Preface

This book grew out of my work in the last decade teaching, researching, solving problems, and guiding undergraduate research. I hope it will benefit readers who are interested in advanced problem solving and undergraduate research. Math students often have trouble the first time they confront advanced problems like those that appear in Putnam competitions and in the journals of the Mathematical Association of America. And while many math students would like to do research in mathematics, they don't know how to get started. They aren't aware of what problems are open, and when presented with a problem outside of the realm of their courses, they find it difficult to apply what they know to this new situation. Historically, deep and beautiful mathematical ideas have often emerged from simple cases of challenging problems. Examining simple cases allows students to experience working with abstract ideas at a nontrivial level — something they do little of in standard mathematics courses. Keeping this in mind, the book illustrates creative problem solving techniques via case studies. Each case in the book has a central theme and contains kernels of sophisticated ideas connected to important current research. The book seeks to spell out the principles underlying these ideas and to immerse readers in the processes of problem solving and research.

The book aims to introduce students to advanced problem solving and undergraduate research in two ways. The first is to provide a colorful tour of classical analysis, showcasing a wide variety of problems and placing them in historical contexts. The second is to help students gain mastery in mathematical discovery and proof. Although one proof is enough to establish a proposition, students should be aware that there are (possibly widely) various ways to approach a problem. Accordingly, this book often presents a variety of solutions for a particular problem. Some reach back to the work of outstanding mathematicians like Euler; some connect to other beautiful parts of mathematics. Readers will frequently see problems solved by using an idea that might at first have seemed inapplicable, or by employing a specific technique that is used to solve many different kinds of problems. The book emphasizes the rich and elegant interplay between continuous and discrete mathematics by applying induction, recursion, and combinatorics to traditional problems in classical analysis.

Advanced problem solving and research involve not only deduction but also experimentation, guessing, and arguments from analogy. In the last two decades, computer algebra systems have become a useful tool in mathematical research. They are often used to experiment with various examples, to decide if a potential result leans in the desired direction, or to formulate credible conjectures. With the continuing increases of computing power and accessibility, experimental mathematics has not only come of age but is quickly maturing. Appealing to this trend, I try to provide in the book a variety of accessible problems in which computing plays a significant role. These experimentally discovered results are indeed based on rigorous mathematics. Two interesting examples are the WZ-method for telescoping in Chapter 9 and the Hermite-Ostrogradski Formula for searching for a new formula for π in Chapter 18.

In his classic "How to Solve It", George Pólya provides a list of guidelines for solving mathematical problems: learn to understand the problem; devise a plan to solve the problem; carry out that plan; and look back and check what the results indicate. This list has served as a standard rubric for several generations of mathematicians, and has guided this book as well. Accordingly, I have begun each topic by categorizing and identifying the problem at hand, then indicated which technique I will use and why, and ended by making a worthwhile discovery or proving a memorable result. I often take the reader through a method which presents rough estimates before I derive finer ones, and I demonstrate how the more easily solved special cases often lead to insights that drive improvements of existing results. Readers will clearly see how mathematical proofs evolve — from the specific to the general and from the simplified scenario to the theoretical framework.

A carefully selected assortment of problems presented at the end of the chapters includes 22 Putnam problems, 50 MAA Monthly problems, and 14 open problems. These problems are related not solely to the chapter topics, but they also connect naturally to other problems and even serve as introductions to other areas of mathematics. At the end of the book appear approximately 80 selected problem solutions. To help readers assimilate the results, most of the problem solutions contain directions to additional problems solvable by similar methods, references for further reading, and to alternate solutions that possibly involve more advanced concepts. Readers are invited to consider open problems and to consult publications in their quest to prove their own beautiful results.

This book serves as a rich resource for advanced problem solving and undergraduate mathematics research. I hope it will prove useful in students' preparations for mathematics competitions, in undergraduate reading courses and seminars, and as a supplement in analysis courses. Mathematicians and students interested in problem solving will find the collection of topics appealing. This book is also ideal for self study. Since the chapters are independent of one another, they may be read in any order. It is my earnest hope that some readers, including working mathematicians, will come under the spell of these interesting topics.

This book is accessible to anyone who knows calculus well and who cares about problem solving. However, it is not expected that the book will be easy reading for many math students straight out of first-year calculus. In order to proceed comfortably, readers will need to have some results of classical analysis at their fingertips, and to have had exposure to special functions and the rudiments of complex analysis. Some degree of mathematical maturity is presumed, and upon occasion one is required to do some careful thinking.

Acknowledgments

I wish to thank Gerald Bryce and Christopher Kennedy. Both read through the entire manuscript with meticulous and skeptical eyes, caught many of my errors, and offered much valuable advice for improvement. This book is much improved because of their dedication and I am deeply grateful to them. Naturally all possible errors are my own responsibility. I also wish to thank Don Albers at the Mathematical Association of America and the members of the Classroom Resource Materials Editorial Board for their kindness, thoughtful and constructive comments and suggestions. I especially wish to thank Susan Staples for carefully reviewing the book and suggesting its current extended title. Elaine Pedreira and Beverly Ruedi have helped me through many aspects of production and have shepherded this book toward a speedy publication. It is an honor and a privilege for me to learn from all of them. My colleagues Martin Bartelt, Brian Bradie and Ron Persky have read the first draft of several chapters and lent further moral support. Much of the work was accomplished during a sabbatical leave in the fall of 2007. My thanks go to Christopher Newport University for providing this sabbatical opportunity. Finally, I wish to thank my wife Ying and children Alex and Abigail for their patience, support and encouragement. I especially wish to thank Alex, who drew all of the figures in the book and commented on most of chapters from the perspective of an undergraduate.

Comments, corrections, and suggestions from readers are always welcome. I would be glad to receive these at hchen@cnu.edu. Thank you in advance.