Introduction

Through the ostensibly infallible process of logical deduction, Euclid of Alexandria (ca. 300 B.C.) derived a colossal body of geometric facts from a bare minimum of genetic material: five *postulates* — five simple geometric assumptions that he listed at the beginning of his masterpiece, the *Elements*. That Euclid could produce hundreds of unintuitive theorems from five patently obvious assumptions about space, and, still more impressively, that he could do so in a manner that precluded doubt, sufficed to establish the *Elements* as mankind's greatest monument to the power of rational organized thought. As a logically impeccable, tightly wrought description of space itself, the *Elements* offered humanity a unique anchor of definite knowledge, guaranteed to remain eternally secure amidst the perpetual flux of existence — a rock of certainty, whose truth, by its very nature, was unquestionable.

This universal, even transcendent, aspect of the *Elements* has profoundly impressed Euclid's readers for over two millennia. In contrast to all explicitly advertised sources of transcendent knowledge, Euclid never cites a single authority and he never asks his readers to trust his own ineffably mystical wisdom. Instead, we, his readers, need not accept anything on faith; we are free and even encouraged to remain skeptical throughout. Should one doubt the validity of the Pythagorean Theorem (*Elements* I.47), for example, one need not defer to the reputation of "the great Pythagoras". Instead, one may satisfy oneself in the manner of Thomas Hobbes, whose first experience with Euclid was described by John Aubrey, in his *Brief Lives*, in the following words.

> He was 40 years old before he looked on Geometry; which happened accidentally. Being in a Gentleman's library, Euclid's *Elements* lay open, and 'twas the 47 *E. Libri I.* He read the Proposition. *By G* —, says he, (he would now and then sweare an emphaticall Oath by way of emphasis) *this is impossible*! So he reads the Demonstration of it, which referred him back to such a Proposition; which proposition he read. That referred him back to another, which he also read. *Et sic deincips* and so on that at last he was demonstratively convinced of that truth. This made him in love with Geometry.

The *Elements* was an educational staple until the early twentieth century. So long as reading it remained a common experience among the educated, Euclid's name was synonymous with demonstrable truth[1]. It is not an exaggeration to assert that Euclid was the envy of both philosophy

[1] At the very least, the demonstrations in the *Elements* were acknowledged as the strongest possible sort of which the rational mind is capable. "It is curious to observe the triumph of slight incidents over the mind: — What incredible weight they have in forming and governing our opinions, both of men and things — that trifles, light as air, shall waft a belief into the soul, and plant it so immovably within it – that Euclid's demonstrations, could they be brought to batter it in breach, should not all have power to overthrow it." (Laurence Sterne, *The Life and Opinions of Tristram Shandy.* Book IV, Ch. XXVII)

and theology. In his *Meditations*, Descartes went so far as to base his certainty that God exists on his certainty that Euclid's 32nd proposition is true. This was but a single instance out of many in which theology has tried to prop itself up against the rock of mathematics. Euclid's *Elements*, for all its austerity, appeals to a deep-seated human desire for certainty. This being the case, any individual with the impertinence to challenge Euclid's authority was certain to inspire reactions of both incredulity and scorn.

But how exactly *can* one challenge Euclid's authority? Euclid asks us to accept nothing more than five postulates, and all else follows from pure logic. Therefore, if there is anything to challenge in the *Elements*, it can only be in the postulates themselves. The first four seem almost too simple to question. Informally, they describe the geometer's tools: a straightedge, a compass, and a consistent means for measuring angles. The fifth postulate, however, is of a rather different character:

> That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

This is Euclid's famous *parallel postulate*, so called because it forms the basis for his theory of parallels, which, in turn, forms the basis for nearly everything else in geometry. Modern geometry texts almost invariably replace this postulate with an alternative, to which it is logically equivalent: *given a line and a point not on it, there is exactly one line that passes through the point and does not intersect the line*. Particularly when expressed in this alternate form, the parallel postulate does strike most as "self-evident", and thus beyond question for any sane individual. It would seem, therefore, that Euclid has no significant weaknesses; his geometry is *the* geometry — impregnable, inevitable, and eternal.

The timeless, almost icy, perfection that characterizes Euclid's work made it not only a logical masterpiece, but an artistic one as well. In this latter aspect, commentators often singled out the parallel postulate as the unique *aesthetic* flaw in the *Elements*. The problem was that the parallel postulate seemed out of place: it read suspiciously like a theorem — something that Euclid should have *proved* from his earlier postulates, instead of adjoining it to their ranks. This structural incongruity — a postulate that "should be" a theorem — disturbed many mathematicians from antiquity to the 19th century. We may safely presume that Euclid tried and failed to prove the postulate as a theorem. We *know* that Euclid's followers and admirers tried to do as much, hoping to perfect their master's work by polishing away this one small but irritating blemish. Many believed that they had succeeded.

Records of flawed "proofs" rarely survive, as there generally seems no reason to preserve them, so the astonishing number of alleged proofs of the parallel postulate that have come down to us should serve to indicate just how much attention was given to this problem. Proclus, a 5th-century neo-Platonic philosopher, who wrote an extensive commentary on the first book of the *Elements*, describes two attempts: one by Posidonius (2nd century B.C.), the other by Ptolemy (the 2nd-century A.D. author of the *Almagest*, the Bible of geocentric astronomy). Both arguments, Proclus points out, are inadmissible because they contain subtle flaws. After detailing these flaws, Proclus proceeded to give his own proof, thus settling the matter for once and all — or so he thought. Proclus' proof, for all his critical acumen, was just as faulty as those he had criticized.

We have flawed proofs by Aghanis (5th century) and Simplicius (6th-century), two Byzantine scholars. Many others by medieval Islamic mathematicians have survived, including attempts

by al-Jawhari and Thabit ibn Qurra in the 9th century, al-Haytham and Omar Khayyám in the 11th, and Nasir-Eddin al-Tusi in the 13th. There are even a few specimens from medieval Europe, such as those conceived by Vitello in the 13th century and Levi ben Gerson in the 14th. A veritable horde of later Europeans left purported proofs of the postulate (to cite just a few examples: Christopher Clavius in 1574, Pietro Antonio Cataldi in 1604, Giovanni Alfonso Borelli in 1658, Gerolamo Saccheri in 1733, Louis Bertrand in 1788, and Adrien Marie Legendre, who published many attempts between 1794 and 1832). Indeed, in 1763, G.S. Klügel wrote a dissertation examining no less than twenty-eight unsound "proofs" of the postulate. Interestingly, most would-be postulate provers followed Proclus in explicitly criticizing one or more of their predecessors' attempts before giving their own flawed "proof to end all proofs".[2]

Adhering to long-standing custom, Nikolai Ivanovich Lobachevski (1792–1856) began many of his own works on the subject by criticizing the alleged proofs of his immediate predecessor, Legendre. However, instead of forging the chain's next link, Lobachevski suggested that the chain be discarded altogether. He insisted that the parallel postulate *cannot* be proved from Euclid's first four postulates. In this sense, Lobachevski was a great defender of Euclid: he felt that Euclid was fully justified in assuming the parallel postulate as such; indeed, he believed that Euclid had no other way to obtain it.

In another sense, Lobachevski believed that Euclid was wholly *unjustified* in assuming the parallel postulate, for we cannot be certain that it accurately describes the behavior of lines in physical space. Euclidean tradition declares that it does, but the universe is not obliged to respect humanity's traditional beliefs about space, even those codified by its great authority, Euclid of Alexandria. Lobachevski considered the validity of the parallel postulate an empirical question, to be settled, if possible, by astronomical measurements.

Unorthodoxy quickly led to heresy: proceeding from the assumption that the parallel postulate does *not* hold, Lobachevski began to develop a *new* geometry, which he called *imaginary geometry*[3], whose results contradicted Euclid's own. He first described this strange new world on February 24, 1826, in a lecture at the University of Kazan. His first written publication on the subject dates from 1829. Several others followed, and after a decade of failed attempts to convince his fellow Russians of the significance of his work, he published accounts of it in French (in 1837) and German (in 1840), hoping to attract attention in Western Europe. He found none. By the time that he wrote *Pangeometry* (1855), he was blind (he had to dictate the book), exhausted, and embittered. He died the following year.[4]

In fact, although Lobachevski never knew it, his work did find one sympathetic reader in his lifetime: Karl Friedrich Gauss (1777–1855), often classed with Isaac Newton and Archimedes as one of the three greatest mathematicians who have ever lived. Gauss shared Lobachevski's convictions regarding the possibility of an alternate geometry, in which the parallel postulate does

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[2] For detailed descriptions of many alleged proofs of the postulate, consult Rosenfeld (Chapter 2) or Bonola (Chapters 1 and 2).

[3] By the end of his life, he preferred the name *pangeometry*, for reasons that will become clear by the end of *The Theory of Parallels*. Other common adjectives for Lobachevski's geometry are *non-Euclidean* (used by Gauss), *hyperbolic* (introduced by Felix Klein), and *Lobachevskian* (used by Russians).

[4] His French paper of 1837, *Géométrie Imaginaire*, appeared in August Crelle's famous journal, *Journal für die Reine und Angewandte Mathematik* (Vol. 17, pp. 295–320). His German publication of 1840 was *The Theory of Parallels*; its full title is *Geometrische Untersuchungen zur Theorie der Parallelinien* (*Geometric Investigations on the Theory of Parallels*). Lobachevski wrote two versions of *Pangeometry*, one in French and one in Russian.

not hold. He reached these conclusions earlier than Lobachevski, but abstained, very deliberately, from publishing his opinions or investigations. Fearing that his ideas would embroil him in controversy, the very thought of which Gauss abhorred, he confided them only to a select few of his correspondents, most of them astronomers. When Gauss read an unfavorable review of Lobachevski's *Theory of Parallels*, he dismissed the opinions of the reviewer, hastened to acquire a copy of the work, and had the rare pleasure of reading the words of a kindred, but more courageous, spirit. Gauss was impressed; he even sought out and read Lobachevski's early publications in Russian. To H.C. Schumacher, he wrote in 1846, "I have not found anything in Lobachevski's work that is new to me, but the development is made in a different way from the way I had started and, to be sure, masterfully done by Lobachevski in the pure spirit of geometry."

True to his intent, Gauss' radical thoughts remained well-hidden during his lifetime, but within a decade of his death, the publication of his correspondence drew the attention of the mathematical world to non-Euclidean geometry. Though the notion that there could be two geometries did indeed generate controversy, the fact that Gauss himself endorsed it was enough to convince several mathematicians to track down the works of the unknown Russian whom Gauss had praised so highly. Unfortunately, Lobachevski reaped no benefit from this interest; he was already dead by that time, as was the equally obscure Hungarian mathematician, János Bolyai (1802–1860), whose related work also met with high praise in Gauss' correspondence.

Bolyai had discovered and developed non-Euclidean geometry independently of both Lobachevski and Gauss. He published an account of the subject in 1832, but it had essentially no hope of finding an audience: it appeared as an appendix to a two-volume geometry text, written by his father, Farkas Bolyai, in Latin. Farkas Bolyai, who had known Gauss in college, sent his old friend a copy of his son's revolutionary studies. Gauss' reply — that all this was already known to him — so discouraged the young János, that he never published again, and even ceased communicating with his father, convinced that he had allowed Gauss to steal and take credit for his own discoveries. Father and son were eventually reconciled, but Bolyai was doubly disheartened some years later to learn that his own *Appendix* could not even claim the honor of being the first published account of non-Euclidean geometry: Lobachevski's earliest Russian paper antedated it by several years.

As mathematicians began to re-examine the work of Lobachevski and Bolyai, translating it into various languages, extending it, and grappling with the philosophical problems that it raised, they changed the very form of the subject in order to assimilate it into mainstream mathematics. By 1900, non-Euclidean geometry remained a source of wonder, but it had ceased to be a controversial subject among mathematicians, who were now describing it in terms of differential geometry, projective geometry, or Euclidean "models" of the non-Euclidean plane. These developments and interpretations helped mathematicians domesticate the somewhat nightmarish creatures that Lobachevski and Bolyai had loosed upon geometry. Much was gained, but something of great psychological importance was also lost in the process. The tidy forms into which the subject had been pressed scarcely resembled the majestic full-blooded animal that Lobachevski and Bolyai had each beheld, alone, in the deep dark wild wood.

Today, in 2011, the vigorous beast is almost never seen in its original habitat. Just as we give toy dinosaurs and soft plushy lions to children, we give harmless non-Euclidean toys, such as the popular Poincaré disc model, to mathematics majors. We take advanced students to the zoo of differential geometry and while we are there, we pause — briefly, of course — to point

out a captive specimen of hyperbolic geometry, sullenly pacing behind bars of constant negative curvature.

If we are to understand the meaning of non-Euclidean geometry — to understand why it wrought such important changes in mathematics — we must first recapture the initial fascination and even the horror that mathematicians felt when confronted with the work of Lobachevski and Bolyai. This, however, is difficult. The advent of non-Euclidean geometry changed the mathematical landscape so profoundly that the pioneering works themselves were obscured in the chaos of shifting tectonic plates and falling debris. Mathematical practices of the early 19th century are not the same as those of the early 21st. The gap of nearly two centuries generally precludes the possibility of a sensitive reading of Lobachevski's works by a modern reader. This book is an attempt to rectify the situation, by supplying the contemporary reader with all of the tools necessary to unlock this rich, beautiful, but generally inaccessible world. But where does one start?

Gauss left us nothing to work with. Bolyai's *Appendix* is out of the question; his writing is often terse to the point of incomprehensibility. Lobachevski is far clearer, but he too makes heavy demands on his readers. Perhaps we should read his earliest works? In 1844, Gauss described them (in a letter to C.L. Gerling) as "a confused forest through which it is difficult to find a passage and perspective, without having first gotten acquainted with all the trees individually." At the other chronological extreme, Lobachevski's final work, *Pangeometry*, is inappropriate for beginners since it merely summarizes the elementary parts of the subject, referring the reader to *The Theory of Parallels*, his German book of 1840, for proofs. *Pangeometry* does make a logical second book to read, but the book that it leans upon, *The Theory of Parallels*, remains the best point of ingress for the modern mathematician.

Accordingly, the following pages contain a new English version of *The Theory of Parallels*, together with mathematical, historical, and philosophical commentary, which will expand and explain Lobachevski's often cryptic statements (which even his contemporaries failed to grasp), and link his individual propositions to the related work of his predecessors, contemporaries, and followers. Resituated in its proper historical context, Lobachevski's work should once again reveal itself as an exciting, profound, and revolutionary mathematical document.

The complete text of Lobachevski's *Theory of Parallels* appears **twice** within the pages of this book. In the appendix, it appears as a connected whole, in its first English translation since Halsted's in 1891. In the body of the book, the complete text appears a second time, but broken into more than 100 pieces; I have woven my illumination around these hundred-odd pieces. Lest there be any confusion as to whose voice is speaking at any given place in the book, Lobachevski's words have been printed in red, while everything else is printed in black.

A Note to the Reader

> "Begin at the beginning," the King said, very gravely,
> "and go on till you come to the end: then stop."
> —*Alice in Wonderland,* Chapter 12

Although following the King of Hearts' advice may be the most rigorous way to read *Lobachevski Illuminated*, it is hardly the only way. Beginning at the beginning is always a sensible idea, but one need not feel compelled to "go on" through the details of each and every auxiliary proof that I provide along the way. Many readers, first-time readers in particular, will simply want an overview of Lobachevski's accomplishments and methods. If you are such a person, then you should feel free to skip any technical proofs that threaten to divert you from the main narrative thread.

Ideally, one should at least read the *statements* of the propositions that I prove in the notes. What one chooses to do with them will then vary from reader to reader. Some will want to try coming up with their own proofs. Others will simply read mine. Still others will take the statements on faith and move on, confident in the knowledge that the proofs are there, patiently waiting, should they ever need to be consulted. All of these are reasonable approaches.

Of course, the further one travels into the counterintuitive non-Euclidean countryside, the less confidence one will have in dismissing anything as "obvious", "trivial", or "a mere technicality". If you have never left the Euclidean world before, then be forewarned: you are about to embark on a thoroughly disorienting (but strangely exhilarating) journey. Some readers will be more comfortable taking one tiny step at a time into this new land, mapping the terrain slowly and carefully, proving everything in detail, until even its most alien features take on a kind of unexpected familiarity. Others will charge boldly ahead, skipping many proofs, eager to reach the dark heart of the matter as quickly as possible; they will get there, of course, but are likely to find themselves so thoroughly befuddled that they will almost certainly want to go back and carefully retrace their steps so as to make some retrospective sense of the strange sights they have beheld. Again, both approaches are fine, and are ultimately a matter of individual psychology. An approach somewhere between these two extremes is probably best.

Of course, there will be some readers who will want considerably *more* technical detail than I've provided. In particular, some may desire a rigorously-argued Hilbert-style examination of the foundations of geometry. Others won't stop even there, and will want to dig into the primal matter of logic itself. But as with its readers, so a book's author must draw the line somewhere.

I have laid out the meal. Fall to it and eat. Only you can decide what to put on your plate.