



A ‘Mod’ern Mathematical Adventure in

CALL OF DUTY BLACK OPS

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Mathematics shows up in the most surprising places. Would you ever think to look in an action-packed, first-person-shooter video game? Well, check out *Call of Duty: Black Ops*, released in November 2010 with more than 25 million copies sold worldwide. Although most of the game involves killing people on various intricately designed maps, there is some interesting mathematics hidden in the game.

The setting for the “Call of the Dead” map, available as an additional purchase, is a shipwreck off of a cold, desolate Siberian coast. The objective is to

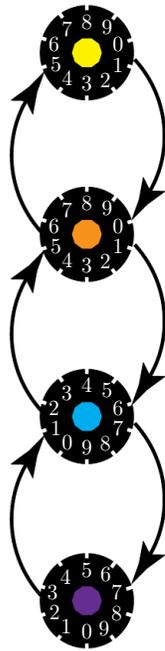
survive rounds of Nazi zombies that attack; however, in this map there is something special: an Easter egg. An Easter egg in a video game is a hidden message, puzzle, or treasure that is not a part of the typical gameplay. As we will see, the Easter egg on this map can be solved with a little help from linear algebra and modular arithmetic! We develop a technique that will solve this puzzle and a broader class of more challenging puzzles.

WHAT’S THE PROBLEM?

On the frigid tundra is a dark lighthouse with four floors, each with a single, colored dial; from top to



Figure 1, above. A dial in the lighthouse.
Figure 2, right. Turning a dial turns the neighboring dials.



bottom they are yellow, orange, blue, and purple (see figures 1 and 2). Each dial contains the numbers zero through nine and can be turned only in the counter-clockwise direction (so that the numbers increase). The goal is to turn the four dials so that they read 2, 7, 4, and 6 from top to bottom. Getting this solution is one step in solving the Easter egg.

As one might expect, aligning the dials is not as easy as it sounds—because what is the fun in that? Not only are zombies attacking as the player tries to find the solution, but each time the player turns a dial, it will turn one or two other dials by the same amount (see figure 2). Turning the yellow dial turns the orange dial, the orange dial turns the yellow and blue dials, the blue dial turns the orange and purple dials, and the purple dial turns the blue dial.

The starting positions of the dials from top to bottom are 8, 8, 4, and 5. However, turning the dials before starting to solve the puzzle does not prevent the player from solving it later—it just means that the player will have a new starting configuration. Our technique will produce the solution for any starting configuration.

AN ALGEBRAIC APPROACH

The solution involves solving a system of equations using modular arithmetic.

Definition. Let m be a positive integer. We say that two integers a and b are congruent modulo m if there is an integer k such that $a - b = km$, and if this is the case, we write $a \equiv b \pmod{m}$.

For example, this idea of congruencies allows us to distinguish between even and odd numbers. Let $m = 2$. We see that $12 \equiv 0 \pmod{2}$, because $12 - 0 = 6 \cdot 2$. Similarly, every even number is congruent to $0 \pmod{2}$ because an even number is a multiple of two. On the other hand, every odd number is congruent to $1 \pmod{2}$. We see that $131 \equiv 1 \pmod{2}$ because $131 - 1 = 65 \cdot 2$. Thus, every integer is congruent to either 0 or $1 \pmod{2}$.

To solve our Easter egg, we will work modulo 10 because there are 10 digits on the dials, and so every integer is congruent to a number 0 through 9. We can also perform arithmetic operations in this setting. If the dial starts on 8 and we turn it two clicks, we obtain $8 + 2 = 10$, but because $10 \equiv 0$, this shows up as 0 on the dial.

Let x , y , z , and w be the number of clicks we turn each dial, listed from top to bottom. The introduction of these variables enables us to create an equation for each of the four dials. Consider the top dial on the lighthouse. At the beginning of the game, it shows an 8. We turn it x times, but it also turns each time the orange dial is turned. Thus, it turns $x + y$ clicks to $8 + x + y$, and we want this to equal $2 \pmod{10}$. This gives the first (yellow) equation. Using the starting values of 8, 8, 4, and 5 and the desired ending values of 2, 7, 4, and 6, we create the entire system of equations:

$$\begin{aligned} & \text{■ } 8 + x + y \equiv 2 \pmod{10}, \\ & \text{■ } 8 + x + y + z \equiv 7 \pmod{10}, \\ & \text{■ } 4 + y + z + w \equiv 4 \pmod{10}, \\ & \text{■ } 5 + z + w \equiv 6 \pmod{10}. \end{aligned}$$

We know from linear algebra that a typical system of equations has zero, one, or infinitely many solutions. But, because we are working with integers modulo 10, there are only 10^4 candidate solutions. So a system like this could have 0, 1, 10, 100, 1,000, or 10,000 solutions. Fortunately, our system has a unique solution modulo 10. Let's find it!

First, we subtract the yellow equation from the orange equation to find $z \equiv 5 \pmod{10}$. Then we plug this value of z into the purple equation and find that $w \equiv -4 \pmod{10}$. Wait! Remember that we cannot use negative numbers because we can't turn the dial clockwise; luckily, we are doing calculations modulo 10 and $w \equiv -4 \equiv 6 \pmod{10}$. Thus, we must turn the yellow dial six clicks (we could also turn it 16 or 26 times, but that would give the zombies a lot of time to catch us).

A similar procedure gives $x \equiv 5 \pmod{10}$ and $y \equiv 9 \pmod{10}$. This means that we turn the yellow dial five clicks, the orange dial nine clicks, the blue dial six

clicks, and the purple dial five clicks.

Notice that our analysis did not take into account the order in which we turn the dials. Once we get a solution, the order does not matter! In fact, if we turn the dial on the top floor two clicks and a zombie arrives, we could race to another floor, turn that dial, then, when the coast is clear, return to the top floor to turn it the remaining three clicks.

A MATRIX APPROACH

We know that we can express a system of linear equations as a matrix equation of the form $A\bar{v} = \bar{b}$ and that doing so lets us harness the power of linear algebra. We can do the same for a system of equations such as ours, but the modular arithmetic adds some interesting details.

We start by converting the system of equations into a matrix equation of the form $\bar{b}_1 + A\bar{v} = \bar{b}_2$,

$$\begin{bmatrix} 8 \\ 8 \\ 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \equiv \begin{bmatrix} 2 \\ 7 \\ 4 \\ 6 \end{bmatrix} \pmod{10},$$

then we subtract \bar{b}_1 from both sides to obtain the matrix equation $A\bar{v} \equiv \bar{b}$:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \equiv \begin{bmatrix} -6 \\ -1 \\ 0 \\ 1 \end{bmatrix} \equiv \begin{bmatrix} 4 \\ 9 \\ 0 \\ 1 \end{bmatrix} \pmod{10}.$$

Notice that we can reduce \bar{b} modulo 10 if we want the entries to be between 0 and 9.

Ordinarily we would solve for the variables x , y , z , and w by multiplying both sides of the equation by A^{-1} . We know that a matrix with real entries has an inverse if, and only if, the determinant is nonzero, and for us $\det(A) = -1$ (check it!). But we are solving this matrix equation modulo 10 and not every such matrix has an inverse—even when the determinant is nonzero. Before we can discuss when a matrix is invertible modulo 10, we need to address the question of when an integer has a multiplicative inverse modulo 10.

Let $\mathbb{Z}_{10} = \{0, 1, \dots, 9\}$ denote the set of integers modulo 10 (in an abstract algebra course we call such a set a *ring*). An element m in \mathbb{Z}_{10} has a *multiplicative inverse* if there is an integer r such that $mr \equiv 1 \pmod{10}$. For example, $9 \cdot 9 \equiv 81 \equiv 1 \pmod{10}$; therefore, 9 is its own multiplicative inverse. Also, $7 \cdot 3 \equiv 21 \equiv 1 \pmod{10}$; therefore, 7 and 3 are multiplicative inverses. Not every number in \mathbb{Z}_{10} has a multiplicative inverse (hint: think about 2).

The following result from number theory tells us exactly which numbers have an inverse. We will state it for the more general case of \mathbb{Z}_n .

Theorem 1. *An element, m , of \mathbb{Z}_n has a multiplicative inverse if, and only if, the greatest common divisor of m and n is 1.*

So 1, 3, 7, and 9 have inverses in \mathbb{Z}_{10} , but 0, 2, 4, 5, 6, and 8 do not. OK, so we know how to determine if an element of \mathbb{Z}_{10} has an inverse, but what about a matrix with elements from \mathbb{Z}_{10} ?

Theorem 2. *A matrix A with elements in \mathbb{Z}_n has an inverse if, and only if, $\det(A)$ has a multiplicative inverse; that is, if, and only if, the greatest common divisor of $\det(A)$ and n is 1.*

Notice that this is a generalization of the real case; a real matrix C is invertible if, and only if, $\det(C) \neq 0$. But this is equivalent to saying that $\det(C)$ is an invertible real number. The determinant of our matrix A is $-1 \equiv 9 \pmod{10}$. And we know that 9 has an inverse, thus A is invertible. We used a computer algebra system to find that

$$A^{-1} = \begin{bmatrix} 1 & 0 & 9 & 1 \\ 0 & 0 & 1 & 9 \\ 9 & 1 & 0 & 0 \\ 1 & 9 & 0 & 1 \end{bmatrix}.$$

Indeed,

$$\begin{aligned} A^{-1}A &= \begin{bmatrix} 1 & 0 & 9 & 1 \\ 0 & 0 & 1 & 9 \\ 9 & 1 & 0 & 0 \\ 1 & 9 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 10 & 10 & 10 \\ 0 & 1 & 10 & 10 \\ 10 & 10 & 1 & 0 \\ 10 & 10 & 10 & 1 \end{bmatrix} \\ &\equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \pmod{10}. \end{aligned}$$

Now we solve the matrix equation $A\bar{v} \equiv \bar{b}$ by multiplying both sides by A^{-1} ,

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 & 9 & 1 \\ 0 & 0 & 1 & 9 \\ 9 & 1 & 0 & 0 \\ 1 & 9 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 9 \\ 0 \\ 1 \end{bmatrix} \equiv \begin{bmatrix} 5 \\ 9 \\ 45 \\ 86 \end{bmatrix} \equiv \begin{bmatrix} 5 \\ 9 \\ 5 \\ 6 \end{bmatrix} \pmod{10},$$

which agrees with the solution we found using algebra.

There are videos online that show the solution to this dial-turning problem, but they work only with the given initial values. The methods discussed above work

Here are the commands to find the inverse of a matrix modulo n using three popular computer algebra systems.

- Sage: `matrix(Zmod(n), [[1,1,0,0],[1,1,1,0],[0,1,1,1],[0,0,1,1]])^(-1)`
- Mathematica: `Inverse[{{1,1,0,0},{1,1,1,0},{0,1,1,1},{0,0,1,1}}, Modulus -> n]`
- Maple: `Inverse(Matrix([[1,1,0,0],[1,1,1,0],[0,1,1,1],[0,0,1,1]]))mod n`

with any starting values. This method also works for other variations of the problem: a different number of dials, the numbers on the dials are 0 through n for other values of n , or even different relationships between the dials. We illustrate with a more complex example.

Suppose we have four dials just as before, but now turning the yellow dial one click turns the orange and purple dials one click, turning the orange dial turns the purple dial two clicks, turning the blue dial turns the orange dial five clicks, and turning the purple dial turns the yellow dial two clicks and the blue dial three clicks. These relationships are illustrated in figure 3. Again, suppose the dials start at 8, 8, 4, and 5 and the goal is to have them read 2, 7, 4, and 6.

In this case the linear equations are

- $8 + x + 2w \equiv 2 \pmod{10}$,
- $8 + x + y + 5z \equiv 7 \pmod{10}$,
- $4 + z + 3w \equiv 4 \pmod{10}$,
- $5 + x + 2y + w \equiv 6 \pmod{10}$.

Proceeding as before, these yield the matrix equation $A\vec{v} \equiv \vec{b}$:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 1 & 1 & 5 & 0 \\ 0 & 0 & 1 & 3 \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \equiv \begin{bmatrix} 4 \\ 9 \\ 0 \\ 1 \end{bmatrix} \pmod{10}.$$

Because $\det(A) \equiv 33 \equiv 3 \pmod{10}$, A is invertible. A computer algebra system gives

$$A^{-1} = \begin{bmatrix} 7 & 8 & 0 & 6 \\ 8 & 3 & 5 & 9 \\ 9 & 2 & 1 & 9 \\ 7 & 6 & 0 & 7 \end{bmatrix},$$

and hence

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \equiv \begin{bmatrix} 7 & 8 & 0 & 6 \\ 8 & 3 & 5 & 9 \\ 9 & 2 & 1 & 9 \\ 7 & 6 & 0 & 7 \end{bmatrix} \begin{bmatrix} 4 \\ 9 \\ 0 \\ 1 \end{bmatrix} \equiv \begin{bmatrix} 106 \\ 68 \\ 63 \\ 89 \end{bmatrix} \equiv \begin{bmatrix} 6 \\ 8 \\ 3 \\ 9 \end{bmatrix} \pmod{10}.$$

Thus we must turn the yellow dial six clicks, the orange dial eight clicks, the blue dial three clicks, and the purple dial nine clicks.

If you are up for a challenge, try the exercises below.

The answers are given at the bottom of this page.

(1) Suppose there are three dials (top, middle, and bottom) with the digits 0 through 3. Turning the top dial turns the middle dial, turning the middle dial turns the top and bottom dials, and turning the bottom dial turns the middle dial. The dials start at 1, 3, and 2, from top to bottom. The goal is to have the dials read 1, 1, and 1, from top to bottom. How many times must you turn each dial?

(2) Suppose there are three dials (top, middle, and bottom) with the digits 0 through 9. Turning the top dial one click turns the middle dial two clicks, turning the middle dial one click turns the top and bottom dials one click, and turning the bottom dial one click turns the middle dial one click. The dials start at 2, 3, and 4, from top to bottom. The goal is to have the dials read 5, 6, and 7, from top to bottom. How many times must you turn each dial? ■

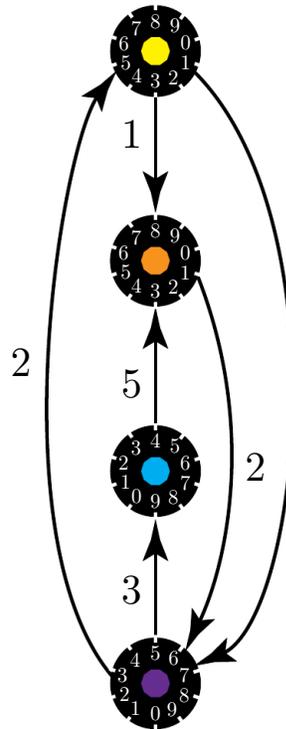


Figure 3. Turning a dial turns other dials in various ways.

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Solutions

- problem has no solution.
 (2) The system of equations has no solution, thus, the
 and two clicks.
 (1) Listed from top to bottom, turn the dials three, one,