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A Mathematical Look at a Medieval Cathedral

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If you wish to design and build a cathedral, you'd better know some mathematics. The application of mathematics has been central to the design and execution of art and architecture from the Classical era through the Middle Ages and still today. The renown of the Greek prescriptive sculptural instructions, the *Canon of Polykleitos*, attests to this. The celebrated Roman architectural and engineering manual, Vitruvius's *De Architectura*, also emphasized the importance of mathematics in fulfilling the purpose of building. Medieval stonemasonry was itself reverently known as the Art of Geometry. Our focus here will be on the mathematics known and used by medieval stonemasons, in particular in the construction of Durham Cathedral in Northeast England.

One of the main applications of mathematics in medieval architecture was practical geometry. Practical geometry did not concern itself with axioms, deductions, theorems and proofs. Its approach was more empirical and time-tested. Generally, medieval masons including master masons would not have been able to read more abstract or speculative mathematical treatises in Latin, even if they were allowed access to them in the libraries of bishops and monasteries. However, a master mason could adeptly and repeatedly apply a few simple geometric operations and tools, such as the mason's large compass, to produce a myriad of sophisticated designs as attested to by extant late medieval design manuscripts, by full-scale working drawings still etched on some church floors and walls, and by the cathedrals themselves.

The basic tools for design were compasses, dividers,

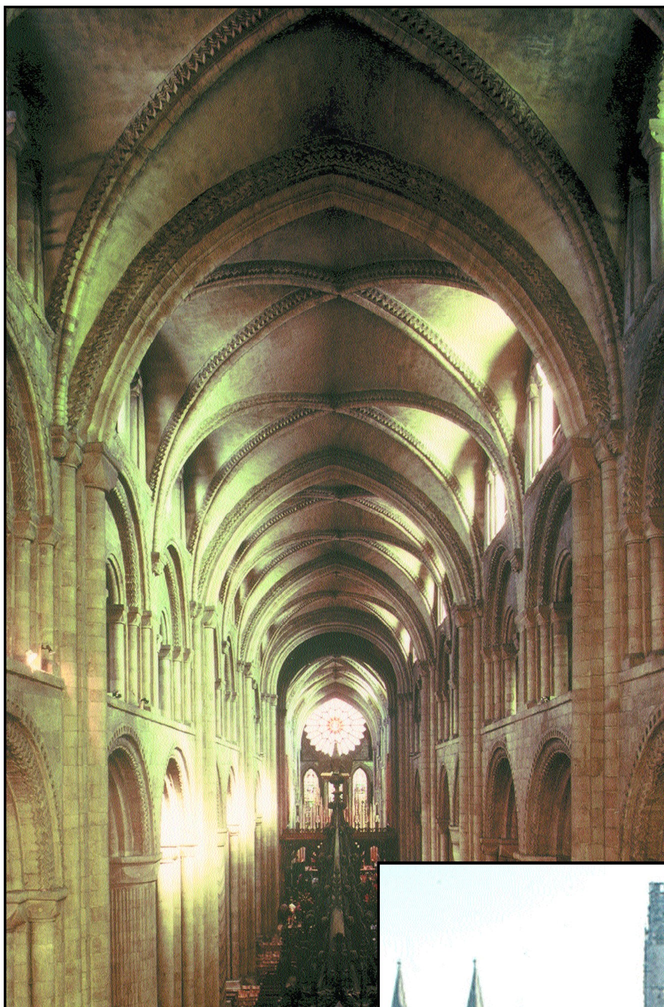


Figure 1. Durham Cathedral: full-length interior view looking toward the liturgical East. Figures 1 and 2, photographs by Peter Coffman.

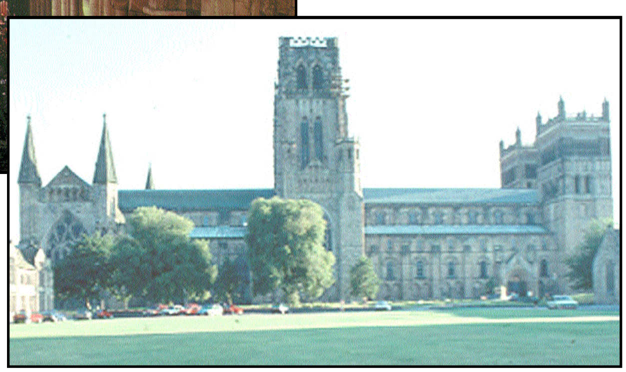
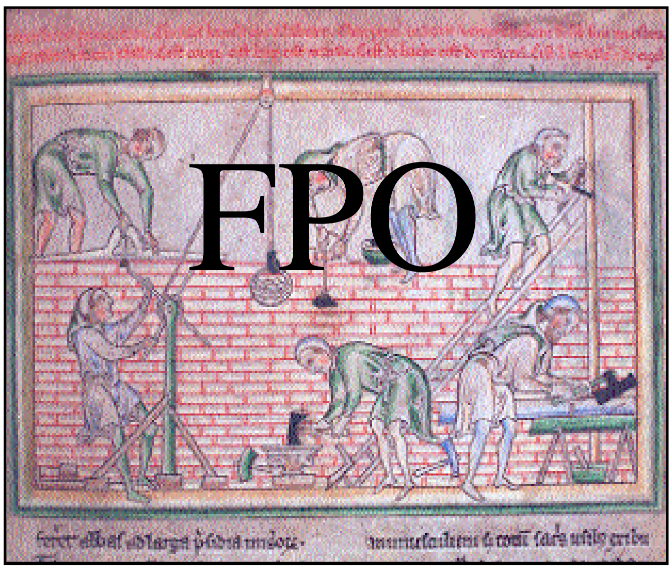
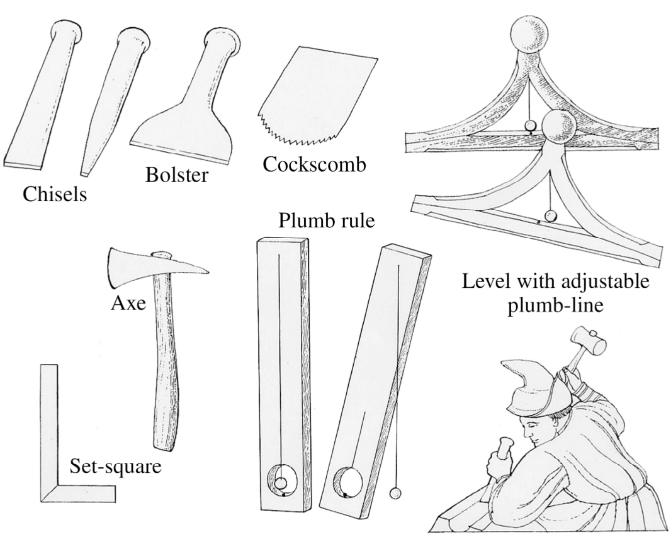


Figure 2. Durham Cathedral exterior, north side.





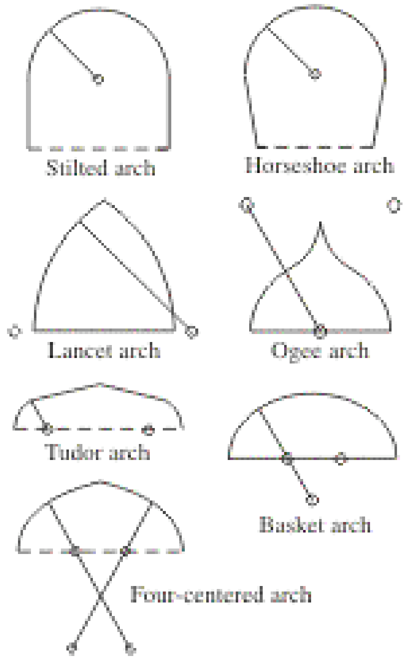


Figure 3. (top left) King Offa and a master mason (holding large compass and set square) instructing stoneworkers at a building site. (middle left) Stonemasons at work on a cathedral wall. This, and the previous image are from *The Book of St. Albans* and reproduced courtesy of the Board of Trinity College Dublin. (bottom left) A stonemason's toolkit, from *Building the Medieval Cathedrals*, Percy Watson. Reprinted with permission from Cambridge University Press.

Figure 4. (above) Various pointed and rounded arch shapes produced with straightedge and compass.

straightedges, rulers, and set squares. See Figure 3. Both small and large compasses and dividers were employed. The large compasses could be up to a meter long. Compasses and dividers were used, of course, for drawing circular arcs, and the latter was also employed to copy or transfer a given length. To implement some larger designs, string and rope could be used to swing out arcs and set out lengths. For drawing straight lines, straightedges, rulers, and set squares would have been employed. However, the central purpose of the set square was, of course, drawing and checking right angles. Simple combinations of compass positions could produce a variety of pointed and rounded arch shapes. See Figure 4.

Hugh of St. Victor (1096–1141) in the twelfth century wrote an influential text, *Practica geometriae*, that made the first distinction between "practical geometry" and "theoretical geometry" in the Latin West. In the history of mathematics, important precursors to this treatise were Euclid's *Elements*, Heron's *Metrica*, the Roman surveyors' *Corpus agrimensorum*, *Ars geometriae* attributed to Boethius, and Gerbert's *Geometria* and the associated anonymous *Geometria incerti auctoris*. Hugh of St. Victor did not actually include the geometry

involved in crafts such as stonemasonry under "practical geometry," only the geometry involved in celestial and terrestrial surveying. Later in the same century, the Spanish philosopher and translator of Arabic, Dominicus Gundissalinus, set forth a schematization of knowledge, *De divisione philosophiae*. Here "practical geometry" was now more broadly defined as involving both the geometric methods of surveying and the crafts. "Theoretical geometry" was basically Euclidean geometry with its axiomatic and deductive proof approach.

We are fortunate to have accounts of consultative building committees that were struck for advice on correcting faults in older buildings or on concerns about new building projects. A central emphasis of these deliberations was that a building should be built with the right proportion and measure for its attractiveness and structural stability. The consultations for Milan Cathedral in 1392 and 1400–1401 are particularly well known and include discussion of whether the building program should continue *ad quadratum* or *ad triangulum* in the elevations. The former expression refers, of course, to using the square and the latter the triangle, and probably, more precisely, the equilateral triangle. In addition to statements from master masons, the assistance of a mathematician named Gabriele Stornaloco was enlisted.

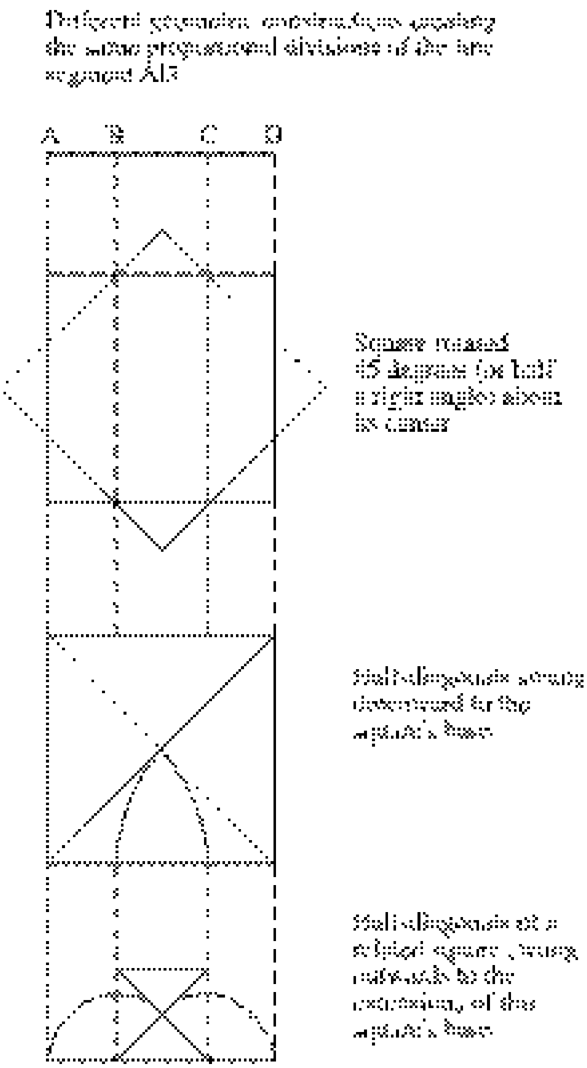


Figure 5. Closely related geometric constructions involving squares, a rotated square, diagonals, and half-diagonals.

The Architectural Ratios

The ratios of the side to the diagonal of a square, the side to the altitude of an equilateral triangle, and the side to the diagonal of a regular pentagon (later known as the 'golden section'), and their (rational) approximations through ratios of simple whole numbers, are fundamental to medieval architectural design. The side of the square to its diagonal, however, has the strongest pedigree in terms of historical documentation in building manuals, diagrams and illustrations. Examples of (rational) approximations for the side to the diagonal of a square are 5:7, 10:14, 12:17, and 24:34. Apart from such (rational) approximations, even simpler whole number relationships 1:1, 1:2, 2:3 and 3:4 were also employed. Another applied mathematical element involved the use of standard measurement units, such as the English Royal foot (0.3048m), and module lengths. The geometry, ratios and measures worked hand in hand throughout the creative process from the building's design, layout on the ground to the ultimate fabrication and assembly in stone and wood manifesting the completed building.

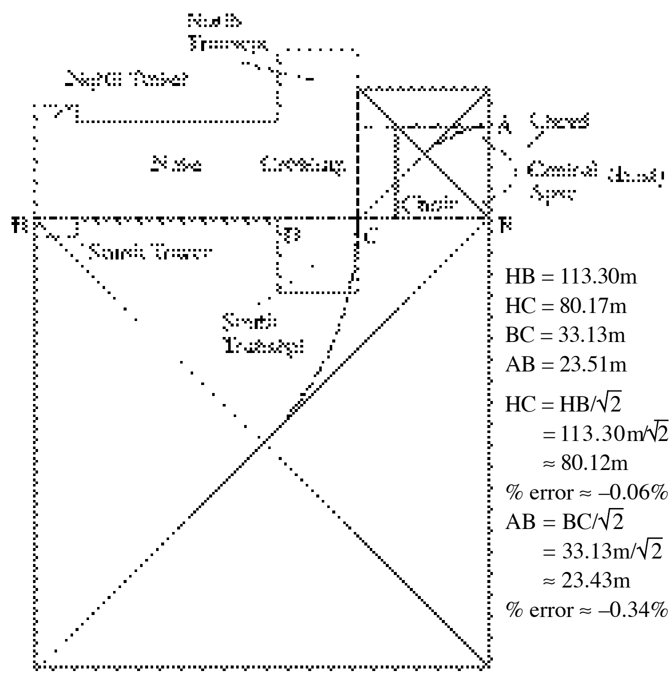


Figure 6. The geometric relationship of the interior length of Durham Cathedral up to the chord of the central apse (HB) and the interior length of the tower, nave and transept (HC), and ii) the interior length (BC) and width (AB) of the choir and east end up to the chord of the central apse. Each of these ratios, *HB* to *HC* and *BC* to *AB*, is as the side to the half-diagonal of a square, i.e., $\sqrt{2}$: 1.

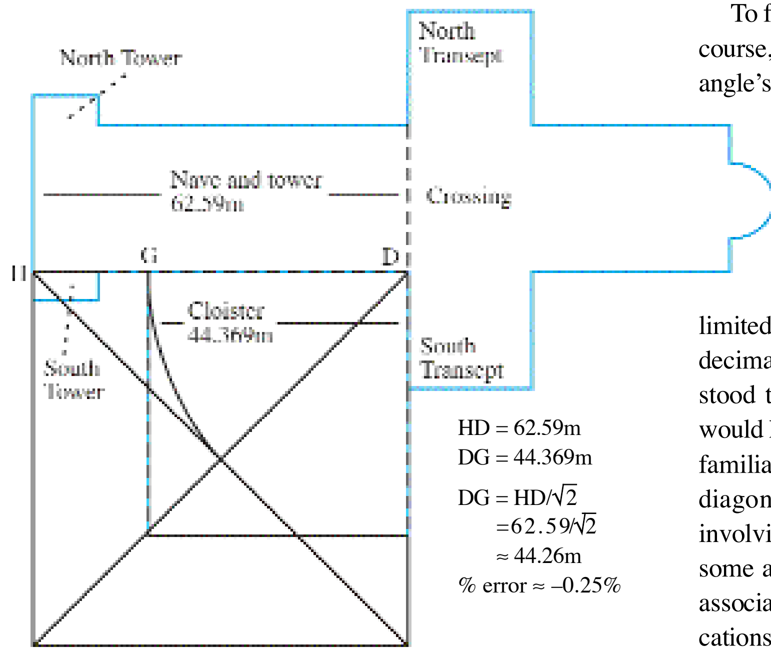


Figure 7. The geometric relationship of the north side of the cloister (DG), and the interior tower and nave (HD) lengths of Durham Cathedral, again we see the side to the half-diagonal of the square.

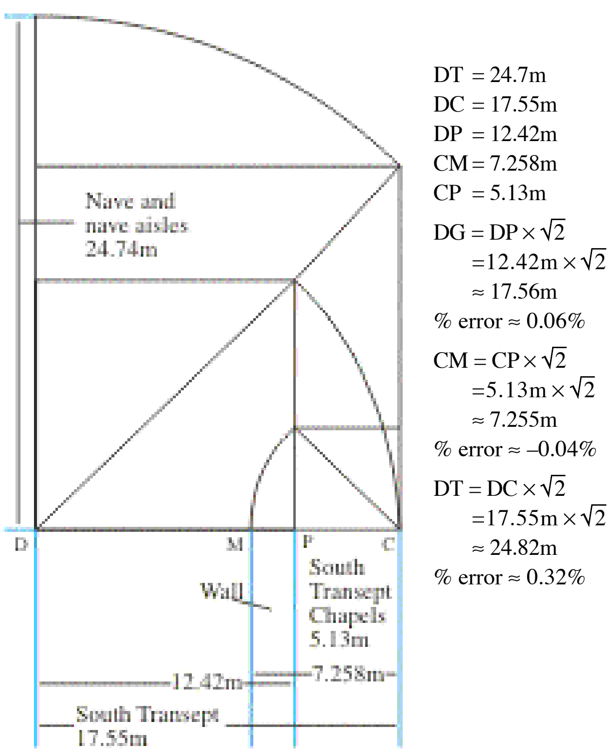


Figure 8. The geometry of the interior ground plan of the south transept, and a relationship with the interior width of the nave and nave aisles of Durham Cathedral, again the ubiquitous $\sqrt{2}:1$ ratio.

To first delve more into the geometry and ratios, one notes, of course, that the square's diagonal to its side, the equilateral triangle's altitude to its side, and the regular pentagon's diagonal to its side are equal to $\sqrt{2}:1$, $\sqrt{3}/2:1$, $(\sqrt{5}+1)/2:1$ (golden section), respectively. Specific proportions may be explained by several competing geometric constructions (e.g. Figure 5). However, during the Middle Ages, mathematical knowledge of the irrational numbers, such as $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$, was quite limited. For example, $\sqrt{2}$ was not understood as a non-repeating decimal expansion beginning 1.414213… as commonly understood today. Indeed, even the mathematical terminology $\sqrt{2}$ would have been foreign to medieval masons, in contrast to their familiarity with the geometric maneuvers involving the square's diagonal and its side, and some (rational) approximations involving whole numbers. Nevertheless, there would have been some awareness of the special quality of these ratios and their associated geometric motifs, and even some sense of the implications of the irrationality of these ratios. This is suggested by the Roman architectural forerunner Vitruvius and his discussion of the application of the side and diagonal of a square. He pays great homage to Plato for stating and showing in *Meno* that the square on the diagonal of another square has twice the area of the smaller square. Vitruvius emphasizes the great utility of this result. He notes that this surmounts an arithmetical impossibility (i.e., writing down the square root of two) with a geometric solution. This ascribes to the ratio of the side of the square to its diagonal a special status—it is a profound principle. Its profundity, association with Plato as noted by Vitruvius, and longstanding traditional use may have given a reverence and prestige to this principle during the medieval period.

Mathematical Rediscovery

The rediscovery of the mathematical schema, including the side of the square and its diagonal, employed at a specific church is a challenging problem within architectural history. As an example, Durham Cathedral, an Anglo-Norman Romanesque church, built 1093–1130/1133, in the northeast of England has many mathematical points of interest. Consider the constructional-geometric procedure for the major lengths of the building and the widths of the transepts. A design motif that was common, though not standard, in the large Anglo-Norman Romanesque churches was basically, in terms of interior lengths, that the west tower/nave (HD in Figure 6) to the west tower/nave/crossing/choir up to the chord of the central east-end apse (HB in Figure 6) is in the same ratio as the side of the square to its diagonal or equivalently, the half-diagonal to the side of the square. A slightly different situation appears at Durham Cathedral. The "cut-point" possibly should be the interior east wall of the transept chapels (C in Figure 6), rather than using the interior west wall of the

Figure 9. God as Geometer: Creation as a mathematical act. Vienna, Österreichische Nationalbibliothek, 2554, fol.1 (frontispiece), Bible Moralisée, Reims, c.1250.

transept or the transept piers (D in Figure 6). The length of the choir up to the chord of the central east-end apse (BC in Figure 6) to the width of the choir (AB in Figure 6) are also in the ratio of the side to the half-diagonal of the square. One of the other common larger scale relationships, for Anglo-Norman churches with attached monasteries including Durham, is that the length of the cloister's side adjoining the nave (DG in Figure 7) to the length of the tower and nave (HD in Figure 7) equals the ratio of the side of the square to its diagonal, or equivalently the 'half-diagonal' of the square to its side. The thorough application of the square's side and diagonal also occurred in the ground plan of the south transept and suggests a relationship between the full interior width of the south transept and the interior width of the nave and its north and south aisles [Figure 8]. These relationships are examples of the application of practical or constructive geometry in the design and laying out of Durham Cathedral.

The Deeper Meaning

Practical-geometric methods were thought necessary for a building's structural stability and the visually harmonious and unifying relationship of its individual parts to the whole. From the point of view of the masons, practical geometry allowed for basic designs to be easily repeated and varied. Also, it would have allowed for easy communication to assistants, and for the straightforward teaching of principles and motifs to apprentices. It was understood to be the correct way as indicated in sources from the Roman Vitruvius to the late medieval German master masons. Further, the three geometric ratios already discussed, and related in modern terms to $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$ have a marked simplicity: they derive from the first three regular polygons. This may have stamped them as fundamental design elements or motifs for masonry .

The meaning and symbolism of the cathedral also applied mathematics. The masons' mathematical knowledge applied in building design had a religious significance. In this regard, an important theme in the study of the sacred significance of medieval architecture is the perception of God's Creation as the exemplary handiwork. Part of this theme is Creation as a mathematical act, following biblical passages such as Wisdom 11:21: "Thou madest all things in measure, number and weight." In medieval manuscript illuminations and drawings there are depictions of God creating the cosmos with the aid of a compass or dividers [Figure 9], and sometimes an equal-armed weighing balance as well. God the Creator and orderer implies the divine origin of geometry and measure applied in creating and ordering medieval architecture. The mathematical elements of design had theological import: the masons and builders were co-creators with God, applying mathematics in the creation of the building, a microcosm, to parallel or mirror, God's mathematical creation of the universe, the macrocosm.

The building of a medieval cathedral, and indeed this period's architecture in general, was highly mathematical in conception and execution. Following Abbot Suger's description of the rebuilding of the eastern part of the Abbey Church of St. Denis (1140–1144) and the use there of mathematical instruments, the building of a medieval church was to follow, like the entirety of Creation, the law of Wisdom 11:21. Further, a widespread and standard encyclopedia of the Middle Ages, the *Etymologiae* of St. Isidore of Seville (560–636) includes discussion of weights and measures and Wisdom 11:21, and architecture, and emphasized:

> Take away number in all things and all things perish.
> Take away computation from the world and everything
> is wrapped in blind ignorance.

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gy, you called such manifolds ‘immunity manifolds.’ Although you say in the introduction that you are not an expert in biology, the idea motivated your nomenclature throughout the thesis. Some immunity manifolds are healthy, some are not. Some even have auto-immune diseases!”

“Just names I gave them to help me describe and understand the mathematical structure. I was doing math, not biology.”

“Perhaps, but your scientists were never able to make sense out of the immune system before and there was so much room for rich and beautiful discoveries to grow out of your theories. I just couldn't wait to get to work on it. You must have seen by now how I was able to bring it to play on some questions regarding the improvement of vaccinations and in just a few months some medical researchers working on the disease scleroderma will discover that they can use Serre duality to…”

“But,” she interrupted, “you made a mistake. I mean, I made a mistake. I was wrong about equation 3.6. The microchimeric subalgebras don't have to be simply connected, and so definition 3.9 just didn't make any sense and…”

“No,” his jaw fell open and he dropped the last little piece of his cookie. “Not equation 3.6! But that was one of my favorite parts. I used that everywhere!”

“Yes, I know. That is how I found you. You see, I caught the mistake just after the *Memoir* was published. It wasn't easy, but I was able to make sure that every copy with the mistake was collected unread and replaced with a corrected version...every copy except the one that was sent separately by private courier here to your house. And that is how I knew…”

“Oh my,” he said, stirring his tea vigorously. “Oh my, how careless of me! We will have to do something about that, won't

Continued from page 15

Art and architecture do need to be seen as separate from mathematics, and traditionally, as mentioned earlier, they went hand in hand as attested in such renowned cases as Polykleitos, the master mason of Durham Cathedral, the consultative committees for Milan Cathedral, Leonardo da Vinci, and Albrecht Dürer. The re-discovery of the mathematics of building a medieval cathedral, stonemasonry, the Art of Geometry, is a worthy part of not only the history of architecture, but also the history of mathematics. ■

For Further Reading

Nicola Coldstream’s *Masons and Sculptors*, Medieval Craftsmen series, British Museum and University of Toronto Press (1991), provides a succinct overview of the history and methods of medieval stonemasonry. Eric Fernie’s “A Beginner's Guide to the Study of Architectural Proportions and Systems of Length” in *Medieval Architecture and its Intellectual Context: Studies in Honour of Peter Kidson*, (Eds. Paul Crossley and Eric Fernie), Hambledon Press (1990), pp. 229–237, gives important guidelines, with examples, for research on recovering geometry, proportions, and historical measurement units in the design and building of medieval architecture. Wolfgang Weimer’s and Gerhard Wetzel’s “A Report on Data Analysis of Building Geometry by Computer,” *Journal of the Society of Architectural Historians*, vol. 53 (1994), pp. 448–460, shows a particular scientific approach with specially designed computer software to analyze medieval building measurements.