Mathematics for Literacy

JAN DE LANGE

(In)numeracy

In 1990, a newspaper reported:

> Yesterday, Monday October 9, AVRO Television paid attention to analphabetism in The Netherlands. From data collected for the transmission, it appeared that no fewer than 1 out of 25 people cannot read or write, that is, cannot read or write a shopping list, cannot follow subtitles on TV, cannot read newspapers, cannot write a letter.
>
> Just imagine, 1 out of 25 people, in a country that sends helpers to developing countries in order to teach their folks reading and writing! 1 out of 25, which means 25% of our citizens.
>
> How many citizens does The Netherlands have? 14 million? That means that in our highly developed country no less than three and a half million cannot read or write.
>
> Aren't you speechless?

Speechless, indeed. Errors such as the one above often are not noticed by our literate, educated citizens. Innumeracy, or the inability to handle numbers and data correctly and to evaluate statements regarding problems and situations that invite mental processing and estimating, is a greater problem than our society generally recognizes. According to Treffers (1991), this level of innumeracy might not be the result of content taught (or not taught) but rather the result, at least in part, of the structural design of teaching practices. "Fixing" this problem, however, requires dealing with several issues: From a mathematical perspective, how do we define literacy? Does literacy relate to mathematics (and what kind of mathematics)? What kind of competencies are we looking for? Are these competencies teachable?

Introduction

Before trying to answer the question "What knowledge of mathematics is important?", it seems wise first to look at a "comfortable" definition of quantitative literacy (QL). Lynn Arthur Steen (2001) pointed out that there are small but important differences in the several existing definitions and, although he did not suggest the phrase as a definition, referred to QL as the "capacity to deal effectively with the quantitative aspects of life." Indeed, most existing definitions Steen mentioned give explicit attention to number, arithmetic, and quantitative situations, either in a rather narrow way as in the National Adult Literacy Survey (NCES 1993):

> The knowledge and skills required in applying arithmetic operations, either alone or sequentially, using numbers embedded in printed material (e.g., balancing a checkbook, completing an order form).

Jan de Lange is Director of the Freudenthal Institute at Utrecht University in The Netherlands. A member of the Mathematical Sciences Education Board (MSEB), de Lange's work focuses on modeling and applications in mathematics education, implementation of mathematics curriculum reform, and assessing student learning in mathematics. De Lange is chair of the Expert Group for Mathematics of OECD's new Program for International Student Assessment (PISA).

or more broadly as in the International Life Skills Survey (ILSS 2000):

> An aggregate of skills, knowledge, beliefs, dispositions, habits of mind, communication capabilities, and problem solving skills that people need in order to engage effectively in quantitative situations arising in life and work.

The problem we have with these definitions is their apparent emphasis on *quantity*. Mathematical literacy is not restricted to the ability to apply quantitative aspects of mathematics but involves knowledge of mathematics in the broadest sense. As an example, being a foreigner who travels a great deal in the United States, I often ask directions of total strangers. What strikes me in their replies is that people are generally very poor in what I call navigation skills: a realization of where you are, both in a relative and absolute sense. Such skills include map reading and interpretation, spatial awareness, "grasping space" (Freudenthal 1973), understanding great circle routes, understanding plans of a new house, and so on. All kinds of visualization belong as well to the literacy aspect of mathematics and constitute an absolutely essential component for literacy, as the three books of Tufte (1983, 1990, 1997) have shown in a very convincing way.

We believe that describing what constitutes mathematical literacy necessitates not only this broader definition but also attention to changes within other school disciplines. The Organization for Economic Cooperation and Development (OECD) publication *Measuring Student Knowledge and Skills* (OECD 1999) presents as part of reading literacy a list of types of texts, the understanding of which in part determines what constitutes literacy. This list comes close, in the narrower sense, to describing many aspects of quantitative literacy. The publication mentions, as examples, texts in various formats:

- Forms: tax forms, immigration forms, visa forms, application forms, questionnaires
- Information sheets: timetables, price lists, catalogues, programs
- Vouchers: tickets, invoices, etc.
- Certificates: diplomas, contracts, etc.
- Calls and advertisements
- Charts and graphs; iconic representations of data
- Diagrams
- Tables and matrices
- Lists
- Maps

The definition Steen used in *Mathematics and Democracy: The Case for Quantitative Literacy* (2001) refers to these as "document literacy," following a definition adopted by the National Center for Education Statistics (NCES).

Against this background of varying perspectives, I chose for "mathematical literacy" a definition that is broad but also rather "mathematical":

> Mathematical literacy is an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments, and to engage in mathematics in ways that meet the needs of that individual's current and future life as a constructive, concerned and reflective citizen (OECD 1999).

This definition was developed by the Expert Group for Mathematics of the Programme for International Student Assessment (PISA), of which I am chair. (I will refer to this document repeatedly below.) Later in this essay I further discriminate between the concepts of numeracy, spatial literacy (SL), quantitative literacy (QL), and mathematical literacy (ML). I also try to build an argument that there is a need for consensus on what constitutes *basic* mathematical literacy as distinct from *advanced* mathematical literacy.

"What Mathematics?" Not Yet the Right Question

In an interview in *Mathematics and Democracy,* Peter T. Ewell (2001) was asked: "'The Case for Quantitative Literacy' argues that quantitative literacy (QL) is not merely a euphemism for mathematics but is something significantly different—less formal and more intuitive, less abstract and more contextual, less symbolic and more concrete. Is this a legitimate and helpful distinction?" Ewell answered that indeed this distinction is meaningful and powerful.

The answer to this question depends in large part on the interpretation of what constitutes good mathematics. We can guess that in Ewell's perception, mathematics is formal, abstract, and symbolic—a picture of mathematics still widely held. Ewell continued to say that literacy implies an integrated ability to function seamlessly within a given community of practice. Functionality is surely a key point, both in itself and in relation to a community of practice, which includes the community of mathematicians. Focusing on functionality gives us better opportunity to bridge gaps or identify overlaps. In the same volume, Alan H. Schoenfeld (2001) observed that in the past, literacy and what is learned in mathematics classes were largely disjointed. Now, however, they should be thought of as largely overlapping and taught as largely overlapping. In this approach, which takes into consideration the changing perception of what constitutes mathematics, mathematics and mathematical literacy are positively not disjointed.

For Schoenfeld, the distinction most likely lies in the fact that as a student he never encountered problem-solving situations, that he studied only "pure" mathematics and, finally, that he never saw or worked with real data. Each of these is absolutely essential for literate citizenship, but none even hints at defining what mathematics is needed for ML, at least not in the traditional school mathematics curricula descriptions of arithmetic, algebra, geometry, and so on.

Again, in *Mathematics and Democracy,* Wade Ellis, Jr. (2001) observes that many algebra teachers provide instruction that constricts rather than expands student thinking. He discovered that students leaving an elementary algebra course could solve fewer real-world problems after the course than before it: after completing the course, they thought that they had to use symbols to solve problems they had previously solved using only simple reasoning and arithmetic. It may come as no surprise that Ellis promotes a new kind of common sense—a quantitative common sense based on mathematical concepts, skills, and know-how. Despite their differences, however, Schoenfeld and Ellis seem to share Treffers' observation that innumeracy might be caused by a flaw in the structural design of instruction.

These several observers seem to agree that in comparison with traditional school mathematics, ML is less formal and more intuitive, less abstract and more contextual, less symbolic and more concrete. ML also focuses more attention and emphasis on reasoning, thinking, and interpreting as well as on other very mathematical competencies. To get a better picture of what is involved in this distinction, we first need to describe what Steen (2001) called the "elements" needed for ML. With a working definition of ML and an understanding of the elements (or "competencies," as they are described in the PISA framework) needed for ML, we might come closer to answering our original question—what mathematics is important?—or formulating a better one.

Competencies Needed for ML

The competencies that form the heart of the ML description in PISA seem, for the most part, well in line with the elements in Steen (2001). The competencies rely on the work of Niss (1999) and his Danish colleagues, but similar formulations can be found in the work of many others representing many countries (as indicated by Neubrand et al. 2001):

1. *Mathematical thinking and reasoning.* Posing questions characteristic of mathematics; knowing the kind of answers that mathematics offers, distinguishing among different kinds of statements; understanding and handling the extent and limits of mathematical concepts.

2. *Mathematical argumentation.* Knowing what proofs are; knowing how proofs differ from other forms of mathematical reasoning; following and assessing chains of arguments; having a feel for heuristics; creating and expressing mathematical arguments.

3. *Mathematical communication.* Expressing oneself in a variety of ways in oral, written, and other visual form; understanding someone else's work.

4. *Modeling.* Structuring the field to be modeled; translating reality into mathematical structures; interpreting mathematical models in terms of context or reality; working with models; validating models; reflecting, analyzing, and offering critiques of models or solutions; reflecting on the modeling process.

5. *Problem posing and solving.* Posing, formulating, defining, and solving problems in a variety of ways.

6. *Representation.* Decoding, encoding, translating, distinguishing between, and interpreting different forms of representations of mathematical objects and situations as well as understanding the relationship among different representations.

7. *Symbols.* Using symbolic, formal, and technical language and operations.

8. *Tools and technology.* Using aids and tools, including technology when appropriate.

To be mathematically literate, individuals need all these competencies to varying degrees, but they also need confidence in their own ability to use mathematics and comfort with quantitative ideas. An appreciation of mathematics from historical, philosophical, and societal points of view is also desirable.

It should be clear from this description why we have included functionality within the mathematician's practice. We also note that to function well as a mathematician, a person needs to be literate. It is not uncommon that someone familiar with a mathematical tool fails to recognize its usefulness in a real-life situation (Steen 2001, 17). Neither is it uncommon for a mathematician to be unable to use common-sense reasoning (as distinct from the reasoning involved in a mathematical proof).

As Deborah Hughes Hallett (2001) made clear in her contribution to *Mathematics and Democracy,* one of the reasons that ML is hard to acquire and hard to teach is that it involves insight as well as algorithms. Some algorithms are of course necessary: it is difficult to do much analysis without knowing arithmetic, for example. But learning (or memorizing) algorithms is not enough: insight is an essential component of mathematical understanding. Such insight, Hughes Hallett noted, connotes an understanding of quantitative relationships and the ability to identify those relationships in an unfamiliar context; its acquisition involves reflection, judgment, and above all, experience. Yet current school curricula seldom emphasize insight and do little to actively support its development at any level. This is very unfortunate. The development of insight into mathematics should be actively supported, starting before children enter school.

Many countries have begun to take quite seriously the problems associated with overemphasizing algorithms and neglecting insight. For example, the Netherlands has had some limited success in trying to reform how mathematics is taught. To outsiders, the relatively high scores on the Third International Mathematics and Science Study (TIMSS) and TIMSS-R by students in the Netherlands appear to prove this, but the results of the Netherlands in the PISA study should provide even more proof.

The Netherlands has been helped in moving away from the strictly algorithmic way of teaching mathematics by the recognition that mathematical abilities or competencies can be clustered: one cluster includes reproduction, algorithms, definitions, and so on; another cluster encompasses the ability to make connections among different aspects or concepts in mathematics to solve simple problems; and a third cluster includes insight, reasoning, reflection, and generalization as key components (de Lange 1992, 1995). In designing curricula and assessments as well as items for international examinations, this clustering approach became a mirror reflecting back to us what we thought constituted good mathematics in the sense of competencies. To a large extent, this approach also prevented the very present danger of viewing the National Council of Teachers of Mathematics (NCTM) goals—reasoning, communication, and connections—as merely rhetoric (Steen 2001). Eventually, this clustering of mathematical competencies found its way into the present OECD PISA study (1999) as well as into a classroom mathematics assessment framework (de Lange 1999) and an electronic assessment tool (Cappo and de Lange 1999).

Finally, we want to make the observation that the competencies needed for ML are actually the competencies needed for mathematics *as it should be taught.* Were that the case (with curricula following the suggestions made by Schoenfeld and Hughes Hallett and extrapolating from experiences in the Netherlands and other countries), the gap between mathematics and mathematical literacy would be much smaller than some people suggest it is at present (Steen 2001). It must be noted, however, that in most countries this gap is quite large and the need to start thinking and working toward an understanding of what makes up ML is barely recognized. As Neubrand et al. (2001) noted in talking about the situation in Germany: "In actual practice of German mathematics education, there is no correspondence between the teaching of mathematics as a discipline and practical applications within a context" (free translation by author).

What Is Mathematics?

To provide a clearer picture of literacy in mathematics, it seems wise to reflect for a moment on what constitutes mathematics. Not that we intend to offer a deep philosophical treatment—there are many good publications around—but it is not unlikely that many readers might think of school mathematics as representing mathematics as a science. Several authors in *Mathematics and Democracy* (Steen 2001) clearly pointed this out, quite often based on their own experiences (Schoenfeld, Schneider, Kennedy, and Ellis, among others). Steen (1990) observed in *On the Shoulders of Giants: New Approaches to Numeracy* that traditional school mathematics picks a very few strands (e.g., arithmetic, algebra, and geometry) and arranges them horizontally to form the curriculum: first arithmetic, then simple algebra, then geometry, then more algebra and, finally, as if it were the epitome of mathematical knowledge, calculus. Each course seems designed primarily to prepare for the next. These courses give a distorted view of mathematics as a science, do not seem to be related to the educational experience of children, and bear no relevance for society. A result of this is that the informal development of intuition along the multiple roots of mathematics, a key characteristic in the development of ML, is effectively prevented. To overcome this misimpression about the nature of mathematics left by such courses, we will try to sketch how we see mathematics and, subsequently, what the consequences can be for mathematics education.

Mathematical concepts, structures, and ideas have been invented as tools to organize phenomena in the natural, social, and mental worlds. In the real world, the phenomena that lend themselves to mathematical treatment do not come organized as they are in school curriculum structures. Rarely do real-life problems arise in ways and contexts that allow their understanding and solutions to be achieved through an application of knowledge from a single content strand. If we look at mathematics as a science that helps us solve real problems, it makes sense to use a phenomenological approach to describe mathematical concepts, structures, and ideas. This approach has been followed by Freudenthal (1973) and by others such as Steen (1990), who state that if mathematics curricula featured multiple parallel strands, each grounded in appropriate childhood experiences, the collective effect would be to develop among children diverse mathematical insight into the many different roots of mathematics. Steen then suggested that we should seek inspiration in the developmental power of five deep mathematical ideas: dimension, quantity, uncertainty, shape, and change. The OECD PISA mathematics expert group has adapted these, creating four phenomenological categories to describe what constitutes mathematics: *quantity*, *space* and *shape*, *change* and *relationships*, and *uncertainty*.

Using these four categories, mathematics content can be organized into a sufficient number of areas to help ensure a spread of items across the curriculum, but also a small enough number to avoid an excessively fine division—which would work against a focus on problems based in real-life situations. Each phenomenological category is an encompassing set of phenomena and concepts that make sense together and may be encountered within and across a multitude of quite different situations. By their very nature, each idea can be perceived as a general notion dealing with a generalized content dimension. This implies that the categories or ideas cannot be sharply delineated vis-à-vis one another. Rather, each represents a certain perspective, or point of view, which can be thought of as possessing a core, a center of gravity, and a somewhat blurred penumbra that allow intersection with other ideas. In principle, any idea can intersect with any other idea. (For a more detailed description of these four categories or ideas, please refer to the PISA framework (OECD 2002).)

Quantity. This overarching idea focuses on the need for quantification to organize the world. Important aspects include an understanding of relative size, recognition of numerical patterns, and the ability to use numbers to represent quantifiable attributes of real-world objects (measures). Furthermore, quantity deals with the processing and understanding of numbers that are represented to us in various ways. An important aspect of dealing with quantity is quantitative reasoning, whose essential components are developing and using number sense, representing numbers in various ways, understanding the meaning of operations, having a feel for the magnitude of numbers, writing and understanding mathematically elegant computations, doing mental arithmetic, and estimating.

Space and Shape. Patterns are encountered everywhere around us: in spoken words, music, video, traffic, architecture, and art. Shapes can be regarded as patterns: houses, office buildings, bridges, starfish, snowflakes, town plans, cloverleaves, crystals, and shadows. Geometric patterns can serve as relatively simple models of many kinds of phenomena, and their study is desirable at all levels (Grünbaum 1985). In the study of shapes and constructions, we look for similarities and differences as we analyze the components of form and recognize shapes in different representations and different dimensions. The study of shapes is closely connected to the concept of "grasping space" (Freudenthal 1973)—learning to know, explore, and conquer, in order to live, breathe, and move with more understanding in the space in which we live. To achieve this, we must be able to understand the properties of objects and the relative positions of objects; we must be aware of how we see things and why we see them as we do; and we must learn to navigate through space and through constructions and shapes. This requires understanding the relationship between shapes and images (or visual representations) such as that between a real city and photographs and maps of the same city. It also includes understanding how three-dimensional objects can be represented in two dimensions, how shadows are formed and interpreted, and what perspective is and how it functions.

Change and Relationships. Every natural phenomenon is a manifestation of change, and in the world around us a multitude of temporary and permanent relationships among phenomena are observed: organisms changing as they grow, the cycle of seasons, the ebb and flow of tides, cycles of unemployment, weather changes, stock exchange fluctuations. Some of these change processes can be modeled by straightforward mathematical functions: linear, exponential, periodic or logistic, discrete or continuous. But many relationships fall into different categories, and data analysis is often essential to determine the kind of relationship present. Mathematical relationships often take the shape of equations or inequalities, but relations of a more general nature (e.g., equivalence, divisibility) may appear as well. Functional thinking—that is, thinking in terms of and about relationships—is one of the fundamental disciplinary aims of the teaching of mathematics. Relationships can take a variety of different representations, including symbolic, algebraic, graphic, tabular, and geometric. As a result, translation between representations is often of key importance in dealing with mathematical situations.

Uncertainty. Our information-driven society offers an abundance of data, often presented as accurate and scientific and with a degree of certainty. But in daily life we are confronted with uncertain election results, collapsing bridges, stock market crashes, unreliable weather forecasts, poor predictions of population growth, economic models that do not align, and many other demonstrations of the uncertainty of our world. Uncertainty is intended to suggest two related topics: data and chance, phenomena that are the subject of mathematical study in statistics and probability, respectively. Recent recommendations concerning school curricula are unanimous in suggesting that statistics and probability should occupy a much more prominent place than they have in the past (Cockroft 1982; LOGSE 1990; MSEB 1993; NCTM 1989, 2000). Specific mathematical concepts and activities that are important in this area include collecting data, data analysis, data display and visualization, probability, and inference.

The Real World

Although we now have "answers" to what constitutes ML, what the needed skills or competencies are, and what mathematics is, we still are not in a position to give an answer to what mathematics is needed for ML. The reason is simple: mathematics curricula have focused on school-based knowledge whereas mathematical literacy involves mathematics as it is used in the real world.

An important part of mathematical literacy is using, doing, and recognizing mathematics in a variety of situations. In dealing with issues that lend themselves to a mathematical treatment, the choice of mathematical methods and representations often depends on the situations in which the problems are presented. Teachers of mathematics often complain that students have difficulty applying the mathematics they have learned in different contexts. As Hughes Hallett (2001) correctly observed, nonscience students often dislike contexts involving physics applications in mathematics because they do not understand the physics. Building from this, I think we need to examine the wisdom of confronting nonscience students with mathematics applications that need specific science literacy at a nonbasic level. As has been pointed out before, to effectively transfer their knowledge from one area of application to another, students need experience solving problems in many different situations and contexts (de Lange 1987). Making competencies a central emphasis facilitates this process: competencies are independent of the area of application. Students should be offered real-world situations relevant to them, either real-world situations that will help them to function as informed and intelligent citizens or real-world situations that are relevant to their areas of interest, either professionally or educationally.

By *situation,* we mean the part of the student's world in which a certain problem is embedded. It is very convenient and relevant to the art of teaching for ML to see situations as having certain distances in relation to the student (de Lange 1995; OECD 1999, 2002). The closest distance is the student's personal life; next is school (educational) life, then work (occupational) and leisure, followed by the local community and society as encountered in daily life. Furthest away are scientific situations. It might be desirable to enlarge the distance domain as the age of the students increases, but not in a strict way.

Steen (2001, 9–15) itemized an impressive list of expressions of numeracy, most of which can be seen as having a certain "distance" from "citizens." Under personal life we include, depending on age, games, daily scheduling, sports, shopping, saving, interpersonal relations, finances, voting, reading maps, reading tables, health, insurance, and so on. School life relates to understanding the role of mathematics in society, school events (e.g., sports, teams, scheduling), understanding data, computers, and so on. Work and leisure involves reasoning, understanding data and statistics, finances, taxes, risks, rates, samples, scheduling, geometric patterns, two- and three-dimensional representations, budgets, visualizations, and so on. In the local community, we see the intelligent citizen making appropriate judgments, making decisions, evaluating conclusions, gathering data and making inferences, and in general adopting a critical attitude—seeing the reasoning behind decisions.

Last, we come to science situations. To function as an intelligent citizen, individuals need to be literate in many fields, not only in mathematics. The use of scientific situations or contexts in mathematics classes should not be avoided per se, but some care must be taken. If we try to teach students the right competencies but use the wrong context, we are creating a problem, not solving it. A good but rather unscientific example concerns work with middle-school students in the United States. The designed lesson sequence had archeology as a context. Archeologists sometimes use rather straightforward but quite unexpected and rather "subjective" mathematical methods in their research—just the kind of mathematics middle school students can handle. The question, therefore, was not whether the students could do the mathematics but whether the context was engaging enough in this short-attention-span society. The students were highly engaged because of the unexpectedness of what they were learning and the relevance of the methods used. As we learned in this instance, connecting to the students' real world can be a complex but highly rewarding journey.

What has become clear in dealing with mathematics in context over the past 25 years is that making mathematics relevant by teaching it in context is quite possible and very rewarding, despite the many pitfalls. We note that much more experience and research is needed, but based on previous experiences we also note that teaching for both mathematical literacy and relevant mathematics at almost the same time might very well prove feasible.

A Matter of Definitions

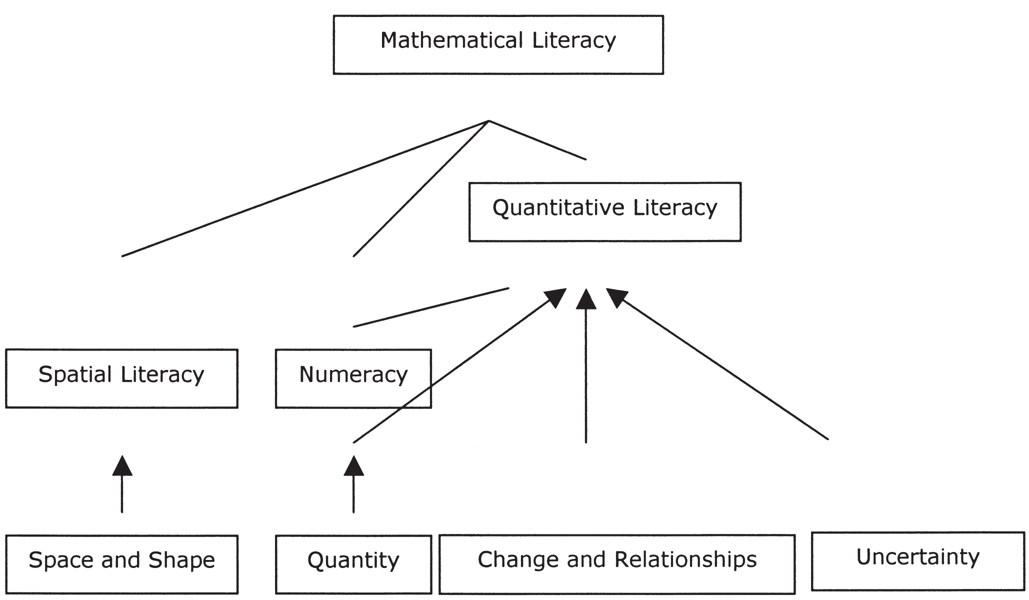
Having set the context, it seems appropriate now to make clear distinctions among types of literacies so that, at least in this essay, we do not declare things equal that are not equal. For instance, some equate numeracy with quantitative literacy; others equate quantitative and mathematical literacy. To make our definitions functional, we connect them to our phenomenological categories.

Spatial Literacy (SL). We start with the simplest and most neglected, spatial literacy. SL supports our understanding of the (three-dimensional) world in which we live and move. To deal with what surrounds us, we must understand properties of objects, the relative positions of objects and the effect thereof on our visual perception, the creation of all kinds of two- and three-dimensional paths and routes, navigational practices, shadows—even the art of Escher.

Numeracy (N). The next obvious literacy is numeracy (N), fitting as it does directly into quantity. We can follow, for instance, Treffers' (1991) definition, which stresses the ability to handle numbers and data and to evaluate statements regarding problems and situations that invite mental processing and estimating in real-world contexts.

Quantitative Literacy (QL). When we look at quantitative literacy, we are actually looking at literacy dealing with a cluster of phenomenological categories: quantity, change and relationships, and uncertainty. These categories stress understanding of, and mathematical abilities concerned with, certainties (quantity), uncertainties (quantity as well as uncertainty), and relations (types of, recognition of, changes in, and reasons for those changes).

Mathematical Literacy (ML). We think of mathematical literacy as the overarching literacy comprising all others. Thus we can make a visual representation as follows:



Advanced Mathematical Literacy and Basic Mathematical Literacy. Another possibly fruitful way to make distinctions within the field of mathematical literacy is to think about the "community of practitioners" in somewhat more detail. Being mathematically literate means different things according to the needs of the community, both as a group and as individuals. It may be a good idea, although well beyond the comfort zone for many, to speak of *basic mathematical literacy* (BML), a level expected of all students up to age 15 or so, independent of their role in society. Individual countries or communities should be able to define in some detail what this actually means in the local culture. After age 15, however, as students begin to think of their future careers, they should, accordingly, acquire *advanced mathematical literacy* (AML), defined by their need to fit into their community of practice. Because of the many different communities of practice in a given society, defining the general content for career-related AML may be unwise, if not impossible. But defining an early career-entry AML for high school students and undergraduates might be appropriate, as might defining a general AML for adult life in society, linking its development, support, and enhancement to continuing education for adults.

Examples: The Mathematics Necessary for Development of ML

Examples from real curricula offer the best illustrations of mathematics that meet at least some of the requirements of ML. It will come as no surprise, perhaps, that the examples offered here, andour frame of reference, will be the Netherlands, but with an eye to the U.S. situation.

In the 1970s, it became increasingly clear that there was a serious mismatch in the Netherlands between what many students needed and what was offered to them in mathematics curricula, a mismatch still present in many curricula. Traditional curricula, for example, include calculus taught in a way that seldom leads to any understanding of its power or usefulness and seldom either develops students' ability to reason with it as a tool or develops it with an eye to mathematical or scientific proof. Few mathematics educators see any merit in this approach (see de Lange 1994). Yet despite the fact that calculus has few easily accessible applications outside the exact sciences, it has survived "for all," albeit in very different shapes, in upper secondary curricula.

This mismatch is particularly acute for the majority of students who do not want to pursue university study in the exact sciences but who need mathematics for economics, biological sciences, language, arts, social sciences, and so on. More generally, this mismatch affects all future members of our society, which depends so heavily on mathematics and technology. In the early 1980s, a specific curriculum was developed in the Netherlands to meet the need for more general, socially relevant mathematical knowledge (ML). The political reasons were simple and clear: all students needed mathematics, but what they needed to study was the mathematics required to function well in society and the concepts and areas relevant to their future work and study. As part of this change, curricula differing in mathematical content, level of formality, context, and even (to a certain extent) pedagogy were created to fit the needs of different clusters of students beyond the age of 14. To convey the nature of this change, we give below some concrete examples of this mathematics, presented in the order of our phenomenological categories.

Quantity

The Defense Budget. In a certain country, the defense budget was $30 million for 1980. The total budget for that year was $500 million. The following year, the defense budget was $35 million, whereas the total budget was $605 million. Inflation during the period between the two budgets was 10 percent.

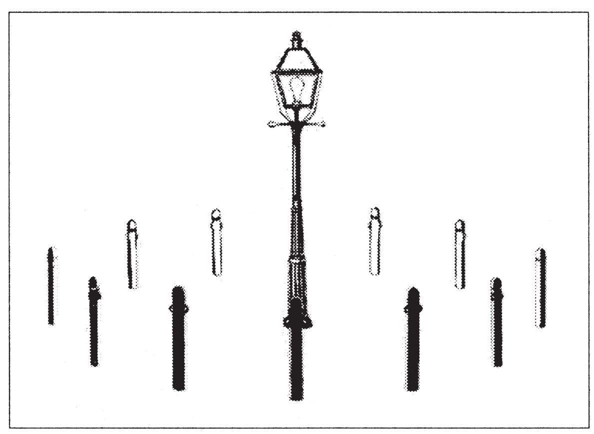
(a) You are invited to hold a lecture for a pacifist society. You want to explain that the defense budget has decreased this year. Explain how to do this.

(b) You are invited to lecture to a military academy. You want to claim that the defense budget has increased this year. Explain how to do this (de Lange 1987).

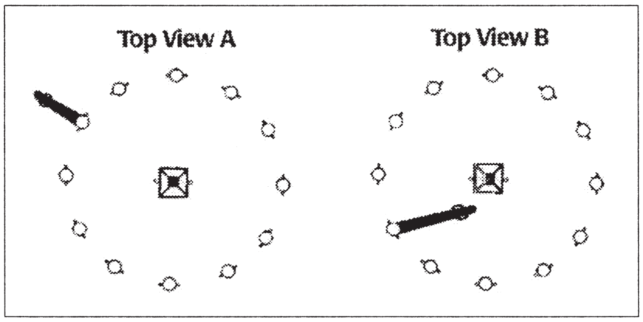
This problem has been thoroughly researched with 16-year-old students. It illustrates very well the third cluster on reflection and insight. Students recognized the literacy aspect immediately and quite often were able to make some kind of generalization; the heart of the solution lies in recognizing that the key mathematical concepts here are absolute and relative growth. Inflation can of course be left out to make the problem accessible to somewhat younger students without losing the key conceptual ideas behind the problem, but doing so reduces the complexity and thus the required mathematization. Another way to make the item simpler is to present the data in a table or schema. In this case, students have no preliminary work to carry out before they get to the heart of the matter.

Space and Shape

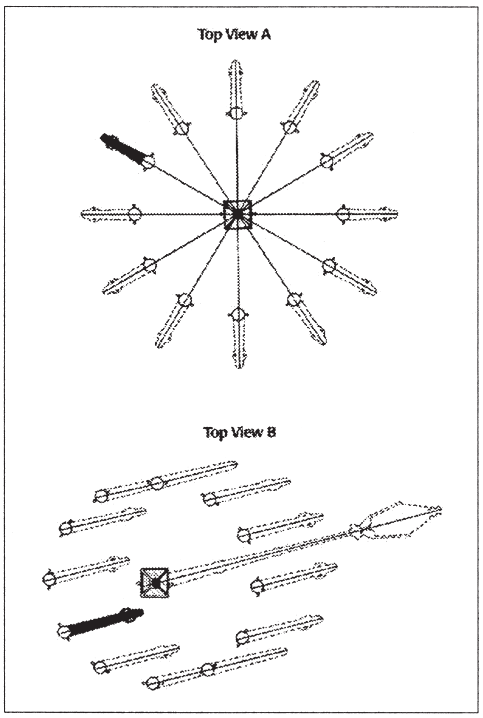
Casting Shadows. We first show an example of basic spatial literacy that reflects a well-known daily experience, but one in which people seldom realize what they see. The variety of shadows cast by the sun (or a light bulb) is an interesting starting point for a wide array of mathematical questions that have a much wider impact than people initially realize. Students first are introduced to a picture of an outdoor lamp surrounded by posts (Feijs 1998):



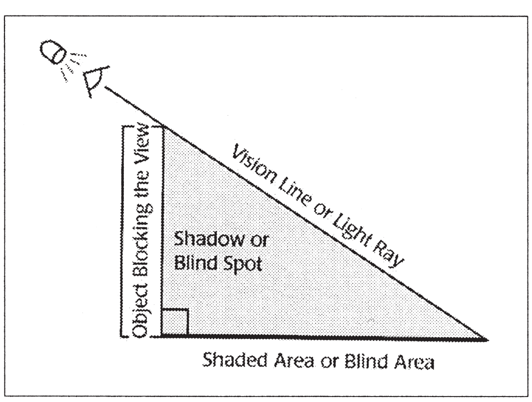
They then are asked to draw the shadows created by the lamp (Top View A) and also the shadows cast by the sun (Top View B):



The answers to both questions are rather straightforward:



They are based on understanding the path of light rays in relation to the objects that are casting shadows:



This example not only illustrates aspects of mathematical literacy as it deals with the world in which we live but also can stimulate thinking about deeper mathematical ideas that are not immediately evident, for example, parallel and central projection, vision lines, blind spots, and ratios.

The second example dealing with space and shape is taken from the curriculum for students preparing for study in the exact sciences.

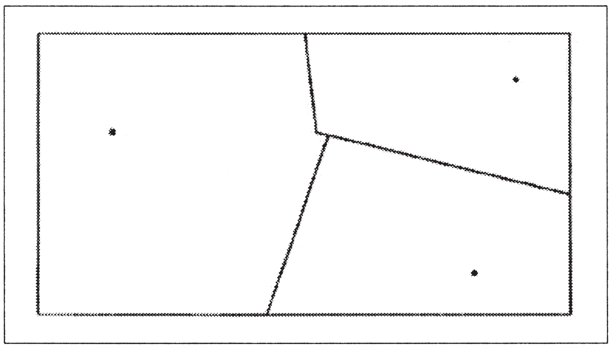
Equal Distances. The economies of some countries depend on their fishing industries. Other countries are interested in the ocean for reasons of oil drilling rights. How do we establish "fair" rules about who gets what and for what reasons? If we are considering a straight canal with two different countries on both sides, it seems obvious: the line through the middle of the canal forms the boundary because it is the line with equal distances to both countries. The political and societal relevance of this and similar questions is immediately clear. But how many people understand the logic and common sense behind the rules?

The part of geometry that deals with equal distances is sometimes called Voronoi geometry. It is an area of mathematics that has relevance to practical (and very often political) problems and also offers experience in useful mathematical reasoning. Unlike the previous example, which illustrated Basic ML, this one is for senior high school students, taken from students' materials at a Dutch high school.

The illustration shows the Netherlands portioned by use of a Voronoi diagram: all cities (city centers) are equidistant to the borders.

(a) What do you know about the distances from A to Middelburg, from A to Den Haag and from A to Den Bosch?

(b) Explain why a Voronoi diagram with three points can never look like the following diagram (Goddijn 1997, taken out of context and order).



These examples form the starting point for a sequence of very interesting space and shape mathematics that is very relevant from a societal point of view (e.g., fishing rights, oil-drilling rights). But Voronoi diagrams have much more to offer in the mathematical sense. For example, although the last question above is not of immediate relevance to fishing or oil-drilling rights, it requires coherent, competent, and consistent reasoning, which is at least as important as the first questions for intelligent citizens to function in their societies.

Change and Relationships

Cheetahs and Horses. Some animals that dwell on grassy plains are safeguarded against attacks by their large size; others are so small that they can protect themselves by burrowing into the ground. Still others must count on speed to escape their enemies.

An animal's speed depends on its size and the frequency of its strides. The tarsal (foot) bone of animals of the horse family is lengthened, with each foot having been reduced to only one toe. One thick bone is stronger than a number of thin ones. This single toe is surrounded by a solid hoof, which protects the bone against jolts when the animal is galloping over hard ground. The powerful leg muscles are joined together at the top of the leg so that just a slight muscle movement at that point can freely move the slim lower leg.

The fastest sprinter in the world is the cheetah. Its legs are shorter than those of a horse, but it can reach a speed of more than 110 km/h in 17 seconds and maintain that speed for more than 450 meters. The cheetah tires easily, however, whereas a horse, whose top speed is 70 km/h, can maintain a speed of 50 km/h for more than 6 km.

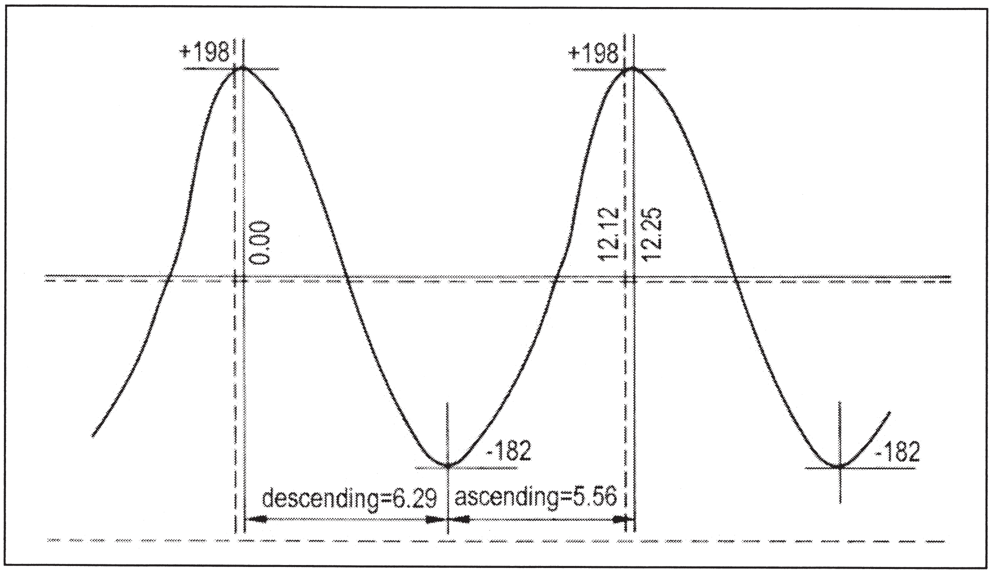
A cheetah is awakened from its afternoon nap by a horse's hooves. At the moment the cheetah decides to give chase, the horse has a lead of 200 meters. The horse, traveling at its top speed, still has plenty of energy. Taking into consideration the above data on the running powers of the cheetah and the horse, can the cheetah catch the horse? Assume that the cheetah will need around 300 meters to reach its top speed. Solve this problem by using graphs. Let the vertical axis represent distance and the horizontal axis time (Kindt 1979, in de Lange 1987).

As Freudenthal (1979) lamented:

> This story of the cheetah seems rather complex. There is an abundance of numbers . . . and nowhere an indication of which operation to perform on which numbers. Indeed, is there anything like a solution? The only question to be answered is, "Does the cheetah catch up with the horse?" It is "yes" or "no"—no numbers, no kilometers, no seconds. Is that a solution in the usual sense? (free translation by author).

According to Freudenthal, this is what mathematics is all about, especially mathematics for ML. This example also shows how we can introduce students to calculus. Calculus needs to be perceived as "the science that keeps track of changes," as a student once characterized it. A qualitative discussion about rates of change can be very illuminating for students and at the same time enable mathematics to contribute to ML. It prevents students from perceiving calculus as that part of mathematics in which "you take the exponent, put it in front, and the new exponent is one less than the original one." Another student in the course in a nonmathematics-related major, who was not very successful in traditional mathematics, answered: "Differentiation is about how to keep check on rates of change." Part of the importance of ML can be seen in the gap between these two answers.

Tides. Natural phenomena should play a vital role in mathematics for ML. For a country like the Netherlands, with 40 percent of its area below sea level, the tides are very important. The following protocol is taken from a classroom of 16-year-olds (de Lange 2000):



Teacher: Let's look at the mean tidal graph of Flushing. What are the essentials?

Student A: High water is 198 cm, and low is –182 cm.

Teacher: And? [pause]

Student A: So it takes longer to go down than up.

Teacher: What do you mean?

Student A: Going down takes 6 hours 29 minutes, up only 5 hours 56 minutes.

Teacher: OK. And how about the period?

Student A: 6 hours 29 and 5 hours 56 makes 12 hours 25 minutes.

Teacher: Now, can we find a simple mathematical model?

Student B: [pause] Maybe 2 sin x.

Teacher: What is the period, then?

Student B: 2 pi, that means around 6.28 hours [pause] 6 hours 18 minutes [pause] oh, I see that it must be 2x, or, no, x/2.

Teacher: So?

Student C: 2 sin (x/2) will do.

Teacher: Explain.

Student C: Well, that's simple: the maximum is 2 meters, the low is –2 meters, and the period is around 12 hours 33 minutes or so. That's pretty close, isn't it?

Teacher: [to the class] Isn't it?

Student D: No, I don't agree. I think the model needs to show exactly how high the water can rise. I propose 190 sin (x/2) + 8. In that case, the maximum is exactly 198 cm, and the minimum exactly –182 cm. Not bad.

Teacher: Any comments, anyone? [some whispering, some discussion]

Student E: I think it is more important to be precise about the period. 12 hours 33 minutes is too far off to make predictions in the future about when it will be high water. We should be more precise. I think 190 sin [(pi/6.2)x] is much better.

Teacher: What's the period in that case?

Student F: 12.4 hours, or 12 hours 24 minutes, only 1 minute off.

Teacher: Perfect. What model do we prefer? [discussion]

Student G: 190 sin [(pi/6.2)x] + 8.

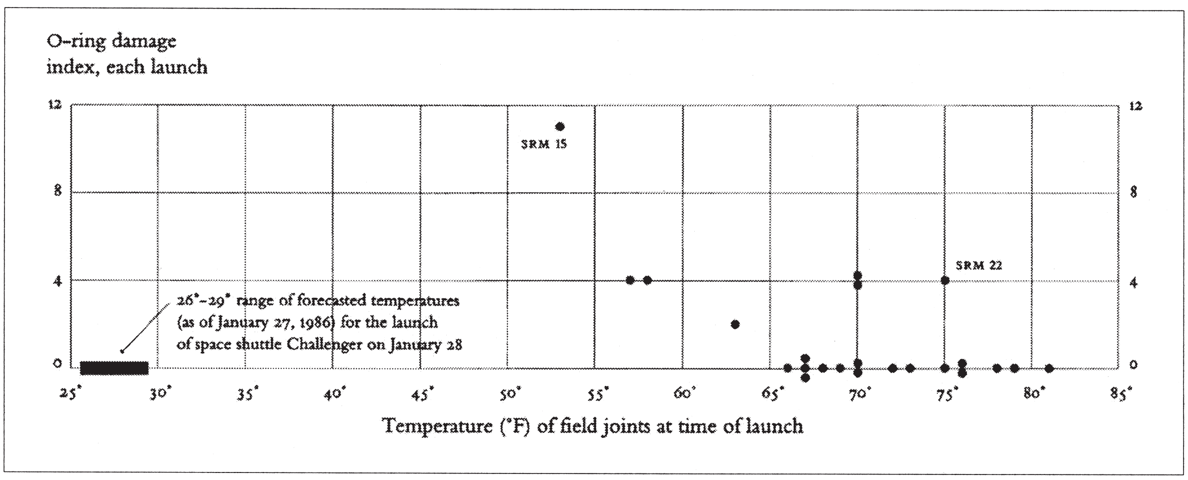
The discussion continued with "What happens if we go to a different city that has smaller amplitudes and where high tides come two hours later? How does this affect the formula? "Why is the rate of change so important?"

Why do we consider this a good example of mathematics for ML? Given the community in which this problem is part of the curriculum, the relevance for society is immediately clear—and the relevance is rising with global temperatures. The relevance also becomes clear at a different level, however. The mathematical method of trial and error illustrated here not only is interesting by itself, but the combination of the method with the most relevant variables also is interesting: in one problem setting we are interested in the exact time of high water, in another we are interested in the exact height of the water at high tide. Intelligent citizens need insight into the possibilities and limitations of models. This problem worked very well for these students, age 16, and the fact that the "real" model used 40 different sine functions did not really make that much difference with respect to students' perceptions.

Uncertainty

Challenger. If we fail to pose problems properly or fail to seek essential data and represent them in a meaningful way, we can very easily drown in data. One dramatic example concerns the advice of the producer of solid rocket motors (SRM) to NASA concerning the launch of the space shuttle Challenger in 1986. The recommendation issued the day before the launch was *not* to launch if the temperature was less than 53 degrees Fahrenheit; the low temperature (29 degrees) that was predicted for the day of the launch might produce risks. As beautifully laid out by Tufte (1997), the fax supporting the recommendation was an excellent example of failed mathematical and common-sense reasoning. Instead of looking at the data on all 24 previous launches, the fax related to only two actual launches (giving temperatures, with ensuing damage to rubber O-rings). NASA, of course, refused to cancel the launch based on the arguments found in the fax. Simple mathematics could have saved the lives of the seven astronauts.

The scientists at Morton Thiokol, producer of the O-rings, were right in their conclusion but were unable to find a correlation between O-ring damage and temperature. Let us look at the problem systematically. The first thing to do if we suspect a correlation is to look at all the data available, in this case, the temperatures at the time of launch for all 24 launches and the ensuing damage to the O-rings. At that point, we then order the entries by possible cause: temperature at launch, from coolest to warmest. Next, for each launch, we calculate the damage to the O-rings and then draw a scatter plot showing the findings from all 24 launches prior to the Challenger.



In this graph, the temperature scale extends down to 29 degrees, visually expressing the extraordinary extrapolation (beyond all previous experience) that had to be made to "see" the launch at 29 degrees. The coolest flight without any O-ring damage was at 66 degrees, some 37 degrees warmer than that predicted for the Challenger; the forecast of 29 degrees was 5.7 standard deviations distant from the average temperature on previous launches. This launch was completely outside the engineering database accumulated in 24 previous flights. The result: the O-rings had already failed before the rocket was launched.

What Mathematics Education Supports the Development of ML?

These examples, taken from the classroom, show what kinds of problems students need to work with to learn good mathematics while at the same time becoming mathematically literate. In a general sense, we agree with Steen (2001) that deploying mathematics in sophisticated settings such as modern work-based tasks gives students not only motivation and context but also a concrete foundation from which they can later abstract and generalize. As these examples show, however, it is not necessary to restrict ourselves to work-based settings. Not every setting lends itself equally to the successful development of mathematical concepts. The importance of choosing appropriate contexts is well documented (e.g., de Lange 1987; Feijs in press), and the issues involved in selecting situations to develop mathematical concepts are quite different from those involved in choosing contexts "just" for application. This is a very tricky area, and much more research is needed before we can make general statements, but the view expressed here is not innovative in any sense. For example, in 1962, some 75 well-known U.S. mathematicians produced a memorandum, "On the Mathematics Curriculum of the High School," published in the *American Mathematical Monthly*:

> To know mathematics means to be able to do mathematics: to use mathematical language with some fluency, to do problems, to criticize arguments, to find proofs, and, what may be the most important activity, to recognize a mathematical concept in, or to extract it from, a given concrete situation (Ahlfors et al. 1962).

It is precisely this "most important activity" that is the essence of the philosophy of mathematics education in the Netherlands, mainly because of the influence of Hans Freudenthal, who in 1968 observed that the goals of teaching mathematics as a socially useful tool could only be reached by having students start from situations that needed to be mathematized. Many mathematics educators and researchers have since supported this view, among them Lesh and Landau (1986), who argued that applications should not be reserved for consideration only after learning has occurred; they can and should be used as a context within which the learning of mathematical concepts takes place.

Our first observation about the mathematics needed to support ML is that *mathematical concepts should be learned through solving problems in appropriate settings,* with opportunities for progressive mathematization and generalization. It should be noted that certain areas in mathematics lend themselves better than others to these purposes. For instance, matrices and graphs, introduced into curricula in the Netherlands in the 1980s, lend themselves very nicely to modeling and representation without the burden of too much specialized language or too many formulas.

A desirable consequence of starting with real settings is the bonus of connected and integrated mathematics. The same problem, but especially the more complex ones, often can be solved in many different ways. Sometimes students choose a more algebraic method, sometimes a more geometric one; sometimes they integrate these, or produce something completely unexpected. Our second observation, then, is that *the mathematics that is taught not only should be connected to other mathematics but also should be embedded in the real world of the student.*

Our third observation is that *the goals of education should not be formulated exclusively in subject areas but should also include competencies.* This holds as well for areas within mathematics: we should not think along the subject lines of arithmetic, algebra, geometry, among others, but about mathematical competencies. This point of view forms the backbone of the PISA Framework for Mathematics, supported by more than 30 countries, including the United States.

The fourth observation is a trivial but important one: *mathematics literacy will lead to different curricula in different cultures.* ML will need to be culturally attuned and defined by the needs of the particular country. This should be kept in mind as we attempt to further determine what mathematics is needed for ML. (We also note that the technology gap will have a serious impact on the type of mathematics competencies that define being mathematically literate in a given country.)

The fifth observation is that, given the goals of ML, *the content of curricula will have to be modernized at least every five to 10 years*. Mathematics is a very dynamic discipline. The culture, and thus the relation between mathematics and society, changes very quickly as well. (In the Netherlands, the curricula for mathematics have a life span of about seven years.)

The sixth observation is that *we might be able to reach some degree of consensus about the meaning of basic mathematical literacy.* A good starting point, as Steen, Schoenfeld, and many others have pointed out, could be the revised NCTM standards (NCTM

2000). The standards, however, stick relatively close to tradition and clearly reflect the difficult process of trying to please everyone. As a result, they can serve only as a starting point, not as a definitive framework for ML.

The seventh observation is that *I can neither describe a curriculum for ML nor even identify the relevant mathematics in any detail*. Some things are clear: from kindergarten on, we should focus on competencies; right from the start we should pay attention not just to arithmetic but to all four phenomenological categories; we should rethink completely the role of algebra; we should design the longitudinal development of mathematical concepts in a very coherent way (at least for students from 4 to 15 years of age); and we should formulate in some detail what it means to be mathematically literate in the basic sense. In this, the results from PISA might help us in a modest way. I also suggest that we should not shy away from new mathematical developments (e.g., discrete dynamic modeling).

The examples I have presented, if properly interpreted and extrapolated, together with these several observations, give the interested reader an impression of the mathematics needed for ML. But designing such a curriculum, let alone teaching it, is a completely different story.

Reflections

I have not answered the question I was asked to address, namely, what mathematics is important for ML? But I have attempted to offer some directions: the desired competencies, not the mathematical content, are the main criteria, and these are different at different ages and for different populations. From a competencies perspective, mathematics for ML can coexist with calculus—or even better, should coexist with a calculus track—but with opportunities to develop intuition, to explore real-world settings, to learn reasoning, and so on. It goes without saying that the line of reasoning I have tried to follow holds for all ages, including university students. We also need mathematicians to become mathematically literate—as such, they are much better prepared to participate in society at large and, even more important, can contribute in a constructive and critical way to the discussion about mathematics education. We all need to understand how important, how essential, ML is for every student, and mathematicians in particular need to understand that ML will contribute to a better perception about what constitutes mathematics and how important that field is to our lives.

We have not addressed several other questions. One of the most important is: How do we teach mathematics for ML? What are the pedagogical arguments and didactics of mathematics for ML? But that question needs an article by itself, as in fact do most of the issues I have discussed here.

But let me end positively. If the experiences in my own country, the experiments we carried out in the United States, and the observations we made worldwide are any indication, there is a good chance that we can achieve ML. The issue is very complex, however, and we have a long and challenging way to go.

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