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CHAOS RULES!

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The "Classical" Chaos Game

The "chaos game" and its multitude of variations provides a wonderful opportunity to combine elementary ideas from geometry, linear algebra, probability, and topology with some quite contemporary mathematics. The easiest chaos game to understand is played as follows. Start with three points at the vertices of an equilateral triangle. Color one vertex red, one green, and one blue. Take a die and color two sides red, two sides green, and two sides blue. Then pick any point whatsoever in the triangle; this is the *seed*. Now roll the die. Depending upon which color comes up, move the seed half the distance to the similarly colored vertex. Then repeat this procedure, each time moving the previous point half the distance to the vertex whose color turns up when the die is rolled. After a dozen rolls, start marking where these points land.

When you repeat this process many thousands of times, the pattern that emerges is a surprise: it is not a "random mess," as most first-time players would expect. Rather, the image that unfolds is one of the most famous fractals of all, the Sierpinski triangle shown in Figure 1. Notice that there are no points in the "missing" triangles in this set. This is why we did not plot the first few points when we rolled the die.

Now here is the observation that leads to the geometry: the Sierpinski triangle consists of three self-similar pieces, each of which is exactly one half the size of the original triangle in terms of the lengths of the sides. These are precisely the numbers that we used to play the game: three vertices and move half the distance to the vertex after each roll. So we can go backwards. Just by looking at the Sierpinski triangle, we can read off the rules of the game we played to produce it.

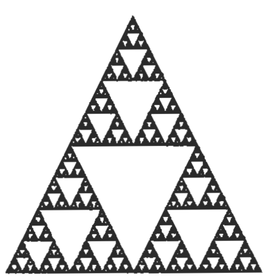

Figure 1: The Sierpinski triangle. The original red, green, and blue vertices are located at the vertices of this image.

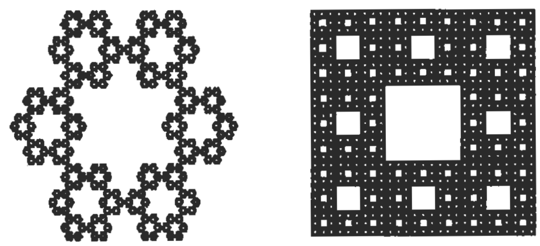

Figure 2a and 2b: The Sierpinski hexagon and carpet.

Other Chaos Games

For a different example of a chaos game, put six points at the vertices of a regular hexagon. Number them one through six and erase the colors on the die. We change the rules a bit here: instead of moving the point half the distance to the appropriate vertex after each roll, we now "compress the distance by a factor of three." By this we mean we move the point so that the resulting distance from the moved point to the chosen vertex is one third the original distance. We say that the *compression ratio* for this game is three.

Again we get a surprise: after rolling the die thousands of times the resulting image is a "Sierpinski hexagon" as shown in Figure 2a. And again we can go backwards: this image consists of six self-similar pieces, each of which is exactly one third the size of the full Sierpinski hexagon—the same numbers we used to design the game. By the way, there is much more to this picture than meets the eye at first: notice that the interior white regions of this figure are all bounded by the well known Koch snowflake fractal!

The Sierpinski triangle and hexagon show that the objects that result from playing these chaos games have interesting topology. Here is an even more intriguing example of this illustration. Play the chaos game with eight vertices: four at the corners of a square and four at the midpoints of the sides. When a compression ratio of three is used, the result is the equally famous Sierpinski carpet shown in Figure 2b. Topologists know that this object contains a homeomorphic copy of every planar, one-dimensional, compact, connected set, no matter how complicated that set is. What this means is that, roughly speaking, every bounded curve, containing any number of branch points, as long as the set is one-dimensional, can be deformed to fit in inside the Sierpinski carpet.

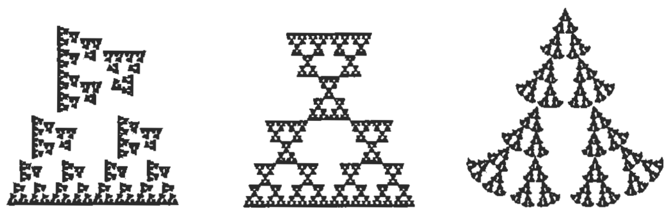

Figure 3a, 3b, and 3c: Sierpinski with rotations.

Here now is a "reverse surprise" (and also an example that does not involve the name Sierpinski). Play the chaos game with four vertices at the corners of a square and a compression ratio of two. After the previous examples, the result of this game is—surprise!—a square. But this is not really a surprise, since the square consists of four self-similar subsquares, each of which is exactly one half the size of the original (in length and width). While the square is not a fractal, it is indeed a self-similar object. An applet called Fractalina can be used to create similar chaos game images. It is available at the Boston University Dynamical Systems and Technology website (`math.bu.edu/DYSYS`).

Fractals

Clearly, self-similarity is only one component in the definition of a fractal. A line segment and a square are self-similar sets, but they are definitely not fractals. The missing ingredient here is fractal dimension: a fractal set must also have fractal dimension that exceeds the set's topological dimension. Without going into details, topological dimension is the "usual" dimension of a set; it is always a nonnegative integer. Fractal dimension gives finer information about the roughness or complexity of a set. Sets like the Sierpinski triangle, hexagon, and carpet have intricate geometries and therefore have fractal dimension larger than one, which is the topological dimension of all three. For more details, consult *Fractals Everywhere* by Barnsley, or *Fractals: A Toolkit of Dynamics Activities* by Choate, Devaney, and Foster. Incidentally, many people believe that a fractal is a set whose fractal dimension is not an integer. This is incorrect: there are many fractals that have integer fractal dimension. The Sierpinski tetrahedron (a tetrahedral analog of the triangle) has fractal dimension two (but topological dimension one).

Rotations

Now let's add rotations to the mix. This is where the geometry of transformations becomes more important. Start with the vertices of a triangle as in the case of the Sierpinski triangle. For the bottom two vertices, the rules are as before: just move half the distance to that vertex when that vertex is called. For the top vertex, the rule is: first move the point half the distance to that vertex, and then rotate the point 90 degrees about the vertex in the clockwise direction. The result of this chaos game is shown in Figure 3a: note that there are basically three self-similar pieces in the fractal, each of which is half the size of the original, but the top one is rotated by 90 degrees in the clockwise direction. Again, as before, we can go backwards and determine the rules of the chaos game that produced the image.

Changing the rotation at the top vertex to 180 degrees yields the image in Figure 3b. This time, the top self-similar piece is rotated 180 degrees. For the fractal in Figure 3c, we rotated twenty degrees in the clockwise direction around the lower left vertex, twenty degrees in the counterclockwise direction around the lower right vertex, and there was no rotation around the top vertex.

Determining the rules of a chaos game that produced a certain image is not easy. In Figure 4 we give you the opportunity to try your hand at this. You must determine the number and locations of the vertices, the compression ratio, and the rotations involved in each case.

Another great source of fun is fractal movie-making. Once you know how to create a single fractal pattern via the chaos game, you can slowly vary some of the rotations, compression ratios, or locations of the vertices to create a fractal movie. We challenge our students to make a movie that is "beautiful" where the underlying rule is hard to figure out. Our students often work for hours to make these animations. Of course, beautiful here means "with a lot of symmetry," so there really is a lot of geometry in this activity. While a magazine may not be the best place to display a movie, several frames from the movie "Dancing Sierpinski" are displayed in Figure 5. Another applet called Fractanimate is available to make these movies at the Boston University Dynamical Systems and Technology website. A number of fractal movies created by students are also posted at this site.

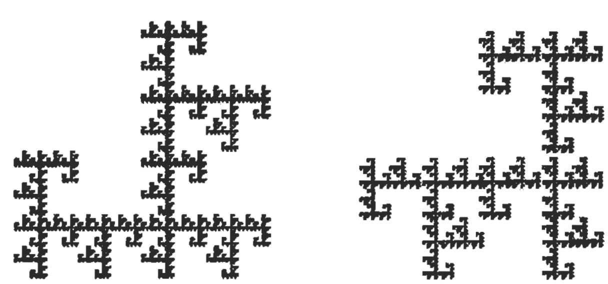

Figure 4: Challenging chaos game images.

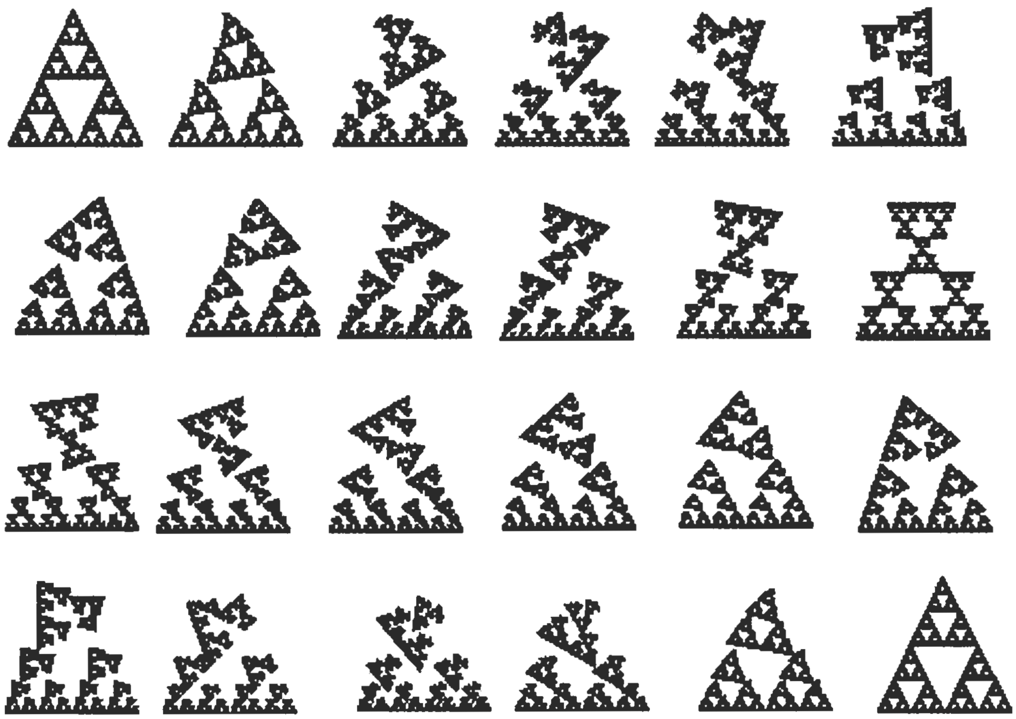

Figure 5: How did we produce this fractal "movie" ?

Probability and Linear Algebra

Up to now, all of the compression ratios we have used in a given chaos game have been the same. When these numbers change, it often becomes necessary to change the probability of choosing a certain vertex. The reason for this is that if a compression ratio at a certain vertex is just slightly larger than one, then we need to choose that vertex over and over in succession in order to fill the entire portion of the fractal corresponding to that vertex. For example, the fractal starfish in Figure 6 was made with just two vertices, one in the upper left corner with compression ratio 5 and one in the center with compression ratio 1.04 and a rotation of 38°. We actually placed 11 vertices all with the same compression ratios and rotations in the center to change the odds of moving toward the center vertex.

All of the chaos games thus far have been specified by giving just the location of the vertices, compression ratio, and rotation. Using linear algebra, we could have specified these rules by providing an affine transformation of the form.

$$T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1/a & 0\\0 & 1/a\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x-x_0\\y-y_0\end{pmatrix}+\begin{pmatrix}x_0\\y_0\end{pmatrix}.$$

Here (x_0, y_0) is the vertex, $a > 1$ is the compression ratio, and θ is the rotation angle. More generally, we could allow any affine transformation, provided that the matrix involved is a contraction. When we allow this, the output of the chaos game produces a much richer collection of fractals, including not only fractals from geometry, but also fractals from nature. The fractal fern in Figure 7 was produced using just four affine transformations which, after some algebraic simplifications, are given by

$$T_1\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0 & 0\\0 & 0.16\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}0\\0\end{pmatrix}$$

$$T_2\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0.85 & 0.04\\-0.04 & 0.85\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}0\\1.64\end{pmatrix}$$

$$T_3\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0.20 & -0.26\\0.23 & 0.22\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}0\\1.6\end{pmatrix}$$

$$T_4\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-0.15 & 0.28\\0.26 & 0.24\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}-0.028\\1.05\end{pmatrix}$$

and probabilities 0.01, 0.85, 0.07, and 0.07 respectively.

Some Applications

Our students never seem to worry about applications of these ideas when they see the fascinating shapes that arise from the chaos game. Nonetheless there are many ways that these are currently being used. One involves data compression. Think about how much data we need to feed into the computer to generate the Sierpinski triangle: just three vertices, a compression ratio of two, and the total number of iterations. That