# Pi Day Is Upon Us Again and We Still Do Not Know if Pi Is Normal

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**Abstract.** The digits of $\pi$ have intrigued both the public and research mathematicians from the beginning of time. This article briefly reviews the history of this venerable constant, and then describes some recent research on the question of whether $\pi$ is normal, or, in other words, whether its digits are statistically random in a specific sense.

**1. PI AND ITS DAY IN MODERN POPULAR CULTURE.** The number $\pi$, unique among the pantheon of mathematical constants, captures the fascination both of the public and of professional mathematicians. Algebraic constants such as $\sqrt{2}$ are easier to explain and to calculate to high accuracy (e.g., using a simple Newton iteration scheme). The constant $e$ is pervasive in physics and chemistry, and even appears in financial mathematics. Logarithms are ubiquitous in the social sciences. But none of these other constants has ever gained much traction in the popular culture.

In contrast, we see $\pi$ at every turn. In an early scene of Ang Lee's 2012 movie adaptation of Yann Martel's award-winning book *The Life of Pi*, the title character Piscine ("Pi") Molitor writes hundreds of digits of the decimal expansion of $\pi$ on a blackboard to impress his teachers and schoolmates, who chant along with every digit.[1] This has even led to humorous take-offs such as a 2013 Scott Hilburn cartoon entitled "Wife of Pi," which depicts a 4 figure seated next to a $\pi$ figure, telling their marriage counselor "He's irrational and he goes on and on." [**22**].

This attention comes to a head on March 14 of each year with the celebration of "Pi Day," when in the United States, with its taste for placing the day after the month, 3/14 corresponds to the best-known decimal approximation of Pi (with 3/14/15 promising a gala event in 2015). Pi Day was originally founded in 1988, the brainchild of Larry Shaw of San Francisco's Exploratorium (a science museum), which in turn was founded by Frank Oppenheimer (the younger physicist brother of Robert Oppenheimer) after he was blacklisted by the U.S. Government during the McCarthy era.

Originally a light-hearted gag where folks walked around the Exploratorium in funny hats with pies and the like, by the turn of the century Pi Day was a major educational event in North American schools, garnering plenty of press.[2] In 2009, the U.S. House of Representatives made Pi Day celebrations official by passing a resolution designating March 14 as "National Pi Day," and encouraging "schools and educators to observe the day with appropriate activities that teach students about Pi and engage them about the study of mathematics." [**23**].[3]

As a striking example, the March 14, 2007 *New York Times* crossword puzzle featured clues, where, in numerous locations, $\pi$ (standing for PI) must be entered at the

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[1]Good scholarship requires us to say that in the book Pi contents himself with drawing a circle of unit diameter.

[2]Try `www.google.com/trends?q=Pi+` to see the seasonal interest in 'Pi'.

[3]This seems to be the first legislation on Pi to have been adopted by a government, though in the late 19th century Indiana came embarrassingly close to legislating its value, see [**12**, Singmaster, Entry 27] and [**14**]. This MONTHLY played an odd role in that affair.

intersection of two words. For example, 33 across "Vice president after Hubert" (answer: S$\pi$RO) intersects with 34 down "Stove feature" (answer: $\pi$LOT). Indeed, 28 down, with clue "March 14, to mathematicians," was, appropriately enough, PIDAY, while PIPPIN is now a four-letter word. The puzzle and its solution are reprinted with permission in [**15**, pp. 312–313].

**$\pi$ Mania in popular culture.** Many instances are given in [**14**]. They include the following:

1. On September 12, **2012**, five aircraft armed with dot-matrix-style skywriting technology wrote 1000 digits of $\pi$ in the sky above the San Francisco Bay area as a spectacular and costly piece of *piformance* art.
2. On March 14, **2012**, U.S. District Court Judge Michael H. Simon dismissed a copyright infringement suit relating to the lyrics of a song by ruling that "Pi is a non-copyrightable fact."
3. On the September 20, **2005** edition of the North American TV quiz show *Jeopardy!*, in the category "By the numbers," the clue was "'How I want a drink, alcoholic of course' is often used to memorize this." (Answer: What is Pi?).
4. On August 18, **2005**, Google offered 14,159,265 "new slices of rich technology" in their initial public stock offering. On January 29, **2013** they offered a $\pi$ million dollar prize for successful hacking of the Chrome Operating System on a specific Android phone.
5. In the first **1999** *Matrix* movie, the lead character Neo has only 314 seconds to enter the Source. *Time* noted the similarity to the digits of $\pi$.
6. The **1998** thriller "Pi" received an award for screenplay at the Sundance film festival. When the authors were sent advance access to its website, they diagnosed it a fine hoax.
7. The May 6, **1993** edition of *The Simpsons* had Apu declaring "I can recite pi to 40,000 places. The last digit is 1." This digit was supplied to the screenwriters by one of the present authors.
8. In Carl Sagan's **1986** book *Contact*, the lead character (played by Jodie Foster in the movie) searched for patterns in the digits of $\pi$, and after her mysterious experience sought confirmation in the base-11 expansion of $\pi$.

With regards to item #3 above, there are many such "pi-mnemonics" or "piems" (i.e., phrases or verse whose letter count, ignoring punctuation, gives the digits of $\pi$) in the popular press [**12, 14**]. Another is "Sir, I bear a rhyme excelling / In mystic force and magic spelling / Celestial sprites elucidate / All my own striving can't relate." [**13**, p. 106]. Some are very long [**12**, Keith, Entry 59, pp. 560–561].

Sometimes the attention given to $\pi$ is annoying, such as when on August 14th, 2012, the U.S. Census Office announced the population of the country had passed *exactly* 314,159,265. Such precision was, of course, completely unwarranted. Sometimes the attention is breathtakingly pleasurable.[4,5]

**Poems versus piems.** While piems are fun they are usually doggerel. To redress this, we include examples of excellent $\pi$ poetry and song.[6] In Figure 1 we present the much anthologised poem "PI," by Polish poet Wislawa Szymborska (1923–2012) who won

[4] See the 2013 movie at http://www.youtube.com/watch?v=Vp9zLbIE8zo.

[5] A comprehensive Pi Day presentation is lodged at http://www.carma.newcastle.edu.au/jon/piday.pdf.

[6] See also [**12**, Irving Kaplansky's "A song about Pi."].

the 1996 Novel prize for literature [**29**, p. 174]. In Figure 2 we present the lyrics of "Pi" by the influential British singer songwriter Kate Bush [**18**]. The *Observer* review of her 2005 collection *Aerial*, on which the song appears, wrote that it is

> a sentimental ode to a mathematician, audacious in both subject matter and treatment. The chorus is the number sung to many, many decimal places.[7]

The admirable number pi:
three point one four one.
All the following digits are also just a start,
five nine two because it never ends.
It can't be grasped, six five three five, at a glance,
eight nine, by calculation,
seven nine, through imagination,
or even three two three eight in jest, or by comparison
four six to anything
two six four three in the world.
The longest snake on earth ends at thirty-odd feet.
Same goes for fairy tale snakes, though they make it a little longer.
The caravan of digits that is pi
does not stop at the edge of the page,
but runs off the table and into the air,
over the wall, a leaf, a bird's nest, the clouds, straight into the sky,
through all the bloatedness and bottomlessness.
Oh how short, all but mouse-like is the comet's tail!
How frail is a ray of starlight, bending in any old space!
Meanwhile two three fifteen three hundred nineteen
my phone number your shirt size
the year nineteen hundred and seventy-three sixth floor
number of inhabitants sixty-five cents
hip measurement two fingers a charade and a code,
in which we find how blithe the trostle sings!
and please remain calm,
and heaven and earth shall pass away,
but not pi, that won't happen,
it still has an okay five,
and quite a fine eight,
and all but final seven,
prodding and prodding a plodding eternity
to last.

**Figure 1.** "PI," by Wislawa Szymborska

## 2. PRE-DIGITAL HISTORY.

$\pi$ is arguably the only mathematical topic from very early history that is still being researched today. The Babylonians used the approximation $\pi \approx 3$. The Egyptian Rhind Papyrus, dated roughly 1650 BCE, suggests $\pi = 32/18 = 3.16049\ldots$. Early Indian mathematicians believed $\pi = \sqrt{10} = 3.162277\ldots$. Archimedes, in the first mathematically rigorous calculation, employed a clever iterative construction of inscribed and circumscribed polygons to establish that

$$3\ 10/71 = 3.14084\ldots < \pi < 3\ 1/7 = 3.14285\ldots$$

[7]She sings over 150 digits but errs after 50 places. The correct digits occurred with the published lyrics.

Sweet and gentle sensitive man
With an obsessive nature and deep fascination
For numbers
And a complete infatuation with the calculation
Of PI

Oh he love, he love, he love
He does love his numbers
And they run, they run, they run him
In a great big circle
In a circle of infinity

3.1415926535 897932
3846 264 338 3279

Oh he love, he love, he love
He does love his numbers
And they run, they run, they run him
In a great big circle
In a circle of infinity
But he must, he must, he must
Put a number to it

50288419 716939937510
582319749 44 59230781
6406286208 821 4808651 32

Oh he love, he love, he love
He does love his numbers
And they run, they run, they run him
In a great big circle
In a circle of infinity

82306647 0938446095 505 8223...

**Figure 2.** "Pi," by Kate Bush

This amazing work, done without trigonometry or floating point arithmetic, is charmingly described by George Phillips [**12**, Entry 4].

**Life after modern arithmetic.** The advent of modern positional, zero-based decimal arithmetic, most likely discovered in India prior to the fifth century [**4, 27**], significantly reduced computational effort. Even though the Indo-Arabic system, as it is now known, was introduced to Europeans first by Gerbert of Aurillac (c. 946–1003, who became Pope Sylvester II in 999) in the 10th century, and again, in greater detail and more successfully, by Fibonacci in the early 13th century, Europe was slow to adopt it, hampering progress in both science and commerce. In the 16th century, prior to the widespread adoption of decimal arithmetic, a wealthy German merchant was advised, regarding his son's college plans,

> If you only want him to be able to cope with addition and subtraction, then any French or German university will do. But if you are intent on your son going on to multiplication and division—assuming that he has sufficient gifts—then you will have to send him to Italy. [**24**, p. 577]

**Life after calculus.** Armed with decimal arithmetic and modern calculus, 17th-, 18th-, and 19th-century mathematicians computed $\pi$ with aplomb. Newton recorded

16 digits in 1665, but later admitted, "I am ashamed to tell you how many figures I carried these computations, having no other business at the time." In 1844 Dase, under the guidance of Strassnitzky, computed 212 digits correctly in his head [**14**]. These efforts culminated with William Shanks (1812–1882), who employed John *Machin's formula*

$$\frac{\pi}{4} = 4 \arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right), \tag{1}$$

where $\arctan x = x - x^3/3 + x^5/5 - x^7/7 + x^9/9 - \cdots$, to compute 707 digits in 1874. His 1853 work to 607 places was funded by 30 subscriptions from such notables as Rutherford, De Morgan (two copies), Herschel (Master of the Mint and son of the astronomer) and Airy.[8]

Alas, only 527 digits were correct (as Ferguson found nearly a century later in 1946 using a calculator), confirming the suspicions of De Morgan at the time, who asserted that there were too many sevens in Shanks' published result (although the statistical deviation was not as convincing as De Morgan thought [**26**]). A brief summary of this history is shown in Table 1. We note that Sharp was a cleric, Ferguson was a school teacher, and Dase a "kopfrechnenner." Many original documents relating to this history can be found in [**12**].

**Table 1.** Brief chronicle of pre-20th-century $\pi$ calculations

| | | | |
|---|---|---|---|
| Archimedes | 250? BCE | 3 | 3.1418 (ave.) |
| Liu Hui | 263 | 5 | 3.14159 |
| Tsu Ch'ung Chi | 480? | 7 | 3.1415926 |
| Al-Kashi | 1429 | 14 | |
| Romanus | 1593 | 15 | |
| Van Ceulen | 1615 | 39 | (35 correct) |
| Newton | 1665 | 16 | |
| Sharp | 1699 | 71 | |
| Machin | 1706 | 100 | |
| De Lagny | 1719 | 127 | (112 correct) |
| Vega | 1794 | 140 | |
| Rutherford | 1824 | 208 | (152 correct) |
| Strassnitzky and Dase | 1844 | 200 | |
| Rutherford | 1853 | 440 | |
| Shanks | 1853 | 607 | (527 correct) |
| Shanks | 1873 | 707 | (527 correct) |

**Mathematics of Pi.** Alongside these numerical developments, the mathematics behind $\pi$ enjoyed comparable advances. In 1761, using improper continued fractions, Lambert [**12**, Entry 20] proved that $\pi$ is irrational, thus establishing that the digits of $\pi$ never repeat. Then in 1882, Lindemann [**12**, Entry 22] proved that $e^\alpha$ is transcendental for every nonzero algebraic number $\alpha$, which immediately implied that $\pi$ is transcendental (since $e^{i\pi} = -1$). This result settled in decisive terms the 2000-year-old question of whether a square could be constructed with the same area as a circle, using compass and straightedge (it cannot, because if it could then $\pi$ would be a geometrically *constructible number* and hence algebraic).

[8]He had originally intended to present only about 500 places, and evidently added the additional digits while finishing the galleys a few months later [**12**, Entry 20]. Errors introduced in a rush to publish are not new.

**3. THE TWENTIETH CENTURY AND BEYOND.** With the development of computer technology in the 1950s and 1960s, $\pi$ was computed to thousands of digits, facilitated in part by new algorithms for performing high-precision arithmetic, notably the usage of fast Fourier transforms to dramatically accelerate multiplication.

**Ramanujan-type series for $1/\pi$.** Even more importantly, computations of $\pi$ began to employ some entirely new mathematics, such as Ramanujan's 1914 formula

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!\,(1103 + 26390k)}{(k!)^4\, 396^{4k}}, \tag{2}$$

each term of which produces an additional eight correct digits in the result [**16**]. David and Gregory Chudnovsky employed the variant

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!\,(13591409 + 545140134k)}{(3k)!\,(k!)^3\, 640320^{3k+3/2}}, \tag{3}$$

each term of which adds 14 correct digits. Both of these formulas rely on rather deep number theory [**14**] and related modular-function theory [**16**].

**Reduced complexity algorithms [17] for $1/\pi$.** Another key development in the mid 1970s was the *Salamin–Brent algorithm* [**12**, Entries 46 and 47] for $\pi$: Set $a_0 = 1$, $b_0 = 1/\sqrt{2}$, and $s_0 = 1/2$. Then for $k \geq 1$, iterate

$$a_k = \frac{a_{k-1} + b_{k-1}}{2} \qquad b_k = \sqrt{a_{k-1} b_{k-1}}$$

$$c_k = a_k^2 - b_k^2 \qquad s_k = s_{k-1} - 2^k c_k \qquad p_k = \frac{2a_k^2}{s_k}. \tag{4}$$

The value of $p_k$ converges *quadratically* to $\pi$—each iteration approximately *doubles* the number of correct digits.

A related algorithm, inspired by a 1914 Ramanujan paper, was found in 1986 by one of us and Peter Borwein [**16**]: Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Then for $k \geq 0$, iterate

$$y_{k+1} = \frac{1 - (1 - y_k^4)^{1/4}}{1 + (1 + y_k^4)^{1/4}} \tag{5}$$

$$a_{k+1} = a_k (1 + y_{k+1})^4 - 2^{2k+3} y_{k+1} (1 + y_{k+1} + y_{k+1}^2).$$

Then $a_k$ converges *quartically* to $1/\pi$—each iteration approximately quadruples the number of correct digits. Just twenty-one iterations suffices to produce an algebraic number that agrees with $\pi$ to more than six trillion digits (provided all iterations are performed with this precision).

With discoveries such as these, combined with prodigious improvements in computer hardware (thanks to *Moore's Law*) and clever use of parallelism, $\pi$ was computed to millions, then billions, and, in 2011, to 10 *trillion* decimal digits. A brief chronicle of $\pi$ computer-age computations is shown in Table 2.[9]

[9]It is probably unnecessary to note that the Shanks of this table is not the Shanks of Table 1.

**Table 2.** Brief chronicle of computer-age $\pi$ calculations

| | | |
|---|---|---|
| Ferguson | 1945 | 620 |
| Smith and Wrench | 1949 | 1,120 |
| Reitwiesner et al. (ENIAC) | 1949 | 2,037 |
| Guilloud | 1959 | 16,167 |
| Shanks and Wrench | 1961 | 100,265 |
| Guilloud and Bouyer | 1973 | 1,001,250 |
| Kanada, Yoshino and Tamura | 1982 | 16,777,206 |
| Gosper | 1985 | 17,526,200 |
| Bailey | Jan. 1986 | 29,360,111 |
| Kanada and Tamura | Jan. 1988 | 201,326,551 |
| Kanada and Tamura | Nov. 1989 | 1,073,741,799 |
| David and Gregory Chudnovsky | Aug. 1991 | 2,260,000,000 |
| Kanada and Takahashi | Apr. 1999 | 51,539,600,000 |
| Kanada and Takahashi | Sep. 1999 | 206,158,430,000 |
| Kanada and 9 others | Nov. 2002 | 1,241,100,000,000 |
| Bellard | Dec. 2009 | 2,699,999,990,000 |
| Kondo and Yee | Aug. 2010 | 5,000,000,000,000 |
| Kondo and Yee | Oct. 2011 | 10,000,000,000,000 |

## 4. COMPUTING DIGITS OF $\pi$ AT AN ARBITRARY STARTING POSITION.

A recent reminder of the folly of thinking that $\pi$ is fully understood was the 1996 discovery of a simple scheme for computing binary or hexadecimal digits of $\pi$, beginning at an arbitrary starting position, without needing to compute any of the preceding digits. This scheme is based on the following formula, which was *discovered by a computer program* implementing Ferguson's "PSLQ" algorithm [**9, 20**]:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left( \frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right). \tag{6}$$

The proof of this formula (now known as the "BBP" formula for $\pi$) is a relatively simple exercise in calculus. It is perhaps puzzling that it had not been discovered centuries before. But then no one was looking for such a formula.

**How bits are extracted.** The scheme to compute digits of $\pi$ beginning at an arbitrary starting point is best illustrated by considering the similar (and very well known) formula for $\log 2$:

$$\log 2 = \sum_{k=1}^{\infty} \frac{1}{k 2^k}. \tag{7}$$

Note that the binary expansion of $\log 2$ beginning at position $d+1$ is merely the fractional part of $2^d \log 2$, so that we can write (where $\{\cdot\}$ denotes fractional part):

$$\{2^d \log 2\} = \left\{ \left\{ \sum_{k=1}^{d} \frac{2^{d-k} \bmod k}{k} \right\} + \left\{ \sum_{k=d+1}^{\infty} \frac{2^{d-k}}{k} \right\} \right\}. \tag{8}$$

Now note that the numerators of the first summation can be computed very rapidly by means of the binary algorithm for exponentiation, namely the observation, for example, that $3^{17} \bmod 10 = ((((3^2 \bmod 10)^2 \bmod 10)^2 \bmod 10)^2 \bmod 10) \cdot 3 \bmod 10$.

This same approach can be used to compute binary or hexadecimal digits of $\pi$ using (6).

This scheme has been implemented to compute hexadecimal digits of $\pi$ beginning at stratospherically high positions. In July 2010, for example, Tsz-Wo Sze of Yahoo! Cloud Computing computed base-16 digits of $\pi$ beginning at position $2.5 \times 10^{14}$. Then on March 14 (Pi Day), 2013, Ed Karrels of Santa Clara University computed 26 base-16 digits beginning at position one quadrillion [**25**]. His result: 8353CB3F7F0C9ACCFA9AA215F2.

**Beyond utility.** Certainly, there is no need for computing $\pi$ to millions or billions of digits in practical scientific or engineering work. A value of $\pi$ to 40 digits is more than enough to compute the circumference of the Milky Way galaxy to an error less than the size of a proton. There are certain scientific calculations that require intermediate calculations to be performed to higher than standard 16-digit precision (typically 32 or 64 digits may be required) [**3**], and certain computations in the field of experimental mathematics have required as high as 50,000 digits [**6**], but we are not aware of any "practical" applications beyond this level.

Computations of digits of $\pi$ are, however, excellent tests of computer integrity—if even a single error occurs during a large computation, almost certainly the final result will be in error, resulting in disagreement with a check calculation done with a different algorithm. For example, in 1986, a pair of $\pi$-calculating programs using (4) and (5) detected some obscure hardware problems in one of the original Cray-2 supercomputers.[10] Also, some early research into efficient implementations of the fast Fourier transform on modern computer architectures had their origins in efforts to accelerate computations of $\pi$ [**2**].

## 5. NEW TECHNIQUES TO EXPLORE NORMALITY AND RELATED PROPERTIES.

Given an integer $b \geq 2$, a real number $\alpha$ is said to be *b-normal* or *normal base b* if every $m$-long string of base-$b$ digits appears in the base-$b$ expansion of $\alpha$ with limiting frequency $1/b^m$. It is easy to show via measure theory that almost all real numbers are $b$-normal for every $b \geq 2$ (a condition known as *absolute normality*), but establishing normality for specific numbers has proven to be very difficult.

In particular, no one has been able to establish that $\pi$ is $b$-normal for any integer $b$, much less for all bases simultaneously. It is a premier example of an intriguing mathematical question that has occurred to countless schoolchildren as well as professional mathematicians through the ages, but which has defied definitive answer to the present day. A proof for any specific base would not only be of great interest worldwide, but would also have potential practical application as a provably effective pseudorandom number generator. This ignorance extends to other classical constants of mathematics, including $e$, $\log 2$, $\sqrt{2}$, and $\gamma$ (Euler's constant). Borel conjectured that all irrational algebraic numbers are absolutely normal, but this has not been proven in even a single instance, to any base.

Two examples where normality has been established are Champernowne's number $C_{10} = 0.12345678910111213\ldots$ (constructed by concatenating the positive integers), which is provably 10-normal, and Stoneham's number $\alpha_{2,3} = \sum_{k\geq 0} 1/(3^k 2^{3^k})$, which is provably 2-normal—see below [**10, 11, 28**]. One relatively weak result for algebraic numbers is that the number of 1-bits in the binary expansion of a degree-$D$ algebraic number $\alpha$ must exceed $Cn^{1/D}$ for all sufficiently large $n$, for a positive number $C$ that

[10] Cray's own tests did not find these errors. After that, these $\pi$ algorithms were included in Cray's test suite in the factory.

depends on $\alpha$ [**8**]. Thus, for example, the number of 1-bits in the first $n$ bits of the binary expansion of $\sqrt{2}$ must exceed $\sqrt{n}$.

In spite of these intriguing developments, it is clear that more powerful techniques must be brought to bear on the question of normality, either for $\pi$ or other well-known constants of mathematics, before significant progress can be achieved. Along this line, modern computer technology suggests several avenues of research.

**Statistical analysis.** One approach is simply to perform large-scale statistical analyses on the digits of $\pi$, as has been done, to some degree, on nearly all computations since ENIAC. In [**7**], for example, the authors empirically tested the normality of its first roughly four trillion hexadecimal (base-16) digits using a Poisson process model, and concluded that, according to this test, it is "extraordinarily unlikely" that $\pi$ is not 16-normal (of course, this result does not pretend to be a proof).

**Graphical representations.** Another fruitful approach is to display the digits of $\pi$ or other constants graphically, cast as a random walk [**1**]. For example, Figure 3 shows a walk based on one million base-4 pseudorandom digits, where at each step the graph moves one unit east, north, west, or south, depending on the whether the pseudorandom iterate at that position is 0, 1, 2, or 3. The color indicates the path followed by the walk—shifted up the spectrum (red-orange-yellow-green-cyan-blue-purple-red) following an HSV scheme with $S$ and $V$ equal to one. The HSV (hue, saturation, and value) model is a cylindrical-coordinate representation that yields a rainbow-like range of colors.

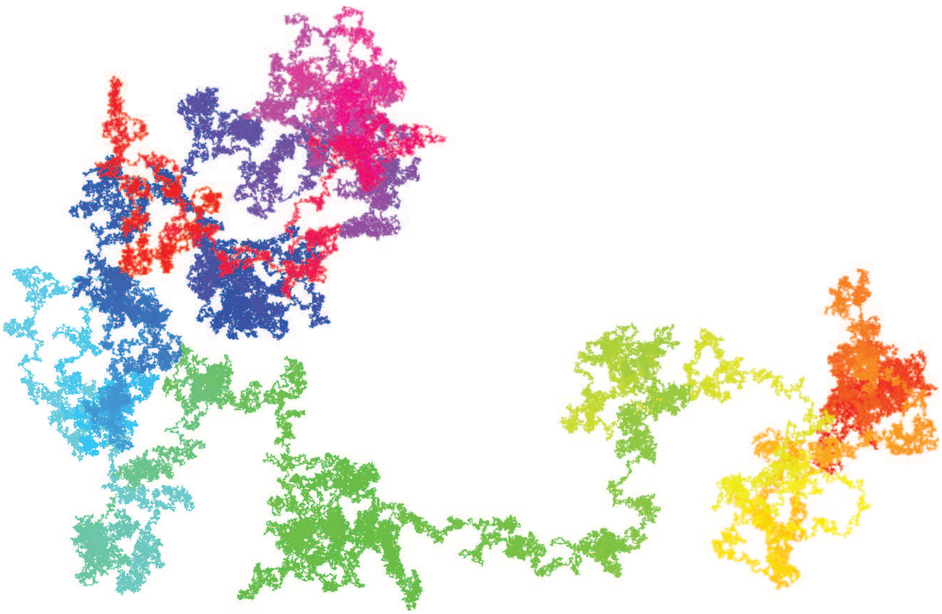

**Figure 3.** A uniform pseudorandom walk

Figure 4 shows a walk on the first 100 billion base-4 digits of $\pi$. This may be viewed dynamically in more detail online at `http://gigapan.org/gigapans/106803`, where the full-sized image has a resolution of $372{,}224 \times 290{,}218$ pixels (108.03 billion pixels in total). This is one of the largest mathematical images ever produced and, needless to say, its production was by no means easy [**1**].

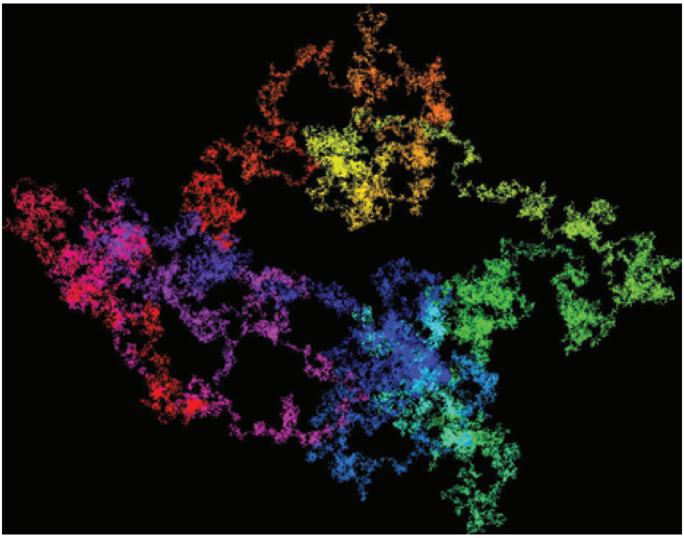

**Figure 4.** A walk on the first 100 billion base-4 digits of $\pi$

Although no clear inference regarding the normality of $\pi$ can be drawn from these figures, it is plausible that $\pi$ is 4-normal (and thus 2-normal), since the overall appearance of its graph is similar to that of the graph of the pseudorandomly generated base-4 digits.

**The Champernowne numbers.** We should emphasize what a poor surrogate for randomness the notion of normality actually is. The base-$b$ *Champernowne* number, $C_b$, is formed by concatenating the natural numbers base $b$ as a floating-point number in that base. It was the first type of number proven to be normal and fails stronger normality tests [**1**]. Thus,

$$\begin{aligned} C_b &:= \sum_{k=1}^{\infty} \frac{\sum_{m=b^{k-1}}^{b^k-1} m b^{-k\left[m-(b^{k-1}-1)\right]}}{b\sum_{m=0}^{k-1} m(b-1)b^{m-1}} \\ C_{10} &= 0.123456789101112\ldots \\ C_4 &= 0.1231011121320212223\ldots_4 . \end{aligned} \tag{9}$$

In Figure 5 we show how far from random a walk on a normal number may be—pictorially or by many quantitative measures—as illustrated by $C_4$.[11]

**Stoneham numbers.** This same tool can be employed to study the digits of Stoneham's constant, namely

$$\alpha_{2,3} = \sum_{k=0}^{\infty} \frac{1}{3^k 2^{3^k}}. \tag{10}$$

This constant is one of the few that is provably 2-normal (and thus $2^n$-normal, for every positive integer $n$) [**10, 11, 28**]. What's more, it is provably *not* 6-normal, so that it is

[11]The subscript four denotes a base-four representation.

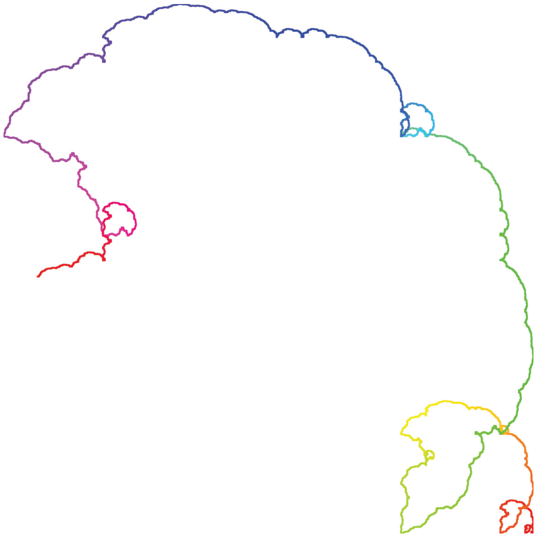

**Figure 5.** A walk on Champernowne's base-4 number

an explicit example of the fact that normality in one base does not imply normality in another base [**5**]. For other number bases, including base 3, its normality is not yet known one way or the other.

Figures 6, 7, and 8 show walks generated from the base-3, base-4, and base-6 digit expansions, respectively, of $\alpha_{2,3}$. The base-4 digits are graphed using the same scheme mentioned above, with each step moving east, north, west, or south, according to whether the digit is 0, 1, 2, or 3. The base-3 graph is generated by moving unit distance at an angle $0, \pi/3$, or $2\pi/3$, respectively, for 0, 1, or 2. Similarly, the base-6 graph is generated by moving unit distance at angle $k\pi/6$ for $k = 0, 1, \ldots, 5$.

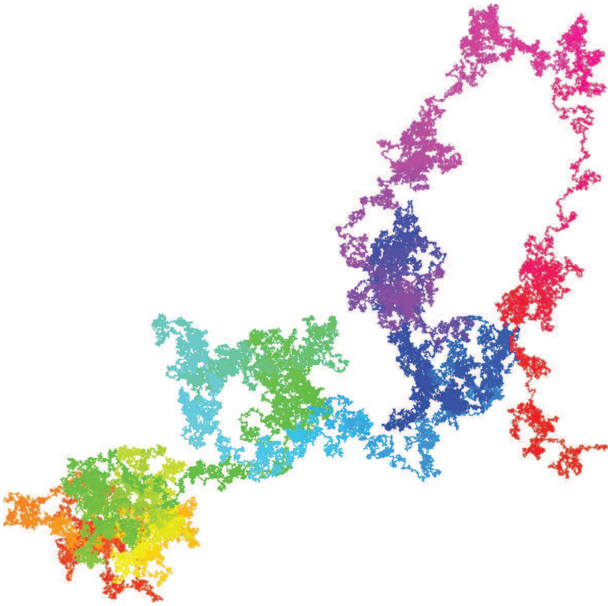

**Figure 6.** A walk on the base-3 digits of Stoneham's constant ($\alpha_{2,3}$)

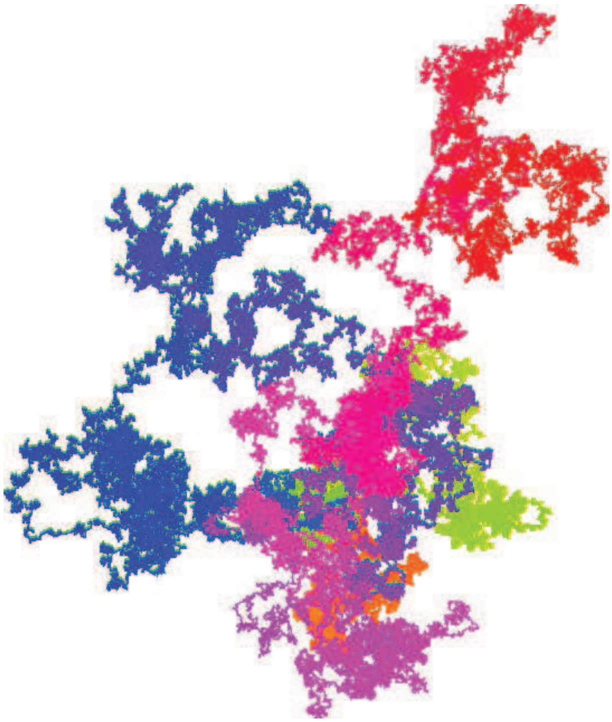

**Figure 7.** A walk on the provably normal base-4 digits of Stoneham's constant ($\alpha_{2,3}$)

**Figure 8.** A walk on the abnormal base-6 digits of Stoneham's constant ($\alpha_{2,3}$)

From these three figures, it is clear that while the base-3 and base-4 graphs appear to be plausibly random (since they are similar in overall structure to Figures 3 and 4), the base-6 walk is vastly different, mostly a horizontal line. Indeed, we discovered the fact that $\alpha_{2,3}$ fails to be 6-normal by a similar empirical analysis of the base-6 digits—there are long stretches of zeroes in the base-6 expansion [**5**]. Results of this type are given in [**1**] for numerous other constants besides $\pi$, both "man-made" and "natural."

Such results certainly do not constitute formal proofs, but they do suggest, often in dramatic form, as we have seen, that certain constants are not normal, and can further be used to bound statistical measures of randomness. For example, remarkable structure was uncovered in the normal Stoneham numbers [**1**]. Moreover, many related quantitative measures of random walks were examined, as were other graphical representations. Much related information, including animations, is stored at `http://carma.newcastle.edu.au/walks/`.

## 6. OTHER UNANSWERED QUESTIONS.

**Mathematical questions.** There are, of course, numerous other unanswered mathematical questions that can be posed about $\pi$.

1. Are the continued fraction terms of $\pi$ bounded or unbounded? The continued fraction expansion provides information regarding how accurately $\pi$ can be written as a fraction.

2. Is the limiting fraction of zeroes in the binary expansion of $\pi$ precisely $1/2$? Is the limiting fraction of zeroes in the decimal expansion precisely $1/10$? We do not know the answers to these questions even for simple algebraic constants such as $\sqrt{2}$, much less $\pi$.
3. Are there infinitely many ones in the ternary expansion of $\pi$? Are there infinitely many sevens in the decimal expansion of $\pi$? Sadly, we cannot definitively answer such basic questions one way or the other.

**Meta-mathematical questions.** For that matter, there are numerous historical questions that are worth asking, if only rhetorically.

1. Why was $\pi$ not known more accurately in ancient times? It could have been known to at least two-digit accuracy by making careful measurements with a rope.
2. Why did Archimedes, in spite of his astonishing brilliance in geometry and calculus, fail to grasp the notion of positional, zero-based decimal arithmetic? This would have greatly facilitated his computations (and likely would have changed history as well).
3. Why did Indian mathematicians fail to extend their system of decimal arithmetic for integers to decimal fractions? Decimal fraction notation was first developed in the Arabic world in the 12th century. They managed by scaling their results, but missed the obvious.
4. Why did Gauss and Ramanujan fail to exploit their respective identities for $\pi$? After all, the Salamin-Brent quadratically convergent algorithm for $\pi$ is derived directly from some identities of Gauss, and other algorithms for $\pi$ follow from (then largely unproven) formulas of Ramanujan. For that matter, why was the notion of an algorithm, fundamental in our computer age, so foreign to their way of thinking?
5. Why did centuries of mathematicians fail to find the BBP formula for $\pi$, namely formula (6), not to mention the associated "trick" for computing digits at an arbitrary starting position? After all, as mentioned above, it can be proven in just a few steps with freshman-level calculus.

In any event, it is clear that modern computer technology has changed the game for $\pi$. Modern systems are literally billions of times faster and more capacious than their predecessors when the present authors began their careers, and advances in software (such as fast Fourier transforms for high-precision numerical computation and symbolic computing facilities for algebraic manipulations) have improved computational productivity just as much as hardware improvements.

And computers are no longer merely passive creatures. A computer program discovered the BBP formula for $\pi$, as well as similar formulas for numerous other constants. Other formulas for $\pi$ have been discovered by computer in a similar way, using high-precision implementations of the PSLQ algorithm or related integer relation algorithms.

**Two unproven facts.** In some of these cases, such as the following two formulas, proofs remain elusive:

$$\frac{4}{\pi^3} \stackrel{?}{=} \sum_{k=0}^{\infty} \frac{r^7(k)\left(1+14k+76k^2+168k^3\right)}{8^{2k+1}}, \tag{11}$$

$$\frac{2048}{\pi^4} \stackrel{?}{=} \sum_{k-0}^{\infty} \frac{(\frac{1}{4})_k (\frac{1}{2})_k^7 (\frac{3}{4})_k}{(1)_k^9 \, 2^{12k}} \left(21+466k+4340k^2+20632k^3+43680k^4\right), \tag{12}$$

where, in the first (due to Gourevich in 2001), $r(k) = 1/2 \cdot 3/4 \cdot \cdots \cdot (2k-1)/(2k)$, and, in the second (due to Cullen in 2010), the notation $(x)_n = x(x+1)(x+1)\cdots(x+n-1)$ is the *Pochhammer symbol*.

**7. CONCLUSION.** The mathematical constant $\pi$ has intrigued both the public and professional mathematicians for millennia. Countless facts have been discovered about $\pi$ and published in the mathematical literature. But, as we have seen, much misunderstanding abounds. We must also warn the innocent reader to beware of mathematical terrorists masquerading as nice people, in their evil attempt to replace $\pi$ by $\tau = 2\pi$ (which is pointless in any event, since the binary expansion of $\tau$ is the same as $\pi$, except for a shift of the decimal point).[12]

Yet there are still very basic questions that remain unanswered, among them whether (and why) $\pi$ is normal to any base. Indeed, why do the basic algorithms of arithmetic, implemented to compute constants such as $\pi$, produce such random-looking results? And can we reliably exploit these randomness-producing features for benefit, say, as commercial-quality pseudorandom number generators?

Other challenges remain as well. But the advent of the computer might at last give humankind the power to answer some of them. Will computers one day be smarter than human mathematicians? Probably not any time soon, but for now they are remarkably pleasant research assistants.

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[12]See `http://tauday.com/` and `www.pbs.org/newshour/rundown/2013/03/for-the-love-of-pi-and-the-tao-of-tau.html`.

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**Haiku 2:57**

I will come to bed
when all three numbers in the
clock are prime . . . again.

—Submitted by Terry Trowbridge