TRUE GRIT IN REAL ANALYSIS

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> *"When I use a word," Humpty Dumpty said, in a rather scornful tone, "it means just what I choose it to mean - neither more nor less." "The question is," said Alice, "whether you can make words mean so many different things." "The question is," said Humpty Dumpty, "which is to be master - that's all."* Lewis Carroll, *Through the Looking Glass* [7]

The axiomatic or Euclidean description of mathematics is unmatched for clarity and precision. It is the ideal toward which all mathematical theories aspire, and one could assert that no theory is truly mathematical if it cannot be so rendered. At the same time, one of the great obstacles faced by undergraduate mathematics majors is understanding and appreciating this style of presentation. Nowhere is this more apparent than in Real Analysis, a course that is usually taught in the purest Euclidean format and which students find confusing, unmotivated, and uninteresting.

The traditional course in Real Analysis proceeds from definitions to axioms to lemmas to theorems without a sense of narrative, without any grip holds for students. It all seems so slippery. The pieces fall miraculously into place, and most students do not know where to begin asking "why?" or "what if?" The solution is not to abandon the axiomatic ideal, but to recognize this conceptual barrier and to find find ways to overcome it. What this course needs is grit, that sandy irritant that abrades against the student's intuitive understanding, forcing her to struggle with the ideas until they become her own and she can see the purpose of precise and unambiguous definitions that are tailored for the proofs that build upon them. This is an account of my own search for how to put that grit into Real Analysis.

1. Lakatos's Insight

The definitions of Real Analysis lie at the core of student difficulties. Like Humpty Dumpty in *Through the Looking Glass*, mathematicians use words to mean whatever they want, and each word means neither more nor less than its definition. Barbara Edwards [9] has given an account of the difficulties that students encounter when faced with this use of definition. She has found two necessary conditions for students to be able to work successfully with mathematical definitions. The first condition is that students must

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realize that mathematicians are not using definitions as they are usually encountered, as descriptions of entities that already exist. For mathematicians, definitions are prescriptive. The second condition is that students must understand the ideas behind and the reasons for each particular definition, a context in which to place it.

Imre Lakatos has a lot to say about mathematical definitions. In his *Proofs and Refutations* [12] he shows how precise definitions emerge from a process in which patterns are observed, theorems and their proofs are discovered, and then counter-examples are produced, forcing a re-examination of underlying assumptions and definitions. This leads to new proofs and new counter-examples in a cycle of proofs and refutations that eventually produces the highly refined definitions and proofs that are codified in textbooks. While Lakatos's principal illustration comes from the history of Euler's formula for the relationship of the numbers of vertices, edges, and faces in a polyhedron, he uses an appendix to illustrate how this process was central to the development of analysis.

As Lakatos realized, appreciation of this dynamic is essential for understanding modern mathematics, and passage directly to Euclidean rigor is pedagogically indefensible:

> The Euclidean method can, in certain problem situations, have deleterious effects on the development of mathematics. Most of these problem situations occur in growing mathematical theories, where growing concepts are the vehicles of progress, where the most exciting developments come from exploring the boundary regions of concepts, from stretching them, and from differentiating formerly undifferentiated concepts. In these growing theories intuition is inexperienced, it stumbles and errs. There is no theory which has not passed through such a period of growth; moreover, this period is the most exciting from the historical point of view. These periods cannot be properly understood without understanding the method of proofs and refutations, without adopting a fallibilist approach
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> This is why Euclid has been the evil genius particularly for the history of mathematics and for the teaching of mathematics, both on the introductory and the creative levels. [12, p. 140]

I became convinced that senior mathematics students must be exposed to this process, and that in this exposure lies the opportunity to explore the nature of mathematical definition and to wrestle with the problems that led to the modern definitions of such concepts as continuity, uniform convergence, integrability, and measure. Lakatos's account of Cauchy and the concept of uniform convergence became one of the pivotal sections of my own real analysis text, *A Radical Approach to Real Analysis* [4].

2. Cauchy's Error

Cauchy's *Cours d'analyse* [8] is actually a pre-calculus textbook written for college freshmen. Calculus was to follow in the second volume which was never written.[1] It is a rigorous pre-calculus text. As Cauchy stated in his preface, "As for the methods, I have sought to give them all of the rigor than one insists upon in geometry." Here we find the first use of $\varepsilon - \delta$ definitions and the first time the modern definitions of continuity and convergence appear in a textbook.

Infinite series were considered to be part of pre-calculus at this time. Cauchy goes to considerable pains to establish them on a solid foundation. He realizes that one can only prove statements about an infinite sum of functions by looking at the approximations by finite sums, and then rigorously justifying the passage to the limit. The first theorem that he proves about infinite sums of functions draws on his definitions of continuity and convergence. He considers a convergent series of continuous functions,

$$S(x) = f_1(x) + f_2(x) + f_3(x) + \cdots ,$$

and defines $S_n(x)$ to be the sum of the first n functions, $R_n(x)$ to be the remainder:

$$\begin{aligned} S_n(x) &= f_1(x) + f_2(x) + \cdots + f_n(x), \\ R_n(x) &= S(x) - S_n(x) = f_{n+1}(x) + f_{n+2}(x) + \cdots . \end{aligned}$$

From his definition of convergence, he knows that $R_n(x)$ can be made arbitrarily small by taking n sufficiently large. Though he will use the phrase "infinitely small," it is clear from other contexts that he means "arbitrarily small."

Cauchy points out that $S_n(x)$, a finite sum of continuous functions, must be continuous, and then goes on to state:

> Let us consider the changes in these three functions when we increase x by an infinitely small value α. For all possible values of n, the change in $S_n(x)$ will be infinitely small; the change in $R_n(x)$ will be as insignificant as the size of $R_n(x)$ when n is made very large. It follows that the change in the function $S(x)$ can only be an infinitely small quantity. From this remark, we immediately deduce the following proposition:
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> **Theorem I** — *When the terms of a series are functions of a single variable x and are continuous with respect to this variable in the neighborhood of a particular value where the series converges, the sum $S(x)$ of the series is also, in the*

[1] The second volume was never written because his students protested so vociferously against this book. What Cauchy was doing was good mathematics, but it was totally inappropriate for his audience.

> *neighborhood of this particular value, a continuous function of x.*

Cauchy has just proven that any infinite series of continuous functions is continuous.

Five years later, in a footnote to a paper on the binomial series [1826], Niels Henrik Abel wrote about this result, "It appears to me that this theorem suffers exceptions" [1]. Indeed, everyone by then knew it did. Fourier series provide the classic examples of infinite sums of continuous functions that are not themselves continuous.

In my classes, this flawed theorem is an opportunity for students to dissect Cauchy's proof in search of the error. Some of them find it. Most of them do not. That is fine. After Abel's observation, it took the mathematical community more than twenty years to formally identify the missing assumption in Cauchy's proof. What is important is that my students are now actively engaged with the ideas behind the theorem. The context has been prepared for the concept of uniform convergence. Students are now able to recognize its importance and usefulness.

Student reactions to the story of Cauchy and the concept of uniform convergence have been instructive. The first time I sprang this on a class, it was greeted with astonishment. How could a great mathematician have been wrong? One student reacted by stating that if he had known earlier that mathematicians could make such fundamental mistakes, then he never would have chosen mathematics as his major. The common reaction was the question, "Then how do we know what is true in mathematics?" There lies the opportunity to begin training mathematicians.

3. Hawkins' Challenge

In the spring of 1997, I taught the second semester of our Real Analysis course. All twelve of the students had used my text for their first semester. Where do we go from there?

I decided to stick to the historical theme that they had enjoyed and which had worked so well, but now to aim for Measure Theory and the Lebesgue integral. A good historical guide to the Lebesgue integral exists in Thomas Hawkins' *Lebesgue's Theory of Integration: Its Origins and Development* [11]. This is challenging reading for those who know measure theory. It was not at all clear that it would work as a textbook. I supplemented it with Bartle's *The Elements of Integration and Lebesgue Measure* [2].

The class met twice a week for thirteen weeks for an hour and a half each time. We began the semester with Bartle's chapters on Lebesgue measure, spending one class each on outer measure (chapters 11 and 12), measurable sets (chapter 13), Borel sets (chapter 14 and part of 15), approximation and additivity (remainder of chapter 15 and 16), and nonmeasurable sets (chapter 17). The structure of these classes was a mixture of short lecture by me and student presentation of solutions to pre-assigned problems that were based on the material for that day. The pace was intentionally fast. I wanted students to be familiar with the terminology and principal results of measure theory. Depth of understanding would come later.

For the next nine weeks, the class was immersed in Hawkins' book, and I stayed away from the front of the classroom. Since I had twelve students, I had chosen twenty-four nineteenth century mathematicians and twenty-four issues for individuals to explain to the rest of the class. Most days, two students presented reports that included a brief biography of the mathematician and a discussion of the issue that had been assigned. They gave me their reports three days before the class so that I could duplicate them and get them to the other students in advance. In most cases, students had come to my office at least once before the report was due to ask questions and get clarification of the key ideas. In class, the students presenting reports were grilled by me and the other students. Those presenting were also the local experts for questions about the problems that had been assigned for that day.

The students found Hawkins' book extremely frustrating. Since they came into this course without knowing the theorems that late nineteenth century mathematicians had struggled to discover, they began by taking all of Hawkins's assertions at face value. But Hawkins is recounting history, not describing mathematical facts. Repeatedly, after seeing a result that seemed to make sense and seeing an argument that looked reasonable, they would then encounter—two or three pages later—a counter-example. And sometimes it was not really a counter-example, it had just seemed like one to the mathematicians of the time. Hawkins proofs are sketchy, and it takes close reading to distinguish among a proof, a piece of a proof, an outline of a proof, an extended example, and a justification of a "fact" that is later revealed to be wrong. My students constantly felt wrong-footed. More importantly, they realized that they were not alone.

They began to consider themselves part of that community of nineteenth century mathematicians who were wrestling with the concept of the integral. They saw the necessity of Cantor's development of set theory and sympathized with his contemporaries who could make little sense of what he was doing. They found the error in Lipschitz's [13] assertion that any nowhere dense set must have a finite number of limit points. They admired Torsten Brodén's construction [6], built on ideas of Dini, Köpcke, and Cantor, of a function with a bounded, non(Riemann)integrable derivative. They agonized over Harnack's inability to recognize the limitations of outer content. More importantly, my students began to ask their own questions: If a set is closed and has measure zero, can it have positive outer content? As the semester progressed, more and more time was spent on discussion of questions that the students themselves raised.

The issue of outer content versus outer measure emerged as one of the dominant themes of the course. The former uses finite covers. The latter allows countable covers. For twenty years, mathematicians focused on outer content rather than measure as they attempted to extend the Riemann integral to functions unbounded around infinitely many points. My students knew why. Riemann's definition of integrability is expressed in terms of a finite number of subintervals. There is no reason to suspect that a countable number of subintervals is even appropriate. Moreover, in 1885 Axel Harnack "proved" that outer measure is internally inconsistent, that it is dependent on the choice of cover [10]. My students came to realize how much more difficult it is to use outer content, and that it leaves huge gaps in the attempt to extend the integral. After the course was over, one student told me that he had wanted to go back to the nineteenth century and shake some sense into those guys; tell them to stop focusing on finite covers and start looking at countable covers.

In the last weeks of the semester, as we started chapter 5 on Lebesgue integration, I returned to Bartle, using his chapters 2–5 to supplement Hawkins' treatment. The last three reports were on Lebesgue, Baire, and Fubini, and these students were responsible for presenting the proofs of the principal theorems of Lebesgue integration. Borel's measure theory and Lebesgue's integral emerged as shafts of light at the end of a very dark passage. My students embraced Bartle's *Elements of Integration* as they could not have at the beginning of the class. Here at last everything was clear and unambiguous, and they could appreciate the struggle that had gone into establishing this clarity.

For the last week, I turned my students loose on Bartle's "Return to the Riemann Integral" [3], a description of and argument for the Henstock integral. They had little trouble digesting this paper and quickly entered into a debate on the pros and cons of this approach to integration.

This was a tough course, but all twelve students stuck with it. The final exam included writing an essay on the problems associated with the Fundamental Theorem of Calculus and how they were overcome. One of the students began, "Before elaborating on the problems with the Fundamental Theorem and their dissolution, I would only like to note that, perhaps for the first time, I got a very real sense of how plausible the Kuhn-Lakatos etc. picture of scientific and mathematical progress is." Another ended his exam with the comment, "Thanks for the class! It has, indeed, been delightfully bewildering, and a real treat. ... I'll miss it."

4. Postscript

Macalester offers *Topics in Analysis* every other year. The next time it was offered, Spring 1999, I was on sabbatical. In 2001, in response to strong student demand, I taught a course on elliptic curves, using the McKean and Moll book [14]. I returned to the course based on Hawkins' book in 2003 and 2005, but, with fewer students (six and nine, respectively), it never again had quite the same magic.

Part of the challenge of presenting students with material they find confusing and frustrating is tuning it so that they believe that they can succeed in its mastery. By the time of the 2005 class, it was clear that I needed to provide more structure. This was the genesis of my *Radical Approach to Lebesgue's Theory of Integration* [5], providing exercises at a variety of levels of difficulty and clear statements of theorems and their proofs. But I still wanted students to experience that sense of uncertainty and confusion through which the mathematical community had passed in the late 19th century. My book intentionally leads down some of the enticing but ultimately unproductive ways of understanding integration so that those who use it can appreciate the solutions that eventually were found. My 2007 class served as a sounding board for an early draft of this book.

The lesson is not that everyone should use Hawkins' *Lebesgue's Theory of Integration*, or even my own *Radical Approach to Lebesgue's Theory of Integration*, as the textbook for an advanced real analysis course. Rather, it is that if we want to facilitate real and effective learning in our classes, then we must force students to experience some of the confusion and uncertainty that has gone into the creation of mathematics. We must know when to leave them to struggle and when and how to support them as they work through to a personal and meaningful understanding. A textbook or a curriculum is merely a starting point, a framework enabling a good teacher to begin formulating the experiences and challenges that will work in this particular place, at this particular time, with this particular group of students.

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