"**Changes in latitudes, changes in attitudes, nothing remains quite the same...**" —Jimmy Buffet

A Mathematician's Look at **Foucault's Pendulum**

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That was when I saw the Pendulum. The sphere, hanging from a long wire set into the ceiling of the choir, swayed back and forth with isochronal majesty...The time it took the sphere to swing from end to end was determined by an arcane conspiracy between the most timeless of measures: the singularity of the point of suspension, the duality of the plane's dimensions, the triadic beginning of π, the secret quadratic nature of the root, and the unnumbered perfection of the circle itself...Were its tip to graze, as it had in the past, a layer of damp sand spread on the floor of the choir, each swing would make a light furrow, and the furrows, changing direction imperceptibly, would widen to form a breach, a groove with radial symmetry...

So begins Umberto Eco's esoteric conspiratorial adventure tale, *Foucault's Pendulum*. But what are the underlying forces at work behind this phenomenon? Are they mathematical or are they physical? Today, at least, I'd like to argue that they are mathematical. It is simply the curvature of the earth which enables the pendulum's tip to create its single pattern.

The notion of curvature is an interesting and exciting topic that is rich in questions inviting exploration and research. While the definition of curvature of a curve may be familiar to many (the rate of change of the unit tangent vector with respect to arc length), curvature of a surface may not be. For a surface, what should curvature measure? How is it defined to agree with our basic ideas (a plane is not curved, while a sphere has constant curvature)? These are both interesting and inviting questions. But why stop with surfaces? Curvature of higher-dimensional surfaces (manifolds) is an equally fascinating subject which can quickly lead to such appealing topics as Einstein's general theory of relativity, the curvature of spacetime, blackholes, and the like. In fact, one of Einstein's famous equations (not the $E = mc^2$ one) basically equates curvature (a mathematical quantity) with matter (a physical quantity)! It makes sense to tackle a familiar problem from physics using a completely mathematical formulation.

Foucault's Pendulum

In the middle of the nineteenth century, Foucault constructed a pendulum by hanging a 5kg bob from the ceiling of a cellar (about 2m high). After setting the pendulum in motion he observed "The oscillating point is continuously displaced..., which indicates that the deviation of the plane of oscillation takes place in the same sense as the apparent motion of the celestial sphere..." Later on at the Pantheon he repeated this experiment on a grander scale using a 28 kg bob hanging from a height of 67m. Hopefully you have had the opportunity to

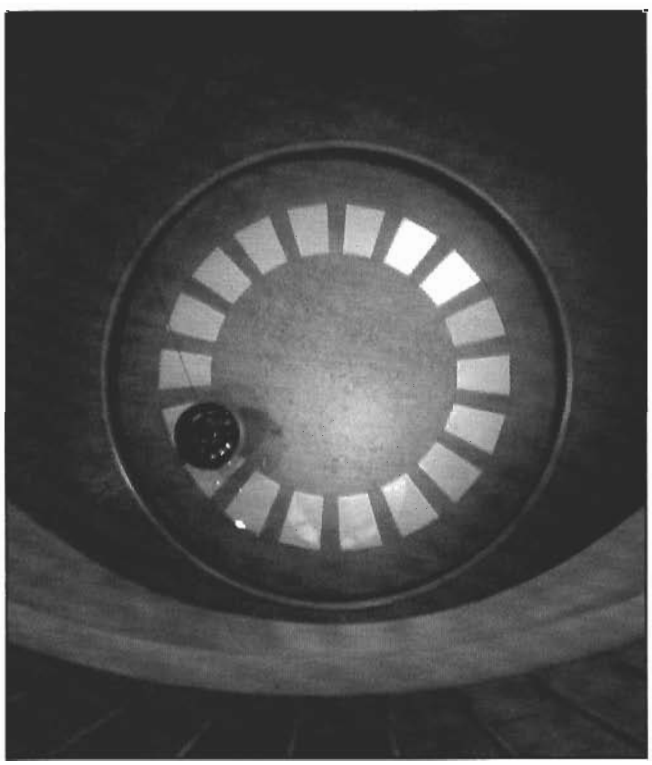

Photo by Jennifer Quinn

The Foucault Pendulum in the atrium of the Hameetman Science Center, Occidental College.

Photo by Jennifer Quinn

A closer look at the pendulum bob.

observe this phenomenon yourself. Can you explain why he observed the plane of oscillation (the direction of the pendulum swing) rotating as time went on?

A couple of simple thought experiments can help replicate this phenomenon if you have never observed it firsthand. First, imagine a pendulum suspended from a great height directly above the North Pole. Give this pendulum just a slight push so that as it swings it almost brushes the ground. Now, assuming that it does not lose any momentum, imagine the pendulum swinging back and forth for a 24-hour period. What happens? Clearly the Earth rotates through an angle of 2π so that an observer standing near the pendulum will observe the direction of swing of the pendulum steadily rotating through an angle of 2π as well (over the same time period).

Now, perform the same thought experiment but this time let's suspend the pendulum directly over a point on the equator (say Quito). Suppose the pendulum is given a slight initial push so that it swings North-South. As the Earth rotates (West-to-East) the direction of the pendulum remains North-South! That is, there is no observable rotation of the plane of oscillation! Obviously there is something different about being at the North Pole and the equator.

The next logical step would be to perform the thought experiment once again for some latitude between the equator and the North Pole. However, at this point it is not as easy to keep track of the position of the pendulum relative to that of the surface of the Earth (try it!). However, one should be able to invoke a continuity argument to claim that since the pendulum does not rotate when positioned at the equator and it rotates through an angle of 2π when positioned at the North pole (over a 24-hour period), it must continuously rotate through greater and greater angles as one positions it in more northern climes. Nevertheless, it is the nature of the increase which is the more interesting question. Is this increase linear? Quadratic? Something else?

The main purpose of this article is to describe the phenomenon of Foucault's pendulum mathematically. Without relying on any incantations of such physical quantities as "coriolis effect" and the like, this paper will show that the phenomenon is just a simple effect due to the curvature of the earth. And we all know that curvature is a mathematical quantity. Isn't it?

Life On a Sphere

Since you are currently reading a mathematics magazine, you are no doubt aware that there are some fundamental differences between a sphere and a plane. For instance,

- In a plane, all triangles have angles which sum to exactly 180° while on the surface of a sphere triangles have angles which sum to greater than 180°.
- In a plane, parallel lines never meet. On a sphere they meet twice! (Just imagine two travelers starting nearby on the equator. If they decide to each walk North they will meet at the North Pole and again at the South Pole.)

These two observations indicate that geometry on a sphere is certainly different then geometry on a plane. However, there are also topological differences as well.

- Imagine a group of people standing in a circle and holding hands. If they were standing on a plane and began to walk "towards" each other, the circle would grow increasingly smaller until everyone met at a point. Or, if they walked away from each other the circle would continue to grow larger and larger (and everyone's arms would have to stretch indefinitely). Now, if this same circle of people were standing on the surface of a sphere they could begin to walk either towards one another or away from one another and eventually they would still meet. (It might take a while and it would require a lot of arm stretching, but it could be done.)
- More simply, just imagine a lone world-traveler always walking in a straight line. In a plane this traveler would keep walking forever without ever passing over the same ground twice. However, on a sphere the traveler would eventually return to her starting point.

Why do these differences exist? Because of CURVATURE! Curvature affects geometry (triangles, notions of parallel) as well as physics (paths that light travel, Einstein's theory of general relativity, and Foucault's pendulum, which we're still getting to). In order to see the effect of curvature on Foucault's pendulum we will first have to learn how to do calculus and geometry on a sphere.

Calculating on a Sphere

Just as in any mathematics course, one needs to be equipped with the appropriate tools in order to do geometry and calculus (differential geometry) on a sphere. For our purposes here, we will need the following tools: appropriate coordinates, basis vectors that are related to our chosen coordinates, and an appropriate notion of derivative.

Coordinates and Basis Vectors

Since the Earth is basically a sphere, it probably makes the most sense to work in spherical coordinates, but let's make one slight modification. Traditionally spherical coordinates use ϕ to measure the angle down from the positive z-axis. For this paper, we'll use ϕ to represent the angle up from the xy-plane and hence will be synonymous with traditional latitude. Now, since we are interested in doing calculus on Earth (our sphere) and only Earth, all points will have the same radial coordinate (call it R) and thus as we travel from point to point on the sphere only the coordinates ϕ and θ will change. Let's think of these as our two coordinates. At this point we have that any point on our sphere can be represented by the position vector

$$R\cos\theta\cos\phi\boldsymbol{i} + R\sin\theta\cos\phi\boldsymbol{j} + R\sin\phi\boldsymbol{k}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ and are the standard basis vectors in $\boldsymbol{R}^3$. As mentioned earlier, as we travel around the sphere the only changes are in θ and ϕ, thus it would make life much easier if we had basis vectors which pointed in the direction of increasing θ and increasing ϕ respectively. A couple of simple partial derivatives will accomplish this. The direction of increasing θ is found by taking the partial derivative of position with respect to θ.

$$\begin{aligned}\frac{\partial}{\partial\theta}(\text{position}) &= \frac{\partial}{\partial\theta}\left(R\cos\theta\cos\phi\boldsymbol{i} + R\sin\theta\cos\phi\boldsymbol{j} + R\sin\phi\boldsymbol{k}\right)\\ &= -R\sin\theta\cos\phi\boldsymbol{i} + R\cos\theta\cos\phi\boldsymbol{j} + 0\boldsymbol{k}.\end{aligned}$$

This vector currently has length $R\cos\phi$ so the unit vector in the direction of increasing θ is

$$\boldsymbol{e}_1 = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j} + 0\boldsymbol{k}.$$

Similarly, by calculating the partial derivative of position with respect to ϕ, one can define the unit vector in the direction of increasing ϕ to be

$$\boldsymbol{e}_2 = -\cos\theta\sin\phi\boldsymbol{i} - \sin\theta\sin\phi\boldsymbol{j} + \cos\phi\boldsymbol{k}.$$

At this time we have our sphere of radius R having coordinates (ϕ, θ) that defines points on the sphere and basis vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, which define directions tangent to the sphere. As we begin to analyze Foucault's pendulum, we will be interested in measuring only those changes which occur tangent to the sphere. Consequently we will wish to ignore all motion orthogonal (or normal) to the surface of the sphere. It should be easy to see that the outward pointing unit normal vector is given by

$$\boldsymbol{n} = \cos\theta\cos\phi\boldsymbol{i} + \sin\theta\cos\phi\boldsymbol{j} + \sin\phi\boldsymbol{k}.$$

The Pendulum

Armed with the necessary tools, we now come back to the original study of the pendulum. As we begin to analyze this motion, keep three important factors in mind:

1. The pendulum swings close to the surface of the sphere through a very small angle and, hence, the direction of swing, which we will call $\boldsymbol{v}$, is tangent to the sphere.
2. The pendulum itself is just trying to go back and forth; it is not trying to twist or rotate. It is the Earth under the pendulum which is rotating! As the Earth rotates, we can think of the pendulum traveling along a circle of constant latitude ($\phi = \phi_0$).
3. All physical forces (e.g. gravity) will be ignored. Remember we are treating this as a geometry problem not a physics problem.

Given these assumptions, we will cast the phenomenon of Foucault's pendulum into mathematical terms.

The Geometry Problem:

Given a unit vector $\boldsymbol{v}$, slide it around a circle of constant latitude such that

1. $\boldsymbol{v}$ is always tangent to the sphere (which means that $\boldsymbol{v}$ can always be expressed as a linear combination of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$), and
2. The direction of $\boldsymbol{v}$ always stays the same, that is, $\boldsymbol{v}$ does not change in the $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ directions (the only directions that we care about).

The above problem is a common one in differential geometry. Using fancy language we want to parallel transport $\boldsymbol{v}$ around a circle of constant latitude. We can then compare what $\boldsymbol{v}$ looks like after we make a complete trip around the Earth (i.e., after 24 hours have elapsed).

Now, as $\boldsymbol{v}$ travels around a circle of constant latitude, only θ changes. Hence we can think of $\boldsymbol{v}$ (the direction of swing) as being parameterized by θ. Furthermore, we are assuming that the pendulum's swing is always tangent to the sphere and hence can be written as a linear combination of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. We have

$$\boldsymbol{v}(\theta) = a(\theta)\boldsymbol{e}_1 + b(\theta)\boldsymbol{e}_2$$

Continued on page 32

Continued from page 21

where a and b are scalars that depend on θ. To see how $\boldsymbol{v}$ changes as it travels around a circle of constant latitude (increasing θ) we need to compute $\frac{\partial}{\partial\theta}\boldsymbol{v}$. First, let's compute the partials of our basis vectors:

$$\begin{aligned}\frac{\partial}{\partial\theta}\boldsymbol{e}_1 &= \frac{\partial}{\partial\theta}\left(-\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j} + 0\boldsymbol{k}\right)\\ &= -\cos\theta\boldsymbol{i} - \sin\theta\boldsymbol{j} + 0\boldsymbol{k}.\end{aligned}$$

Recall, we are only interested in motion tangent to the sphere. Thus, to see how much of this change is tangent to the sphere, let's write it in terms of $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{n}$. One should get (double-check my work here!):

$$\frac{\partial}{\partial\theta}\boldsymbol{e}_1 = 0\boldsymbol{e}_1 + \sin\phi_0\boldsymbol{e}_2 - \cos\phi_0\boldsymbol{n}$$

For the second basis vector we have

$$\begin{aligned}\frac{\partial}{\partial\theta}\boldsymbol{e}_2 &= \frac{\partial}{\partial\theta}\left(-\cos\theta\sin\phi_0\boldsymbol{i} - \sin\theta\sin\phi_0\boldsymbol{j} + \cos\phi_0\boldsymbol{k}\right)\\ &= \sin\theta\sin\phi_0\boldsymbol{i} - \cos\theta\sin\phi_0\boldsymbol{j} + 0\boldsymbol{k}\\ &= -\sin\phi_0\boldsymbol{e}_1 + 0\boldsymbol{e}_2 + 0\boldsymbol{n}.\end{aligned}$$

Since $\boldsymbol{v}$ represents the direction of the motion of the pendulum, then $\boldsymbol{v}$ is not supposed to change (as measured on the sphere). So, we will impose the condition that $\frac{\partial}{\partial\theta}\boldsymbol{v}$, once projected down to the sphere, should be zero (actually the zero vector). Thus, as we start taking derivatives, just keep in mind that any component in the $\boldsymbol{n}$ direction will be ignored and we'll only keep those components in the directions of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. Furthermore, let's use $\nabla\boldsymbol{v}$ to represent $\frac{\partial}{\partial\theta}\boldsymbol{v}$ projected down to the sphere. Thus,

$$\begin{aligned}\nabla\boldsymbol{v} &= \nabla\left(a(\theta)\boldsymbol{e}_1 + b(\theta)\boldsymbol{e}_2\right)\\ &= a'\boldsymbol{e}_1 + a\left(\sin\phi_0\boldsymbol{e}_2\right) + b'\boldsymbol{e}_2 + b\left(-\sin\phi_0\boldsymbol{e}_1\right)\\ &= \left(a' - b\sin\phi_0\right)\boldsymbol{e}_1 + \left(a\sin\phi_0 + b'\right)\boldsymbol{e}_2.\end{aligned}$$

Since we are insisting that $\nabla\boldsymbol{v} = \boldsymbol{0}$, this will only be the case when

$$a' - b\sin\phi_0 = 0$$

and

$$b' + a\sin\phi_0 = 0$$

Which means we have a system of differential equations:

$$\begin{aligned}\frac{da}{d\theta} &= b\sin\phi_0\\ \frac{db}{d\theta} &= -a\sin\phi_0.\end{aligned}$$

We can solve this uniquely once we impose initial conditions. If we assume that the pendulum is started so that it swings east-west (in the θ direction only) we have $\boldsymbol{v}(0) = 1\boldsymbol{e}_1 + 0\boldsymbol{e}_2$ which gives us the initial conditions of $a(0) = 1$ and $b(0) = 0$. To simplify the results, let's let Q be the sine of the latitude $(Q = \sin\phi_0)$. Even if you have never had a course in differential equations, you can probably guess what the solution to the above initial value problem is. Think about it. We are looking for two functions of θ. The first one, $a(\theta)$, has a derivative which is a multiple of the second one, $b(\theta)$, while the derivative of the second function is a multiple of the first. The basic trigonometric functions should quickly come to mind. Together with the initial conditions, one sees that the unknown functions a and b must be

$$\begin{aligned}a(\theta) &= \cos Q\theta\\ b(\theta) &= -\sin Q\theta.\end{aligned}$$

Recall that since the direction of the pendulum's swing is given by $\boldsymbol{v} = a(\theta)\boldsymbol{e}_1 + b(\theta)\boldsymbol{e}_2$, we now know how the direction of the swing moves as the Earth rotates (θ increases). For a more geometrical interpretation of how $\boldsymbol{v}$ changes, exploit the power of matrix multiplication to write

$$\begin{bmatrix}a(\theta)\\ b(\theta)\end{bmatrix} = \begin{bmatrix}\cos Q\theta & \sin Q\theta\\ -\sin Q\theta & \cos Q\theta\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}$$

This matrix multiplication may be familiar from linear algebra. The vector $[a(\theta), b(\theta)]$ is simply the vector $[1, 0]$ after having been rotated through an angle of $Q\theta$. So, after 24 hours of swinging, the pendulum (i.e., $\boldsymbol{v}$) would end up rotating by $2\pi Q$ where Q is the sine of your latitude! A quick reality check shows that this result agrees with our earlier thought experiments. At the equator ($\theta = 0$) there is no oscillation of the pendulum, while at the North Pole ($\theta = \pi/2$) the pendulum completes exactly the same rotation as the Earth! Another interesting observation is that even though one can argue that it is the curvature of the Earth which creates this phenomenon, the radius of the Earth (which affects the magnitude of curvature) does not influence the result. Therefore, you can tell you live on a spherical Earth (as opposed to a flat one) by building such a pendulum and seeing if the plane of oscillation does indeed rotate as Foucault claims, but you can not detect how "curved" your spherical world is. How does one do that? Well... ■

Acknowledgements and Further Reading

Thanks go out to David Scott for inviting me to talk about this material at the University of Puget Sound. The above argument appears in more technical guise in John Oprea's 1997 book *Differential Geometry and its Applications*. The quote on page 19 comes from Rene Dugas' 1955 book *A History of Mechanics*.