The Locus of the Focus of a Rolling Parabola

Anurag Agarwal & James Marengo

October 6, 2009

1 Introduction

The catenary can easily be mistaken for a parabola. Even Galileo made this error. In his *Discorsi* of 1638 [3], he explained that if one were to hang a light chain over two nails at the same level, then the chain would assume the shape of a parabola. Students who've had a course in differential equations know, that in fact, this chain takes the shape of the catenary. They may also know that if the chain is subjected to a uniform load, then the chain assumes the form of a parabola; the shape of a suspension bridge, in which the bridge deck provides the load, is another example [2].

One connection between these two curves is that they arise under the similar physical conditions just mentioned. In this paper we establish a different connection. Suppose one were to roll the parabola $y = x^2$ along the $x-$axis without slipping. How does its focus move? In other words, what is the locus of this focus? It turns out, as we will demonstrate, that the locus is a catenary!

2 Geometrical Description of The Catenary

Suppose we roll the parabola $y = x^2$ along the $x-$axis as shown in Figure 1. We assume that the parabola does not slip as it rolls and we wish to determine the path followed by the focus $F(0, \frac{1}{4})$ of the parabola.

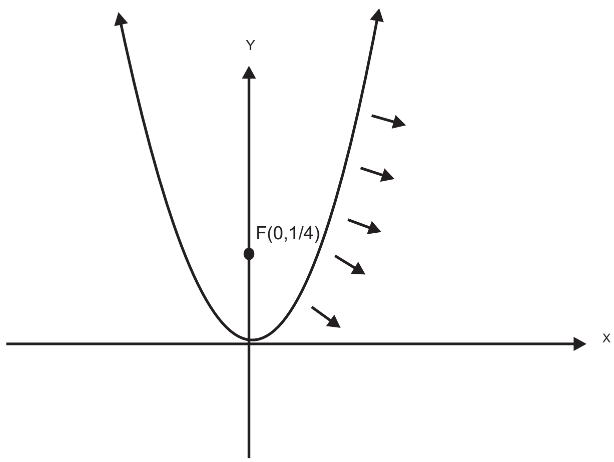

Figure 1. Rolling the parabola $y = x^2$ on the $x-$axis.

To solve this problem we introduce variables as indicated in Figure 2. Here, $\theta_1 = \theta_1(t)$ is the angle between the tangent line to the parabola at $P(t, t^2)$ and the $x-$axis; $\theta_2 = \theta_2(t)$ is the angle between line FP and the $x-$axis; $\alpha(t) = \theta_1 - \theta_2$ is the angle between this tangent line and the line FP. Also, $d = d(t)$ is the length of line segment FP, and $s = s(t)$ is the arc length of the parabola between its vertex $V(0,0)$ and P.

At some point in time the point P will move to the location P' on the $x-$axis in Figure 3. At this instant, F will be at $F'(x, y)$. We will find explicit formulas for the coordinates of F' as functions of t.

Since the parabola rolls without slipping, the length of the line segment VP' is

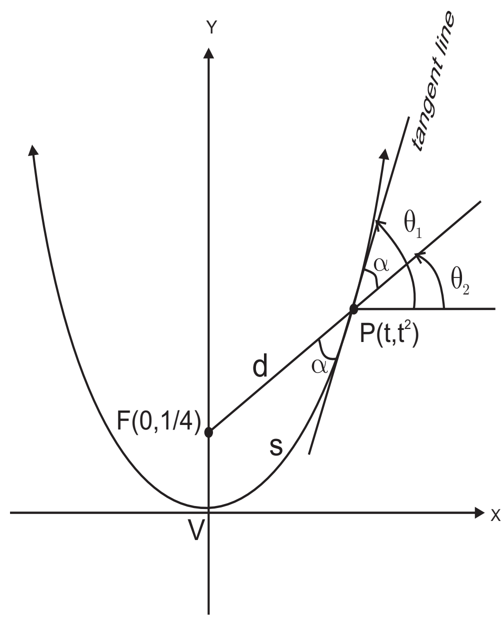

Figure 2.

$s(t)$. The coordinates $x(t)$ and $y(t)$ of F' are given by

$$x(t) = s - d\cos\alpha \qquad \text{and} \qquad y(t) = d\sin\alpha\,. \tag{1}$$

The slope of the tangent line to the parabola at P is

$$\tan\theta_1 = 2t,$$

and the slope of line FP is

$$\tan\theta_2 = \frac{t^2 - \frac{1}{4}}{t}\,.$$

Using standard trigonometry, after simplifying we obtain

$$\tan\alpha = \frac{1}{2t},$$

from which it follows that

$$\cos\alpha = \frac{t}{\sqrt{t^2 + \frac{1}{4}}} \qquad \text{and} \qquad \sin\alpha = \frac{1}{2\sqrt{t^2 + \frac{1}{4}}}\,.$$

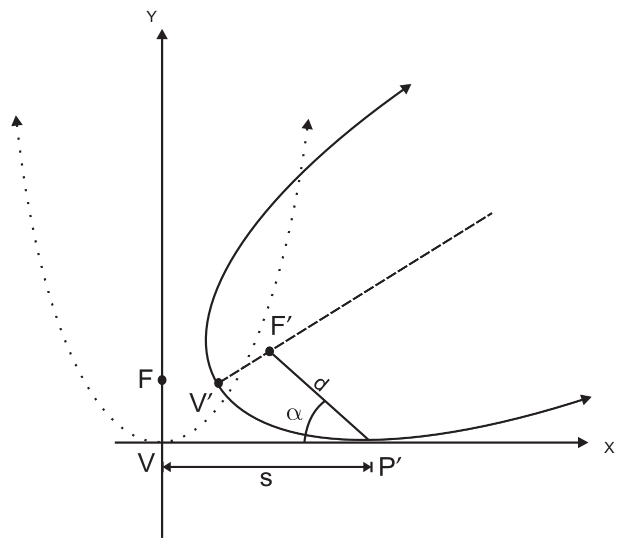

Figure 3.

From Figure 2, using the distance formula, we obtain

$$d = |FP| = t^2 + \frac{1}{4}\,.$$

Finally, we compute $s(t)$ by using the standard formula for arc length [4]. We have

$$\begin{aligned} s(t) &= \int_0^t \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \int_0^t \sqrt{1 + (2x)^2}\, dx\,. \end{aligned}$$

We leave it as an exercise for the reader to show that the value of this integral is given by

$$s(t) = t\sqrt{t^2 + \frac{1}{4}} + \frac{1}{4}\ln\left(2t + 2\sqrt{t^2 + \frac{1}{4}}\right)\,.$$

Since we now have explicit expressions for α, d and s as functions of t we can compute the coordinates (x, y) of F' by substituting these expressions into (1):

$$x(t) = \frac{1}{4} \ln \left(2t + 2\sqrt{t^2 + \frac{1}{4}} \right) \tag{2}$$

$$y(t) = \frac{1}{2}\sqrt{t^2 + \frac{1}{4}}\,. \tag{3}$$

Now that we have expressed x and y in terms of the parameter t, let's try to eliminate t. Solving for t in (2) gives

$$t = \frac{e^{4x} - e^{-4x}}{4}\,.$$

Substituting this last expression in (3), we obtain

$$y = \frac{1}{2}\sqrt{\left(\frac{e^{4x} - e^{-4x}}{4}\right)^2 + \frac{1}{4}}\,.$$

Further simplification yields

$$y = \frac{1}{4}\left(\frac{e^{4x} + e^{-4x}}{2}\right) = \frac{1}{4}\cosh 4x\,.$$

Hence the locus of the focus is the catenary of Figure 4.

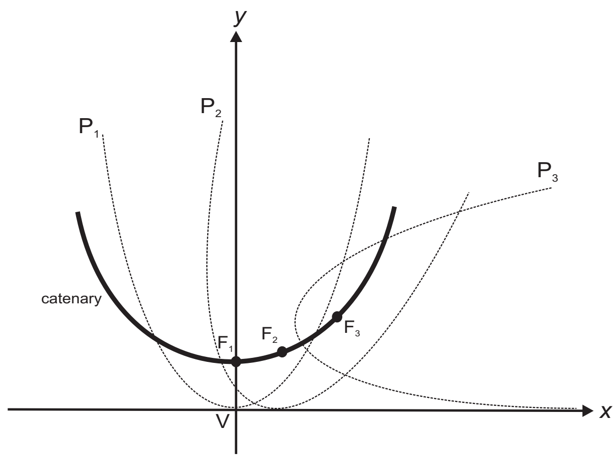

Figure 4.

3 Conclusion

We have demonstrated a geometric connection between the parabola and the catenary—specifically, that the locus of the focus of a parabola which rolls on the $x-$axis without slipping is a catenary.

We close with some related questions for the reader:

1. What is the locus of the focus of the parabola $y = x^2$ as it rolls along some other curve (such as another parabola or perhaps an ellipse or hyperbola) which is tangent to the parabola? It can be shown, for example, that the locus of the focus of the parabola $y = x^2$ as it rolls along the parabola $y = -x^2$ is simply the directrix of the second parabola. That is, the focus of the first parabola moves horizontally (see problem A5 on the 1974 William Lowell Putnam Mathematical Competition [1]).

2. What is the locus of some point on the axis (perhaps the vertex) of the parabola $y = x^2$ as it rolls along the $x-$axis or along some other curve?

3. What if the rolling curve is not a parabola? Can the method discussed above be adapted to find the locus of a point associated with a general smooth curve as this curve rolls along a fixed straight line?

References

[1] G. L. Alexanderson, L. F. Klosinski, and L. C. Larson, *The William Lowell Putnam Mathematical Competition: Problems and Solutions 1965-1984*, Mathematical Association of America, Washington DC, 2004.

[2] J. Bukowski, Christiaan Huygens and the problem of the hanging chain, this JOURNAL, **39** (2008) 2-11.

[3] G. Galilei and S. Hawking, *Dialogues Concerning Two New Sciences*, (trans. Henry Crew and Alfonso deSalvio), Running Press, Philadelphia, PA, 2002.

[4] G. B. Thomas Jr. and R. L. Finney, *Calculus and Analytic Geometry*, 6^{th} ed., Addison-Wesley, Boston, MA, 1984.