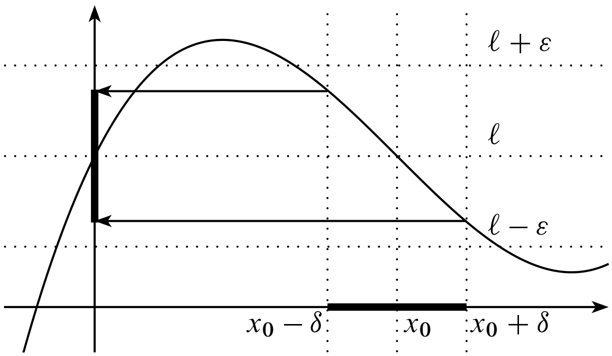
again have a tube around ℓ. But *what* must be inside the ε-tube? For every ε, there must be a δ-neighborhood around x_0, that is, an interval $(x_0 - \delta, x_0 + \delta)$ that represents the idea of points "approaching" x_0. So, if f converges to ℓ as x approaches x_0, for every $\varepsilon > 0$ in the codomain around ℓ, there is a $\delta > 0$ such that any point (except x_0) in $(x_0 - \delta, x_0 + \delta)$ maps into $(\ell - \varepsilon, \ell + \varepsilon)$. Notice that the idea of a limit of the function does not refer to the value of the function at x_0 itself, since we are imaging, like in the case of Zeno's arrow, that that value may not even be known.



Finally we are in a position to give a complete definition of a limit of a function.

***Definition* 4.11**. A function $f : \mathbb{R} \to \mathbb{R}$ has a **limit ℓ at a point** $x_0 \in \mathbb{R}$ if and only if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, for every $x \in \mathbb{R}$ with $0 < \|x - x_0\| < \delta$, $\|f(x) - \ell\| < \varepsilon$.

So having a limit at a point means that for every challenge, ε, there is a response to the challenge, δ, such that points closer than δ to x_0 (but distinct from x_0) are taken to points less than ε from the limit. This definition tells us the meaning of a function converging or having a limit at a point x_0.

This definition of limit is so complicated that it requires some real work to understand why each feature of the definition is necessary. The following exercise is once again basically a copy of an ingenious exercise devised by Carol Schumacher and appearing in *Closer and Closer: Introducing Real Analysis*. This exercise asks you to look at