Problem Section

Editor

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This section features problems for students at the undergraduate and (challenging) high school levels. Problems designated by "S" are especially well-suited for students. Problems designated by "Q" are Quickies, which are answered in this issue.

All problems and/or solutions should be submitted to Andy Liu, Mathematics Department, University of Alberta, Edmonton, Alberta T6G 2G1, Canada. Electronic submissions may also be sent to aliu@math.ualberta.ca. Please include your name, email address, school affiliation, and indicate if you are a student.

Proposals

To be considered for publication, solutions to the following problems should be received by November 10, 2005.

S98. *Proposed by Michael Lanstrum, Cuyahoga Community College, Parma.* Solve the equation $(x^2 - 5x + 5)^{x^2-5x} = 1$.

S99. *Proposed by Mircea Ghita, Flushing.* Let n be a positive integer. Solve the equation

$$\sum_{k=1}^{n} \cos^k kx = \frac{n(n+1)}{2}.$$

S100. *Proposed by Linda Yu, Montreal, Canada.* Lily chooses seven nonnegative numbers with sum 1, and Lala places them around a circle. Lily's score is the highest value among the seven products of adjacent numbers. What is the highest score she can get, with Lala trying to make it as low as possible?

Problem 195. *Proposed by Juan-Bosco Romero Marquez, Universidad de Valladolid, Spain.* Find all solutions (x,y) in positive integers to the equation $(x+y)^x - x^{x+y} = 2$.

Problem 196. *Proposed by Frank Flanigan, San Jose State University, San Jose.* The power series $a_0 + a_1x + a_2x^2 + \cdots$ has nonnegative coefficients not all of which are zero, and is convergent for all real numbers x. Define $b_n = a_n + a_{n-2} + a_{n-4} + \cdots$ where the last term is a_0 or a_1 dependent on the parity of n. Is the power series $b_0 + b_1x + b_2x^2 + \cdots$ also convergent for all real numbers x?

Quickies: (answered on page 34.)

Q4. *Proposed by Karl Havlak, Angelo State University, San Angelo.* Suppose the sum of the 151st term and the 251st term of an arithmetic progression is 10. Determine the sum of the first 401 terms of this progression.

Solutions:

S92. Vidiots and Internuts. *Proposed by the editor.* Moshe Rockshell was hired by the Cybercafé to investigate how much money was taken in during a slow day. The cashier, having absconded with the cash, left behind only the following information.

1. The only customers were a group of 6 vidiots and a group of 13 internuts.
2. The vidiots were in the Cybercafé for twice as many minutes as the internuts.
3. The total bill for the 6 vidiots was the same as the total bill for the 13 internuts.

Based on this and the fact that the Cybercafé charged each customer 20 cents per minute for the first 30 minutes but 30 cents per minutes thereafter, Moshe computed correctly what could have been the total take. Later, when the cashier was apprehended, he admitted that the actual amount was λ times as large as Moshe's figure, where λ was a real number greater than 1. What was the value of λ?

Solution by Pieter J. Torbijn, The Hague, the Netherlands: Suppose the internuts were in the Cybercafé for x minutes where $x \geq 1$. We consider three cases.

1. If $1 \leq x \leq 15$, the total bill for the vidiots would be $6 \cdot 20 \cdot 2x$ cents while that for the internuts would be $13 \cdot 20 \cdot x$ cents. These amounts were clearly unequal.
2. If $16 \leq x \leq 30$, the total bill for the vidiots would be $6(20 \cdot 30 + 30 \cdot (2x - 30))$ cents while that for the internuts would still be $13 \cdot 2 \cdot x$ cents. Equating the two amounts yielded $x = 18$ and the total take could have been \$93.60. This was the figure arrived at by Moshe.
3. If $x \geq 31$, the total bill for the vidiots would still be $6(20 \cdot 30 + 30 \cdot (2x - 30))$ cents while that for the internuts would be $13(2 \cdot 30 + 3 \cdot (x - 30))$ cents. Equating the two amounts yielded $x = 70$ and the actual total take was \$468.00.

Finally,

$$\lambda = \frac{46800}{9360} = 5.$$

Also solved by Armstrong Problem Solvers, Cal Poly Pomona Problem Solving Group, Bill Chen (high school student), Chip Curtis, Lynette Disch (student), Eric Holt (student), Michael Faleski, Yijun Li (student), Mary Washington Problem Solving Group, Abdullah Muhammad, Jacob McMillen (student), Northwestern University Math Problem Solving Group, Austin Outhavong (student), Mike Petty (student), Michael Woltermann and the proposer.

S93. Pythagorean Motion. *Proposed by Will Gosnell.* At the same moment, two particles start respectively from vertices B and C of triangle ABC which has a right angle at C. The particles move at constant speeds and arrive at vertex A at the same moment. If the time of travel is equal to

$$\left(\frac{AB}{AC}\right)^{\frac{3}{2}}$$

and the length of BC is equal to the sum of the constant speeds, determine the time of travel.

Solution by Calcu-Laker Student Problem Solving Group, Mountain Lakes High School, Mountain Lakes.
If t is the time of travel, then

$$BC = \frac{AB}{t} + \frac{AC}{t}$$

so that $(AB + AC)^2 = t^2BC^2 = t^2(AB^2 - AC^2)$. This simplifies to $AB + AC = t^2(AB - AC)$. Since

$$\frac{AB}{AC} = t^{\frac{2}{3}},$$

we have

$$t^{\frac{8}{3}} - t^{\frac{6}{3}} - t^{\frac{2}{3}} - 1 = 0.$$

This factors into

$$\left(t^{\frac{4}{3}} + 1\right)\left(x^{\frac{4}{3}} - t^{\frac{2}{3}} - 1\right) = 0.$$

Seeking positive roots, we have

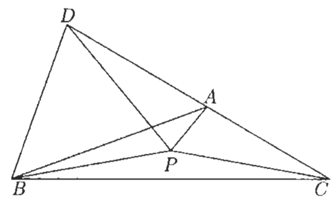
$$t^{\frac{2}{3}} = \frac{1+\sqrt{5}}{2} \text{ and } t = \sqrt{2+\sqrt{5}.}$$

Also solved by Armstrong Problem Solvers, Griffin Brown (student), Cal Poly Pomona Problem Solving Group, Alper Cay, Michael Case (student), Chip Curtis, Yijun Li (student), Mary Washington Problem Solving Group, Hatesh Radia (graduate student), Mark Shattuck (graduate student), Vidyanidhi Vajjhala, Michael Woltermann and the proposer.

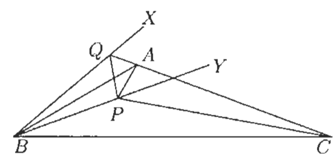
S94. Angle Chasing. *Proposed by Jan van de Craats and the editor.* P is a point inside triangle ABC such that $\angle ABP = \angle PCB = 10°$. Determine $\angle BAP$ if

(a) $\angle PBC = 10°$ and $\angle ACP = 20°$;

(b) $\angle ACP = 10°$ and $\angle PBC = 20°$.



Solution to (a) by Alper Cay, Uzman Private School, Kayseri, Turkey:
Let D be the point on the extension of CA such that $PC = PD$. By the Exterior Angle Theorem, $\angle BPD = 20°+20°+10°+10° = 60°$. Hence BPD is an equilateral triangle, so that $\angle ABD = 60° - 10° = 50°$. By the Angle Sum Theorem, $\angle BAD = 180° - 50° - 60° - 20° = 50°$, so that $AD = BD = PD$. It follows that $\angle PAD = 1/2(180° - 20°) = 80°$, and we have $\angle BAP = 80° - 50° = 30°$.

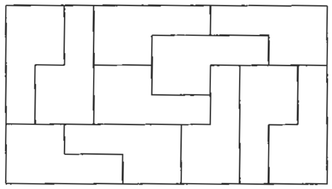


Solution to (b) by David Rhee, student, McNally High School, Edmonton, Canada:
Let Q be the point on the extension of CA such that $\angle QBA = 10°$. Extend BQ to an arbitrary point X and BP to an arbitrary point Y. We have $\angle BQC = 120°$. Since $\angle ABP = \angle PBC$ and $\angle ACP = \angle PCB$, P is the incentre of triangle QBC, so that $\angle BQP = \angle PQC = 60°$. It follows that $\angle CQX = 60°$ also. Now AQ bisects $\angle PQX$ while AB bisects $\angle PBQ$. Hence A is an excentre of triangle BPQ, so that AP bisects $\angle QPY$. Now $\angle QPY = \angle QBP + \angle BQP = 80°$. Hence $\angle APY = 40°$ and $\angle BAP = \angle APY - \angle ABP = 30°$.

Also solved by Cal Poly Pomona Problem Solving Group, Chip Curtis, Mark Shattuck (gradute student), Vidyanidhi Vajjhala, Michael Woltermann and the proposers. All used trigonometry. Alper Cay solved (b) taking D to be the point on the extension of CA such that $CB = CD$. David Rhee solved (a) using the same point Q.

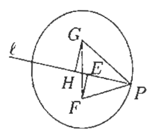
Problem 191. Rectangular Dissection. *Proposed by Sergei Markelov.* A rectangle is dissected into eleven congruent pieces. Prove or disprove that each piece is also a rectangle.

Solution by Bill Chen, student, Illinois Mathematics and Science Academy, Aurora.
The following counterexample shows that the statement is false.



Also solved by Megan Gilroy with Ben Kendrick (students) and the proposer. Both used the same hexomino in the featured counter-example, though their rectangles were assembled in slightly different ways.

Problem 192. An Elliptical Property. *Proposed by Brian Tung.* P is a point on an ellipse and ℓ is the line through P perpendicular to the tangent to the ellipse at P. F and G are the foci of the ellipse, and E and H are the points on ℓ such that EF and GH are perpendicular to ℓ. Prove that $PE \cdot PH$ is constant.



Solution by Eli Maor, Loyola University, Chicago.
Let $\angle EPF = \theta$. Then $\angle HPG = \theta$ also. Now FG is constant. By the Law of the Cosines, so is

$$FG^2 = PF^2 + PG^2 - 2PF{\cdot}PG\cos 2\theta.$$

Now $PF + PG$ is also constant. Hence so is $(PF + PG)^2 = PF^2 + PG^2 + 2PF \cdot PG$. It follows that $2(1-\cos 2\theta)\, PF{\cdot}PG$ is constant. We have

$$PE{\cdot}PH = PF{\cdot}PG\cos^2\theta = 2\,(1-\cos 2\theta)\, PF{\cdot}PG,$$

so that it is also constant.

Also solved by Alper Cay, Cal Poly Pomona Solving Group, Chip Curtis, Mark Shattuck (graduate student), Vidyanidhi Vajjhala, Michael Woltermann and the proposer.

Q4. A Timely Sum. The answer is 2005.

Loose-ends:
The Northwestern University Math Problem Solving Group pointed out that in their featured solution of S89 in the February 2005 issue, brackets were missing from the first sentence, which should have read: "For any real numbers u and v both greater than or equal to 1, we have
$(u+1)/2{\cdot}(v+1)/2+(u-1\ /2{\cdot}(v-1)/2 = (uv+1)/2$. Hence $(uv+1)/2 \geq (u+1)/2{\cdot}(v+1)/2$."

In the April 2005 issue, the solution to Problem 190 was missing its title, which should have been: "The Last Challenge." A solution to the corrected Problem 187(a) was received from Alper Cay. ■

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been reflected by superlative performances in recent Putnam exams.

Since the start of the new millennium, I have been the contest director. Most of the information for the contest can be found on our web pages at `http://www.math.vt.edu/events/competitions/Vtregional/`. In particular, all past exams can be obtained from there, as well as solutions to the more recent ones; it is hoped that solutions to all the exams will eventually be posted. Today we mail out letters of invitation to about 130 universities in the states surrounding Virginia. Even if your university doesn't receive a letter of invitation, you are still welcome to participate. In 2004, we had universities from Arizona, Florida, Georgia, Missouri, and South Carolina participating. If you want to take part, go to our Web page at the beginning of August, when the registration forms for that year should be available. The date of the contest will always be a Saturday at the end of October or beginning of November, to give just enough time to post the results before the closing date for selecting a team for the Putnam competition.

Although we don't encourage high school students to take part, they are eligible to take the exam if a participating university is willing to proctor the exam; on the other hand an undergraduate who already has a degree (say in some other subject) is ineligible. Many of the high school students who have taken part have performed excellently. In fact only twice has a perfect score been obtained in the contest (excluding the two exams in 1977/8) and these were both achieved by high school students. In 1989 Jeffrey Vanderkam performed the feat with 80/80, and in 2002 Daniel Kane did likewise with 70/70. Other notable performances by undergraduates include Mark Roseberry of the University of Louisville who got 79/80 in 1992, and Nicholas Loehr of Virginia Tech who is the only student to win the contest twice, in 1994 and 1997. ■

A Problem from a Previous VTRMC

The VTRC bus company serves cities in the USA. A subset S of the cities is called well-served if it has at least three cities and from every city A in S, one can take a nonstop VTRC bus to at least two different other cities B and C in S (though there is not necessarily a nonstop VTRC bus from B to A or from C to A). Suppose there is a well-served subset S. Prove that there is a well-served subset T such that for any two cities A, B in T, one can travel by VTRC bus from A to B, stopping only at cities in T.

For the solution, visit the *Math Horizons* webpage at `www.maa.org/mathhorizons`.