Does any power of two begin with a seven?
If so, does any power of two begin with 77?

A Dozen Questions About the Powers of Two

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Everyone is familiar with the powers of two: 1, 2, 4, 8, 16, 32, 64, 128, and so on. They appear with surprising frequency throughout mathematics and computer science. For example, the number of subsets of a finite set is a power of two, as too is the sum of the entries of any row of Pascal's triangle. (Mathematically, these two statements are the same!) The largest prime number known today is one less than a power of two, a cube of tofu can be sliced into a maximum of 2^n pieces with n planar cuts, and every even perfect number is the sum of consecutive integers from 1 up to one less than a power of two!

Here I have put together a dozen curiosities all about the powers of two. These puzzles toy with results and ideas from classic number theory and geometry, game theory, and even popular TV culture (one problem is about a variation of the game *Survivor*)! I hope you enjoy thinking about them as much as I did.

1. A Weighty Problem

A woman possesses five stones, each weighing an integral number of kilograms. She claims, with the use of a simple see-saw balance, she can match the weight of any stone you give her and thereby determine its weight. She makes this claim under the proviso that your stone is of integral weight and weighs no more than 31 kilograms.

What are the weights of her five stones?

2. Multiplication without Multiplying

Here's an alternative method to long multiplication: Head two columns with the numbers you wish to multiply. Progressively halve the figures in the left-hand column (ignoring remainders) while doubling the numbers on the right. Continue this operation until the left-hand column is reduced to 1. Delete all rows with an even number in the left-hand column and add all the surviving numbers from the right-hand column. This sum is the desired product.

73	×	23
~~36~~		~~46~~
~~18~~		~~92~~
9		184
~~4~~		~~368~~
~~2~~		~~736~~
1		1472
		1679

$73 \times 23 = 1679.$

Why does this work?

3. Truncated Triangular Numbers

The numbers 5, 12, and 51, for example, can be written as a sum of two or more consecutive positive integers:

$$5 = 2 + 3$$
$$12 = 3 + 4 + 5$$
$$51 = 6 + 7 + 8 + 9 + 10 + 11.$$

Which numbers *cannot* be written as a sum of at least two consecutive positive integers?

4. Survivor

N people, numbered from 1 to N, are stranded on an island. They play the following variation of the TV game *Survivor*:

Members of the group vote whether person number N should survive or be escorted off the island. If 50% or more of the people agree to this person's survival then the game ends here and the N people all take an equal share of a $1,000,000 cash prize. If, on the other hand, the Nth person is voted off the island, the remaining $(N-1)$ people will take a second vote to determine the survival of the $(N-1)$th player (again with a quota of 50%). They do this down the line until a vote eventually passes and a person survives. The cash prize is then shared equally among all the folks remaining after this successful vote.

Assume that all players are greedy, but rational, thinkers; that they will always vote for their own survival, for example, and will vote for the demise of another player provided it does not lead to their own demise as a consequence.

The question here is simple: who survives?

5. Pascal Curiosity

Prove that all entries in the 2^nth row ($n \in \mathbf{N}$) of Pascal's triangle are odd.

6. Checkers in a Circle

Betty places a number of black and white checkers in arbitrary order on the circumference of a circle. (Say Betty lays down N checkers.) Charlie now places new checkers between the pairs of adjacent checkers in Betty's ring: he places a white checker between every two that are the same color, a black checker between every pair of opposite color. He then removes Betty's original checkers to leave a new ring of N checkers in a circle.

Betty then performs the same operation on Charlie's ring of checkers, following the same rules. The two players alternate performing this maneuver over and over again on the circle of checkers before them. Show that if N is a power of two, all the checkers will eventually be white, no matter the arrangement of colors Betty initially puts down.

7. Classic Number Theory

Is $2^{91} - 1$ prime? What about $2^{91} + 1$?

8. De Polignac's Remarkable Conjecture

In the mid-nineteenth century, the French mathematician A. de Polignac made a remarkable observation: It seems that every odd number larger than one can be written as a sum of a power of two and a prime.

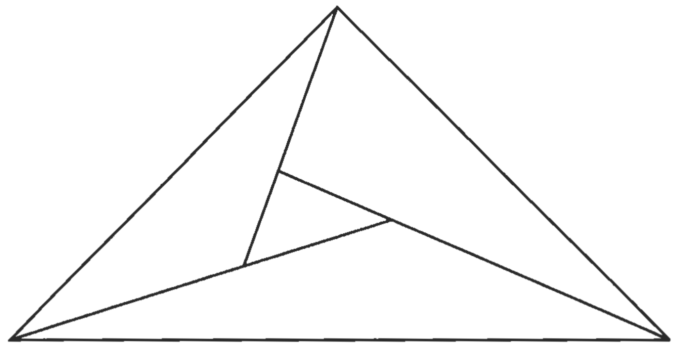
$$3 = 2^0 + 2$$
$$5 = 2^1 + 3$$
$$7 = 2^2 + 3$$
$$\vdots$$
$$53 = 2^4 + 37$$
$$\vdots$$
$$241 = 2^7 + 113$$
$$\vdots$$
$$999999 = 2^{16} + 944463$$

He claimed to have checked this for all odd numbers up to three million! Can you prove de Polignac's conjecture?

9. Stacking Dilemma

Two line segments can sit in one-dimensional space touching in a zero-dimensional subspace, namely, a point:

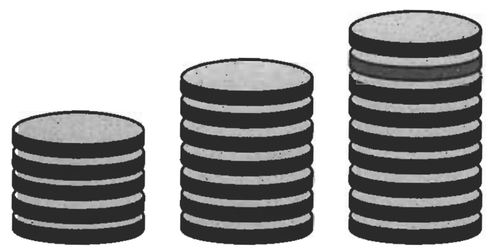
It is possible to arrange four triangles in a plane so that each triangle intersects each other triangle along a one-dimensional line segment of positive length:



Is it possible to arrange eight tetrahedra in three-space so that each tetrahedron meets every other tetrahedron along a portion of surface of positive area?

10. The Game of 5-7-9

Here's a classic game for two players. It is played with three piles of coins, one with five coins, the second with seven and the third with nine coins. At each turn a player picks up as many or as few coins as she chooses from a single pile. The player who picks up the last coin wins.

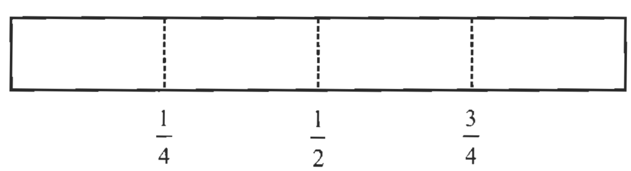


What strategy could the first player employ to guarantee a win?

11. Folding Fractions

It is possible to place a crease mark exactly halfway along a strip of paper simply by folding the paper in half. Then, using this mark as a guide and folding the left and right portions of the paper, we can accurately place creases at positions $\frac{1}{4}$ and $\frac{3}{4}$ respectively.

Is it possible to accurately place a crease mark at position $\frac{5}{7}$ on the paper?



12. Where are the 7's?

Does any power of two begin with a seven? If so, does any power of two begin with 77?

Comments, Answers and Further Questions

1. Her stones weigh 1, 2, 4, 8 and 16 kilograms respectively. As every positive integer can be written uniquely as a sum of distinct powers of two, she can match any weight up to $1+2+4+8+16=31$ kilograms with these stones.

Taking It Further. Using a see-saw balance and a different set of five stones the woman can actually accomplish a much more impressive feat: she can determine the integral weight of any rock you give her weighing up to 243 kilograms! What are the weights of these five different stones?

Hint: The woman no longer claims she can match the weight of your stone, only that she can determine its integral value.

2. Removing the last digit of a number written in base two either divides the number in half or subtracts 1 and then divides by two, depending on whether the number is even or odd. (For example, the binary code for the number 13 is 1101. Deleting the last digit gives 110 which represents 6.) Thus one can determine the binary code for a number simply by repeatedly dividing by 2 (ignoring remainders), and keeping track of whether or not the result is even or odd along the way. We can thus read off the binary code of 73 from the left column of the table as 1001001. This means $73 = 1 + 2^3 + 2^6$, and so

$$73 \times 23 = (1+2^3+2^6) \times 23.$$

Expanding the brackets yields:

$$\begin{aligned} 73 \times 23 = \quad & 1 \times 23 \\ & \cancel{+\ 2 \times 23} \\ & \cancel{+\ 2^2 \times 23} \\ & +\ 2^3 \times 23 \\ & \cancel{+\ 2^4 \times 23} \\ & \cancel{+\ 2^5 \times 23} \\ & +\ 2^6 \times 23. \end{aligned}$$

The desired product is precisely the sum of terms (resulting from doubling the number 23 multiple times) that correspond to the placement of ones in the binary code of the number 73, namely, the placement of odd terms in the left column. This method works for any pair of natural numbers you wish to multiply.

Comment. Computers perform multiplication in this way. The halving and doubling operations are trivial when all numbers are expressed in base 2.

3. All numbers *except* the powers of two can be written as a sum of at least two consecutive integers. If N is a number of the form

$$N=(n+1)+(n+2)+\cdots+(n+k)$$

for $n, k \in \mathbf{N}$ with $k \geq 2$, then

$$N = kn + \frac{k(k+1)}{2} = \frac{1}{2} \cdot k(2n+k+1).$$

If k is odd, then $(2n+k+1)$ is even and it follows that k is an odd factor of N. If, on the other hand, k is even, then $(2n+k+1)$ is an odd factor of N. Either way, N possesses an odd factor and so cannot be a power of two.

Conversely, all numbers possessing an odd proper factor, can be written as a sum of two or more consecutive numbers. Suppose $N = ab$ with $a, b \in \mathbf{N}$, $b \geq 1$, $a \geq 3$, and a odd. Then

$$\begin{aligned} N &= b+b+b+b+\cdots+b \qquad a \text{ times} \\ &= (b-(a-1)/2)+\cdots+(b-1) \\ &\quad +b+(b+1)+\cdots+(b+(a-1)/2). \end{aligned}$$

If $b-(a-1)/2 > 0$ we are done. If not, the first few terms of this sum are negative and will cancel the first few positive terms in the latter part of this sum. We need to show that, after cancellation, at least two positive terms survive. Simple algebra verifies that indeed $-(b-(a-1)/2) \leq b+(a-1)/2-2$.

Taking it Further. Classify those natural numbers that can be expressed as a sum of at least *three* consecutive positive integers.

4. One person on the island ($N=1$) will certainly vote for himself and so survive. With two people on the island ($N = 2$ case), player 1 will not vote for player 2's survival (he'll survive without her) but player 2 will. Thus player 2 survives and both folks share the prize. With three players ($N=3$ case), players 1 and 2 will vote for player 3's demise (they're fine without her) and even voting for herself, player 3 will not garner enough votes to survive. Players 1 and 2 will again share the prize.

Consider the game with four people on the island. Player 4 will certainly vote for his survival. So too will player 3, for without player 4, the previous analysis shows player 3 will not survive. Even though players 1 and 2 will vote against player 4, player 4 has enough votes to survive, and thus all four players stay on the island and share the prize.

For a game with $N<8$ players, players 1 through 4 have no need to vote for the survival of higher numbered players. Players 5, 6, and 7 are therefore doomed to leave the island. We need the addition of an eighth player in the game to ensure their (and this eighth player's) survival.

In this way we see that the number of people who survive our game *Survivor* is the largest power of two less than or equal to the number of people initially on the island.

Taking it Further. How do the results of the game change if players must attain *strictly more* than 50% of the votes in order to survive?

5. We are asked to show that each entry of the 2^nth row of Pascal's triangle is congruent to 1 (mod 2). Regard the top row of Pascal's triangle as an infinite string of zeros except for a single "1" in the 'center.' Then every entry in the remaining rows of the triangle is the sum of the two nearest terms in the row above it. Working modulo 2, the first five rows of Pascal's triangle are thus:

$$\begin{array}{ccccccccccccccc} \ldots 0 & & 0 & & 0 & & 1 & & 0 & & 0 & & 0 \ldots \\ \ldots & 0 & & 0 & & 1 & & 1 & & 0 & & 0 & & \ldots \\ \ldots 0 & & 0 & & 1 & & 0 & & 1 & & 0 & & 0 \ldots \\ \ldots & 0 & & 1 & & 1 & & 1 & & 1 & & 0 & & \ldots \\ \ldots 0 & & 1 & & 0 & & 0 & & 0 & & 1 & & 0 \ldots \end{array}$$

Certainly all the terms of the first, second, and fourth rows of the triangle are 1 (mod 2). Also, the two 1's on the fifth row are sufficiently spaced apart to generate their own copies of the first four rows of the triangle. We thus obtain a row of eight 1's in the eighth row of Pascal's triangle. The ninth row then consists of two single 1's which are sufficiently far apart to generate their own copies of the first eight rows of Pascal's triangle, ending with a sixteenth row which is nothing but 1's; and so on. An induction argument shows that all 2^n entries of the 2^nth row of Pascal's triangle are indeed congruent to 1 (mod 2).

Taking it Further. Prove that $\binom{2^n}{k}$ is even for $n, k \in \mathbf{N}$ with $1 < k < 2^n$. Is $\binom{2n}{n}$ ever odd for $n \geq 1$?

6. Break the circle and line the checkers in a row, noting that the first and last checkers are 'adjacent.' Represent this row of checkers as a string of 0's and 1's, where '0' represents a white checker, and '1' a black checker. The transformation described in the checker game creates a new string of 0's and 1's where each entry in the second row is the sum of the two nearest entries from the top row (with the appropriate interpretation for the end digits given the 'wrap around' effect for the string).

Suppose N is a power of 2. Consider a game starting with a single black checker. Due to the cyclic symmetry of the game, we may as well assume the black checker is placed at the beginning of the string, and thus the game can be represented by a string of the form:

$$1000\ldots0.$$

Note that $N-1$ applications of the transformation generate the first N rows of Pascal's triangle modulo 2 (the 'wrap around' effect does not affect the formation of this initial portion of the triangle). By question 6, all entries of the final row are 1. Thus after $N-1$ steps, all checkers in the checker game will be black. One more application of the transformation yields nothing but white checkers.

An arbitrary game can be thought of as a superposition of individual games involving single black checkers. For example, the game represented by 0110 is a superposition, in some sense, of the games 0100 and 0010. The checker game transformation commutes with 'exclusive or' binary addition (that is, binary addition *without* carrying 1's). As 0100 and 0010 both converge to 0000, it follows that 0110 converges to 0000+0000=0000. This argument shows that all checker games, involving a power of two number of checkers, yield nothing but white checkers in at most that many steps.

Taking It Further. Can anything be said for games with N checkers where N is not a power of two?

7. First note that $2^{91} = (2^7)^{13}$. The identities

$$x^{13} - 1 = (x-1)(x^{12} + x^{11} + \cdots + x + 1)$$

and

$$x^{13} + 1 = (x+1)(x^{12} - x^{11} + \cdots - x + 1)$$

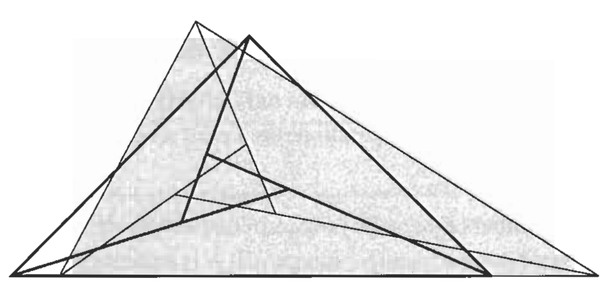
show that $2^7 - 1 = 127$ and $2^7 + 1 = 129$ are factors of $2^{91} - 1$ and $2^{91} + 1$ respectively. (So too are 8191 and 8193.) Thus neither number is prime.

Comment. Numbers of the form $2^n - 1$ are called *Mersenne Numbers.* If n is composite, the above argument generalizes to show that $2^n - 1$ too must be composite. If n is prime, however, the situation is less clear. For example, $2^n - 1$ is prime for $n = 2, 3, 5, 7, 13, 17$ and 19, but not for $n = 11$. Nonetheless, Mersenne numbers have proven to be a rich source of large prime numbers, the largest known today being $2^{6972593} - 1$.

Taking It Further. Is $2^{91} - 3$ prime? What about $2^{91} - 5$ and $2^{91} - 7$?

8. Don't bother! It isn't true. De Polignac missed the number 127 which cannot be written as a sum of a power of two and a prime. (One only need check this for the powers 2^n with $n = 0, \ldots, 6$.) If only de Polignac noticed this slip, he could have saved himself literally months of very hard work!

9. To arrange eight tetrahedra in three-space in the appropriate way, use the following diagram of eight triangles in a plane. Connect the four shaded triangles to a point hovering below the plane of the drawing, and the four unshaded triangles to a point hovering above the plane of the drawing. This yields eight suitably touching tetrahedra.



Taking It Further. Prove it is impossible to arrange *five* triangles in a plane that meet pairwise in a line segment of positive length. Is it possible to arrange *nine* tetrahedra in three-space in the appropriate way?

10. Express the numbers of coins in each pile in binary notation and write these numbers as rows of a table:

5	=		1	0	1
7	=		1	1	1
9	=	1	0	0	1

Notice there are an odd number of 1's appearing in the ones, twos and eights columns of the table. To guarantee a win, the first player should always move so as to produce an even number of 1's in each column. Her first turn is thus to convert the number 1001 into 0010, that is, to reduce the pile of nine coins to two coins. Her opponent will then be forced to create a table with an odd number of 1's in at least one column, and therefore a game with at least one remaining coin. Player 1 always operating this way thus offers her opponent no hope of ever winning.

Taking It Further. *Reverse 5-7-9* is played the same way except this time the person who picks up the last coin loses. Does either player have a best strategy in this variation of the game?

11. Placing a crease mark halfway between two previously constructed crease marks can only ever produce fractions of the form a/N, where N is a power of two. And conversely, every fraction of this form can be created via a finite sequence of such folding operations (see below). Thus there is no (finite) procedure for constructing the fraction $5/7$. However, as every number can be approximated arbitrarily closely by a fraction of the form indicated (just truncate its binary decimal expansion for example), it is possible to place a crease at any position we choose with any desired degree of precision.

Alternatively, one can use the following procedure for placing a crease arbitrarily close to the position $5/7$. Begin by making a guess as to where this fraction lies along the strip and place a crease at this location. Now fold the right portion of the strip in half to place a crease halfway between the guess and the right end of the paper. This produces a crease mark at position $6/7$ with the error reduced in half. Now fold from the left to produce a crease at position $3/7$ with error reduced in half yet again, and then, finally, fold from the right to produce a crease mark at position $5/7$ with one eighth the original error. Repeating the "RLR" sequence of folds multiple times yields a sequence of crease marks that rapidly converge to the true $5/7$ position.

Taking it Further 1. It is no coincidence that the sequence of folds "RLR" mimics the binary representation 101 of the number five. Show that if N is one less than a power of two, then the sequence of left and right folds that hones in on the fraction a/N is precisely the binary expansion of a read backwards with '1' representing 'right' and '0' representing 'left.'

Taking it Further 2. Consider a fraction of the form $a/2^n$. Write a as an n-digit binary number (you may need to place zeros at the beginning), and read this binary expansion backwards as a sequence of instructions with '1' and '0' representing 'right' and 'left' again. Show that if you follow these instructions through just once the final crease mark formed lies precisely at position $a/2^n$.

Taking it Further 3. Let q be *any* fraction in the interval [0, 1]. Show that, with a square sheet of paper, it *is* possible to produce, in a finite number of folds, a creased line segment precisely q units long.

12. We seek integers n and k such that

$$7 \times 10^k \leq 2^n < 8 \times 10^k,$$

that is,

$$\log_{10} 7 + k \leq n \log_{10} 2 < \log_{10} 8 + k.$$

If $\{x\}$ denotes the fractional part of the number x, we thus are being asked to find a value n for which

$$\log_{10} 7 \leq \{n \log_{10} 2\} < \log_{10} 8.$$

Working on the interval [0, 1) with 'wrap around' effect (that is, working modulo 1), we hope to find a multiple of $\log_{10} 2$ that falls within the segment $[\log_{10} 7, \log_{10} 8)$, which is about 0.058 units long. It is easy to show that $\log_{10} 2$ is an irrational number (an equation of the form $2^b = 10^a$ can only hold if both exponents are zero) and consequently no two distinct multiples of $\log_{10} 2$ have the same fractional part. Thus, of the first 21 multiples of $\log_{10} 2$, at least two if them must lie within a distance of $1/20 = 0.05$ from each other (considering the wrap around effect). Call these multiples $m \log_{10} 2$ and $(m+q)\log_{10} 2$. It then follows that the multiples $(m+q)\log_{10} 2$ and $(m+2q)\log_{10} 2$ are also within this distance of each other, as too are $(m+2q)\log_{10} 2$ and $(m+3q)\log_{10} 2$, and so on. Creeping along this way, we must eventually hit upon a multiple of $\log_{10} 2$ that lies in the interval $[\log_{10} 7, \log_{10} 8)$. This shows that powers of two beginning with a seven do exist. (The diligent reader may have discovered that 2^{46} is the first power of two which begins with a 7.)

We can say more: The above argument shows that the multiples of $\log_{10} 2$ form a dense set in the interval [0, 1) and so there are infinitely many multiples that lie in any given segment. Thus there are infinitely many powers of two that begin with seven and, in some sense, we can say that 7 occurs as a first digit 5.8% of the time! Similarly, $\log_{10} 78 - \log_{10} 77 \approx 0.56\%$ of the powers of two begin with '77' and there are infinitely many powers of two that begin with any prescribed (finite) set of digits! (For example, there are infinitely many powers of two that begin with the first billion digits of pi!)

Acknowledgments and Further Reading

Several of these puzzles explore classic ideas from number theory. The interested reader can look at Jay R. Goldman's beautiful text *The Queen of Mathematics, A Historically Motivated Guide to Number Theory* (A. K. Peters, Ltd., 1998), for example, to learn more about this fascinating and challenging subject. I first learned of the classification of truncated triangular numbers as a result discovered by eight year old Mit'ka Vaintrob of New Mexico. He followed a geometric approach, literally truncating triangular arrays of coins or buttons to create trapezoids with at least two rows. His analysis of those numbers which can be expressed as a sum of three consecutive integers is a remarkable achievement. Gengzhe Chang and Thomas Sederberg give a complete analysis of the circular checkers game in their wonderful book *Over and Over Again* (Mathematical Association of America, 1997).

Question 9, with its obvious extensions, is a very old and extremely hard question. It is known that it is always possible to arrange 2^n high-dimensional "simplices" in n-space that meet, pairwise, in regions of $(n-1)$-space of positive volume, but it is not at all clear whether one can do more. It has long been known that is impossible to arrange five triangles in a plane in this way, but it wasn't until 1991 that J. Zaks was able to prove the impossibility of arranging nine tetrahedra in three-space. Analysis of higher-dimensional spaces is still an area of open research. See Martin Aigner and Günter M. Ziegler's *Proofs from the Book* (Springer-Verlag, 1999), Chapter 13, for a discussion on this fascinating topic.

The 5-7-9 game, of course, is a specific instance of the famous game Nim. One can explore the subject of *nimbers* in John H. Conway and Richard K. Guy's *The Book of Numbers* (Copernicus, 1996). Changing the value of the quota in the game *Survivor* leads to some very interesting mathematics; my colleague Charles Adler and I are currently writing about some curious results from this game. ■

Errata

The Bridges of Königsberg/Kaliningrad: A Tale of One City in the April, 2001 *Math Horizons* contained some errors. The following replaces paragraphs three and four.

Euler converted the puzzle to a combinatorial question which he then solved. Thus he became the first mathematician ever to publish a paper on graph theory! He created a mathematical model of the puzzle by replacing each of the four land areas by a vertex and each bridge by an edge joining two vertices, as shown in Figure 2. Then the problem became to begin at any one vertex, v, traverse consecutive edges just once each until every edge has been used, and end at v.

Now we say that a graph or multigraph G is *eulerian* and has an eulerian trail if this puzzle can be solved for G. Euler answered this question by proving that a graph is eulerian if and only if it is connected and every vertex has even degree (is incident with an even number of edges).

The author credit should read: Frank Harary is Professor Emeritus of Mathematics at the University of Michigan and of Computer Science at New Mexico State University.