Euler-Cauchy Using Undetermined Coefficients

Doreen De Leon
Department of Mathematics, California State University, Fresno
Fresno, CA 93740-8001

The Euler-Cauchy equation is often one of the first higher order differential equations with *variable* coefficients introduced in an undergraduate differential equations course. Putting a nonhomogeneous Euler-Cauchy equation on an exam in such a course, I was surprised when some of my students decided to apply the method of undetermined coefficients, which is guaranteed to work only for constant-coefficient equations, and obtained the correct answer! It turns out that we *can* find a particular solution to this equation using a substitution similar to the standard method of undetermined coefficients, if the right-hand side function is of a certain type, *without* using variation of parameters or transforming the equation to a constant-coefficient equation and then applying undetermined coefficients.

Such a solution is possible because of the fact, mentioned in many differential equations textbooks, that the Euler-Cauchy equation may be transformed by a change of variables into a constant-coefficient equation by simply defining $t = e^z$, if we assume $t > 0$. Thus, if the right-hand side function $f(t)$ is a monomial, then $f(e^z)$ is an exponential function; or if the right-hand side function $f(t)$ is the product of a monomial and a nonnegative integer power of $\ln(t)$, then $f(e^z)$ is the product of a monomial and an exponential function. And, since the new equation is a constant-coefficient equation, the method of undetermined coefficients can be applied, prescribing a solution that is an exponential function, in the first case, and the product of a polynomial and an exponential function in the second case. This leads to a method of undetermined coefficients for the original equation.

First, consider the second order Euler-Cauchy equation with a monomial right-hand side function,

$$t^2 y'' + a t y' + b y = A t^{\alpha}, t > 0. \tag{1}$$

If we suppose that $\alpha \in \mathbb{R}$ is not a root of the characteristic equation, then the above discussion indicates that we should try as our particular solution $y_p = C t^{\alpha}$. Plugging y_p into (**??**) gives

$$(\alpha(\alpha - 1) + a\alpha + b) C t^{\alpha} = A t^{\alpha}.$$

Since we have assumed that $t > 0$ and α is not a root of the characteristic equation, we can solve directly for C.

But, what if α is, in fact, a root of the characteristic equation? As mentioned above, the Euler-Cauchy equation can be transformed into a constant-coefficient equation by means of the transformation $t = e^z$. This means that our first guess for the particular solution would be $y_p(z) = C e^{\alpha z}$. But, since α is a root of the characteristic equation, we need to multiply by z

until $y_p(z)$ is no longer a solution to the complementary equation. Multiplication by z in the guess for the particular solution for the transformed equation translates into multiplication by $\ln(t)$ in the particular solution for (**??**), suggesting a particular solution of the form of a constant multiple of t^α and a power of $\ln(t)$. We can verify by direct substitution that this is the correct form of the solution.

These ideas are summarized in the following theorem.

Theorem 1. *For the second order Euler-Cauchy problem,*

$$t^2y'' + aty' + by = At^\alpha, t > 0,$$

where $\alpha \in \mathbb{R}$, a particular solution is of the form

(i) $y_p(t) = Ct^\alpha$*, provided that α is not equal to any root of the characteristic equation, or*

(ii) $y_p(t) = Ct^\alpha(\ln(t))^i$*, if α is equal to a root of the characteristic equation, where i is the multiplicity of the root.*

For the more complicated equation

$$t^2y'' + aty' + by = At^\alpha(\ln(t))^n, t > 0, \tag{2}$$

where $\alpha \in \mathbb{R}$ and n is a nonnegative integer, a similar analysis leads to the following theorem.

Theorem 2. *For the second order Euler-Cauchy problem,*

$$t^2y'' + aty' + by = At^\alpha(\ln(t))^n, t > 0,$$

where $\alpha \in \mathbb{R}$ and n is a nonnegative integer, a particular solution is of the form

$$y_p(t) = (C_0 + C_1\ln(t) + ... + C_n(\ln(t))^n)\, t^\alpha.$$

In fact, the above method will lead to a solution using undetermined coefficients for the following types of functions, as well:

(1) $A\cos(k\ln t)$ or $A\sin(k\ln t)$,

(2) $At^\alpha\cos(k\ln t)$ or $At^\alpha\sin(k\ln t)$, and

(3) $At^\alpha(\ln(t))^n\cos(k\ln t)$ or $At^\alpha(\ln(t))^n\sin(k\ln t)$.

You should, of course, verify this.

By the principle of superposition, the above results can be applied to Euler-Cauchy equations whose right-hand sides are sums of such functions, simply by applying the appropriate result to each term on the right-hand side. Here is an example.

Example: Find a general solution of $t^2y'' - 4ty' + 4y = 4t^2(\ln(t))^2 - t, t > 0$.

- Complementary solution: Solve $t^2y'' - 4ty' + 4y = 0$ to obtain $y_c = c_1t + c_2t^4$.

- Particular solution: Find a solution of $t^2y'' - 4ty' + 4y = 4t^2(\ln(t))^2 - t$.

  The particular solution takes the form $y_p = y_{p_1} + y_{p_2}$. Since the first function is $4t^2(\ln(t))^2$, by Theorem **??** the first component of y_p, y_{p_1}, is $\left(A + B(\ln(t)) + C(\ln(t))^2\right)t^2$. The particular solution corresponding to the second function, t, is determined using Theorem **??**. Since 1 is a simple root of the characteristic equation, the second component of y_p, y_{p_2}, is $Dt\ln(t)$. So, $y_p = \left(A + B(\ln(t)) + C(\ln(t))^2\right)t^2 + Dt\ln(t)$. Plug y_p into the differential equation, collect terms, and equate coefficients to obtain $A = -3, B = 2, C = -2$, and $D = \frac{1}{3}$, so

$$y_p = \left(-3 + 2\ln(t) - 2(\ln(t))^2\right)t^2 + \frac{1}{3}t\ln(t).$$

General solution: $y = y_c + y_p$, so

$$\boxed{y(t) = c_1t + c_2t^4 + \left(-3 + 2\ln(t) - 2(\ln(t))^2\right)t^2 + \frac{1}{3}t\ln(t)}.$$

It is straightforward to generalize the approach descrbed in this paper to higher order Euler-Cauchy equations.

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