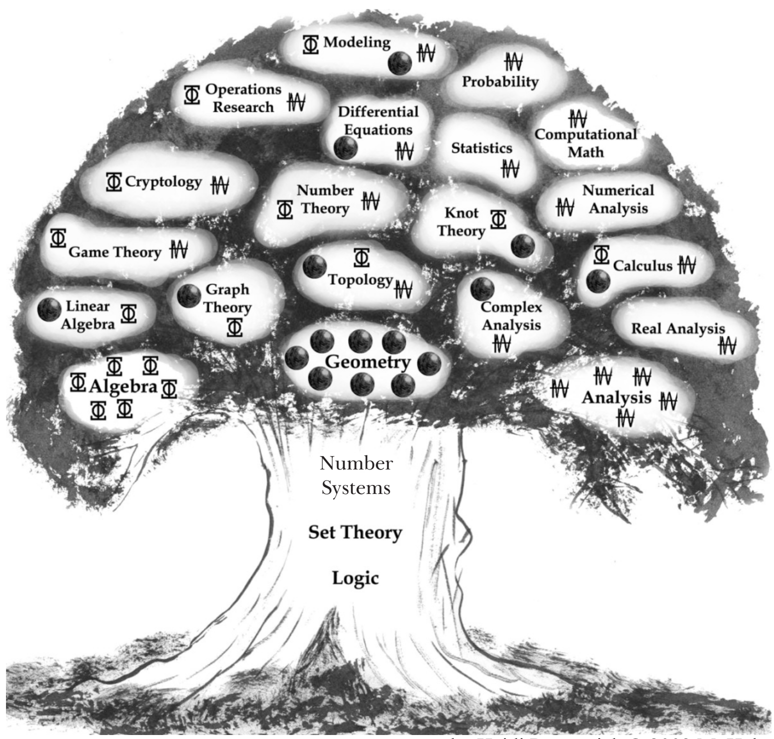
0

Mathematics

0.1 The Tree of Mathematics

The field of mathematics is composed of many subfields, some of which you have studied, and others of which you may not even be aware. These areas are logically related to each other in various ways. One way to look at mathematics is as a tree:



tree by Heidi Ruesswick © 2003 M. Hale

The tree starts at the bottom, the "trunk," and shows dependencies and connections among the fields. You can learn about some areas in any order, but to obtain a sound theory we begin at ground level. This tree is somewhat abbreviated for simplicity. A list of the "official" fields of mathematics appears at the end of this chapter.

You can see that the three main branches of mathematics—algebra, geometry, and analysis—grow out of three basic areas. **Logic** comprises the rules by which mathematicians operate, the "grammar" of the language. **Set theory** provides the vocabulary. And the **Number Systems** comprise the most basic content from which the various branches grow. The course material in this book will acquaint you with the segments on the trunk so that you can make the climb into the canopy.

It is hoped that readers of this book will gain the following:

- facility in interpreting and using mathematical notation;
- a background in elementary logic and practice in reasoning;
- experience with sets and set notation;
- practice in constructing proofs and in evaluating the proofs of others;
- an introduction to the subject matter and activities of mathematics, including the analysis of examples, formulation of conjectures, and reading and writing of proofs;
- an introduction to the professional culture inhabited by mathematicians;
- and, an eagerness to do more mathematics.

The best progress toward these goals will come from a combination of reading the book and talking with your professor and classmates.

0.2 What Is Mathematics?

Before reading on, you may want to think about what mathematics is to you. There is no universally agreed-upon answer, so you can't be wrong.

A simple definition of mathematics is the study of numbers. Your first experiences with math probably bear out this view. But what about geometry? The subject matter there seems to be space. And remember those sequences on standardized tests: $2, 3, 5, 8, 13, 21, \ldots$ what comes next? This is a math question about patterns. Further, if you open an advanced mathematical textbook, you will probably see very few numbers; the statements all seem to be about letters! For example, "If x and y are real numbers, then $x + y = y + x$" or "If $x \geq 0$ then $\sqrt{x} \in \mathbb{R}$." Mathematics includes the study of *properties* of numbers, and properties of other objects as well: sets, relations, functions, vectors, triangles, etc.

So the study of mathematics encompasses more than just the concept of number. Another definition of mathematics is that it is the study of logical relations between statements. If A is true, what must follow? This view is certainly less restrictive, and encompasses all of the above questions. My own opinion is that a really informative description of mathematics is more inclusive: it treats numbers, space, patterns, and logical relationships. Even the most abstract areas of mathematics deal with concepts which at least were inspired by these entities. In addition, we should include the many ways in which already developed mathematics is used to help us understand the world.

Let us define mathematics as the field that is concerned with three major activities:

- logical structure,
- the application of logic to discovering theorems about numbers, space, patterns, and other related structures, and
- the application of these theorems to other fields.

For those interested in word origins, "mathematics" derives from the Greek root *mathema*, meaning "something learned" or "science."

Many mathematicians, famous and otherwise, have had their say about what they thought mathematics was. A few of their words are given as food for thought.

> "*There are three ruling ideas . . . [to which] . . . every mathematical truth admits of being referred; these are the three cardinal notions, of Number, Space and Order.*"
>
> —J. J. Sylvester (1844)[1]

Sylvester identifies arithmetic as the study of number in the abstract, geometry as the study of space, and algebra as the study of order.

> "*Mathematics is the science which draws necessary conclusions.*"
>
> —Benjamin Pierce (1881)[2]

Mr. Pierce must have had sympathy with the pure side of mathematics. In contrast, we have the following poetic view from the applied contingent.

> " *. . . [Mathematics] is the language in which . . . the pages of the universe are written.*"
>
> —J. J. Sylvester[3]

More on pure and applied math is found in the next section.

> "*Mathematics in general is fundamentally the science of self-evident things.*"
>
> —Felix Klein (1902)[4]

I know several students who might disagree.

[1]#110 in [Moritz].
[2]#120 in [Moritz].
[3]#1206 in [Moritz].
[4]#123 in [Moritz].

"... *security, certainty, truth, beauty, insight, structure, architecture, ... one great glorious thing.*"

—Paul Halmos (1985)[5]

0.3 Pure vs. Applied

What about the distinction between pure and applied mathematics? Already the terminology seems to assign to half of the mathematical tree a preferred status. It is certainly true that among the theorists you will find those who view applied mathematicians as heathens of a sort; and some of the latter view much of pure mathematics as empty and purposeless. But the truth is that the two approaches complement each other in a symbiotic process which builds an amazing edifice of theoretical results and practical applications. And in many fields and research papers, it is not so easy to draw a line between what is pure and what is applied. The division between the two might better be described as a fractal[6] boundary.

Despite their connections, there are differences in approach between pure and applied mathematics. Each field of pure mathematics consists of a set of propositions related by logical inference. The content of these propositions is a collection of objects which may have mental antecedents in the real world, but which have no concrete meaning. For example, most high school students take Euclidean geometry. That pure field considers objects such as points, lines, and spheres, which are only idealizations of real objects such as grains of sand, horizons, and planets. The pure mathematician creates an abstract universe around such objects, inventing new ones and proving theorems, without regard for whether the new objects and the theorems represent anything real. The real objects only provide inspiration for the mental exercise.

In contrast, the interests of applied mathematicians are directed toward the real objects. For example, an applied mathematician may ask how the gritty grains of sand found in a fluid might change its flow properties. The motions of planets are calculated by astronomers who predict eclipses, and by NASA[7] scientists who calculate the best times to launch a rocket headed for Mars. Applied mathematicians may use the results of pure mathematics in a modeling process, but their ultimate goal is explanation, prediction, or control of the physical world.

There have been some useful observations on the differences in the two branches, and in the mathematicians who choose to work mainly in one branch. Each of the following is of course a vast simplification.

Logic and intuition: Pure mathematicians are inclined to use logic and intuition about equally, relying on intuition to create mathematics and logic to verify and explain the results. Applied mathematicians use intuition more and logic less. *Truth:* An applied mathematician wants to know the truth of statements p and q, taken separately; a theorist wants to know the truth of $p \Rightarrow q$ (p implies q). *Approximation:* Approximation is tolerated

[5] Quoted in [Albers 1985].

[6] Fractals are discussed in Famous Mathematical Objects, section 5.1.

[7] The National Aeronautical and Space Administration

more in applications than in theory. *Axiomatics:* A pure mathematician will begin with a few axioms and deduce many facts. An applied mathematician (or physicist) begins with the facts and attempts to find the axioms. *Solutions:* A pure mathematician seeks a proof of a conjecture, or a counterexample to show it false. He or she desires a complete accounting of the possibilities: which are in category A and which in category B? On the other hand, an applied mathematician usually has a much smaller scope. When the machine or process works, the problem is solved.

On the lighter side, this joke has made the rounds for years, in several forms.

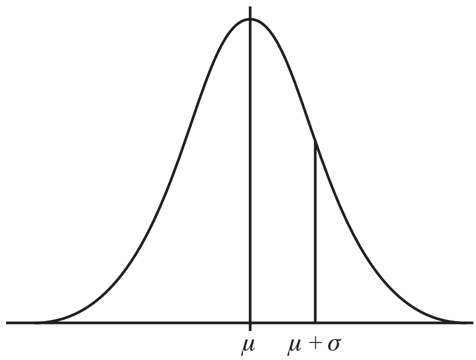
> A mathematician arrives at his car in a dark parking lot, only to find he has dropped his key. If he's an applied mathematician, he retraces his steps, looking at the ground for the key. If he's a pure mathematician, he realizes he cannot see the ground, so he cleverly (?) finds a lighted area and searches there for whatever he can find.

Some of you may be interested in computer science as well as mathematics. Donald Knuth, inventor of the math typesetting system $\TeX$, was interviewed in [Albers, 1985] as a mathematician and computer scientist. He compares those two fields in the following way. Mathematicians seem to be distinguished by their use of geometric reasoning and the way they think about infinity. They find beauty in the most general arguments. Computer scientists have a particular interest in the states of a changing process, and they exhibit a tolerance for more cases than would be considered elegant by a mathematician. Both fields use a large degree of abstraction, and both frequently use formulas.

Before we leave the issue of theory vs. application, we should mention some of the many examples of "purely pure" mathematics that were developed, seemingly just for fun, years (sometimes centuries) before science found a use for them. The conic sections studied by Apollonius (pronounced "ap-ple-OH-nee-us") around 200 BC were just the curves Kepler needed in 1609 to describe the elliptic orbits of the planets, and provided an important key to Newton's law of gravitation later in the same century. Non-Euclidean geometry, invented by Lobachevsky ("low-ba-SHEV-skee") and others in the mid-19$^{\text{th}}$ century, was the right setting for Einstein's theory of general relativity in the 20$^{\text{th}}$. The theory of matrices, thought strange and awkward at the time of its development by Cayley around 1858, helped Heisenberg to solidify his explanation of quantum mechanics in the 1920s. And in a particularly ironic twist, mathematical logic, whose development allowed the separation of the pure from the applied, and on which pure math is based, is being used a century later in the theories of computing machines and computer languages.

There is a memorable story related by the great physicist Eugene Wigner in his much quoted talk "The Unreasonable Effectiveness of Mathematics in the Natural Sciences."[8] Two high school friends meet years later and talk about their jobs. One is a statistician working on population trends. He shows his friend a paper he has written, which contains a graph of the normal curve, used so often in statistics. The friend is quite surprised that so much could be known and expressed mathematically; in fact, he suspects that his old classmate may be pulling his leg. He asks for an explanation of the symbols, μ, σ, and so on: mean, standard deviation, etc. Then he comes upon π. "What is that?" he asks. "Oh," says

[8] [Wigner]



The Normal curve: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$

the statistician, "this is pi ... the ratio of the circumference of the circle to its diameter." ... Put yourself in the friend's place: what connection can there be between a population and the circumference of a circle?! But there it is, the normal curve is fundamental to all of statistics.

In their classic book *What is Mathematics?* Courant and Robbins express the hope that mathematics will embrace both pure and applied work equally. In the prelude to the book they define mathematics and explain its power by means of a metaphor that includes both cooperation and contention between the theoretical and applied aspects.

> "Mathematics as an expression of the human mind reflects the active will, the contemplative reason, and the desire for aesthetic perfection. Its basic elements are logic and intuition, analysis and construction, generality and individuality. Though different traditions may emphasize different aspects, it is only the interplay of these antithetic forces and the struggle for their synthesis that constitute the life, usefulness, and supreme value of mathematical science."[9]

0.4 What Kind of People Are Mathematicians?

We're all pretty nerdy, right? Rumpled, chalk-covered clothes, fly-away hair, absent-minded, head in the clouds, socially challenged, pocket protectors. Oops, it's the engineers with the pocket protectors. Here is a joke I heard at the annual math meetings:

> Q: How do you tell an introverted mathematician from an extroverted mathematician?
>
> A: An extroverted mathematician looks at *your* shoes while he's talking to you.

And yes, mathematicians are male, aren't they?

There are traits that are common to many mathematicians, but these traits do not include any of the above stereotypes. That is not to say the stereotypes don't have their expressions

[9] [Courant]

in certain individuals. However, a reading of the valuable books *Mathematical People, More Mathematical People*[10] and *101 Careers in Mathematics*[11] will convince you that a diverse crowd of otherwise normal people has chosen mathematics as both profession and hobby, and give you an idea of how wide the possibilities are for math majors.

I have known quite a few mathematicians, mostly academic of course, but also some who work for government and industry, and others who work in non-mathematical fields (their "day jobs"), but enjoy pursuing mathematics as a hobby. I find the group as a whole interesting and energetic, and I am proud to be a member. So, how would I describe the "typical" mathematician?

First, most mathematicians I know love doing mathematics, and have often designed their jobs to accommodate their urge to do it. Whether it is research, teaching, or applications in business, industry, and government, the vocation is sometimes hard to distinguish from the avocation.

Mathematicians are drawn to questions, especially those that have verifiable solutions. This may explain why math students often feel inundated by them. Interesting questions and elegant solutions are regarded as beautiful to the mathematical eye. Many mathematicians also enjoy puzzles and games of various types: chess, bridge, Scrabble®, and others require reasoning similar to mathematics.

Questions, in general, fall into two broad types: those that eventually yield definite answers and those whose answers are always relative and incomplete. Mathematics deals with the former. Questions such as "What is the solution to this equation?" and "What structures allow equations to be solved?" have precise answers. The answers may be short or long, easy to find or difficult, but the type of answer expected is well-defined. Sometimes the hardest part is to *make* the question less vague. This is what mathematicians do, and they find these kinds of questions and answers satisfying. The other type of question, such as "Does nuclear power adequately balance commercial and environmental interests?" is also interesting to pursue, but does not have one definite answer agreed to by everyone. While mathematics can play a role in finding political and environmental answers to questions such as this, it is not the type of question that occupies the day-to-day efforts of research mathematicians.

Speaking of questions and problems, the goal is to solve them. But it's not really done until you tell someone. Mathematicians usually enjoy talking about math, whether it's their own work or someone else's. Go to a conference (Yes, go! Students are welcome.) and you will hear almost as much mathematics out in the halls as you will in the lecture rooms. A group of mathematicians at lunch may cover the (paper) napkins and place mats with pictures and formulas. Most academic mathematicians teach as part of their jobs, and most find it rewarding.

One last trait, some say an occupational hazard, is that mathematicians speak, write, and think in very precise language. If you eavesdrop on a mathematical conversation, you'll hear a lot of strange words, and familiar words used in new ways. For example, did you know that the words "group," "set," "category," and "class" all have different meanings? The technical jargon serves the need for precision. Mathematicians accept this need, and

[10] [Albers 1985 and 1990]

[11] [Sterrett]

they are accepting of a high level of structure in general. The rules of logic bind a mathematician to approach and judge mathematical work in a particular way.

Successful mathematicians seem to me to have an ability to focus, to become completely absorbed in a problem. Many routine (new) problems take an hour or more to solve. And it is possible for one person to work for years on a question. Sometimes this is because it is a particularly "fruitful" question, eluding a complete solution but spawning many other worthwhile questions which do get solved. And sometimes, the problem is just plain hard. Andrew Wiles worked intensively for seven years on his solution (achieved in 1994) to Fermat's ("fair-MAH") Last Theorem[12], and it took months to check, revise, and re-check the proof. Even professional mathematicians need not be driven to this extent, but there is a dedication and persistence required for mathematical work.

Aside from the tendencies listed above, I have found that mathematicians are as varied a group as any other profession. Look around your classroom. The student next to you may not look like Einstein (a physicist who was just average in math), but with hard work and a love for the subject, she just may turn into a mathematician.

0.5 Mathematics Subject Classification

The American Mathematical Society (see Professional Organizations, section 7.4) has classified the subfields of mathematics for purposes of publications and member services. Each paper published in a recognized mathematics journal receives one or more classifications as to subject, with further refinement into subsubfields. Each mathematician identifies his or her work with one or more of these fields. My own area is topology, and my Ph.D. thesis has classification 57N20: Manifolds and Cell Complexes, Topological Manifolds, Infinite Dimensional Manifolds. The list is revised now and then to reflect changes in the discipline. Here is the most recent version, formulated in 2000. It is only a rough division. It is estimated that there are now over 3,000 recognized areas of mathematical research. There is a more complete breakdown at [AMS].

[12] One of the most famous problems, unsolved for over 350 years, it has been called the holy grail of mathematics. The zeal of Professor Wiles is perhaps understandable. More will be said about Fermat's Last Theorem in section 7.2.

The 2000 Mathematics Subject Classification

00 General
01 History and biography
03 Logic and foundations
05 Combinatorics
06 Order, lattices, ordered algebraic structures
08 General algebraic systems
11 Number theory
12 Field theory and polynomials
13 Commutative rings and algebras
14 Algebraic geometry
15 Linear and multilinear algebra, matrix theory
16 Associative rings and algebras
17 Nonassociative rings and algebras
18 Category theory, homological algebra
19 K-theory
20 Group theory and generalizations
22 Topological groups, Lie groups
26 Real functions
28 Measure and integration
30 Functions of a complex variable
31 Potential theory
32 Several complex variables and analytic spaces
33 Special functions
34 Ordinary differential equations
35 Partial differential equations
37 Dynamical systems and ergodic theory
39 Difference and functional equations
40 Sequences, series, summability
41 Approximations and expansions
42 Fourier analysis
43 Abstract harmonic analysis
44 Integral transforms, operational calculus
45 Integral equations
46 Functional analysis
47 Operator theory
49 Calculus of variations, optimal control, optimization
51 Geometry
52 Convex and discrete geometry
53 Differential geometry
54 General topology
55 Algebraic topology
57 Manifolds and cell complexes
58 Global analysis, analysis on manifolds
60 Probability theory, stochastic processes
62 Statistics
65 Numerical analysis
68 Computer science
70 Mechanics of particles and systems
74 Mechanics of deformable solids
76 Fluid mechanics
78 Optics, electromagnetic theory
80 Classical thermodynamics, heat transfer
81 Quantum theory
82 Statistical mechanics, structure of matter
83 Relativity and gravitational theory
85 Astronomy and astrophysics
86 Geophysics
90 Operations research, mathematical programming
91 Game theory, economics, social and behavioral sciences
92 Biology and other natural sciences
93 Systems theory, control
94 Information & communication, circuits
97 Mathematics education