles in this article and to the hypercubing mailing list.

Burkard Polster (Mathologer) produced wonderful introductory videos to MagicCube4D and MagicTile "Cracking the 4D Rubik's Cube with simple 3D tricks" (*youtu.be/yhPH1369OWc*) and "Can you solve THE Klein Bottle Rubik's Cube?" (*youtu.be/DvZnh7-nslo*)

The following papers are freely available online:

H. J. Kamack and T. R. Keane, "The Rubik Tesseract," (1982) *http://bit.ly/RubikTess.*

John Stillwell, "The Story of the 120-cell," *Notices of the AMS* 48, no. 1 (2001): 17–24.

Carlo H. Séquin, Jaron Lanier, and UC CET, "Hyperseeing the Regular Hendecachoron," *Proc. ISAMA* (2007): 159–166.

Roice Nelson is a software developer with a passion for exploring mathematics through visualization. He enjoys spending time with his soul mate Sarah and their three cats, and prefers traveling on two or fewer wheels.

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