Gauss, Landen, Ramanujan, the Arithmetic-Geometric Mean, Ellipses, π, and the *Ladies Diary*

GERT ALMKVIST, *University of Lund*

BRUCE BERNDT*, *University of Illinois*

GERT ALMKVIST received his Ph.D. at the University of California in 1966 and has been at Lund since 1967. His main interests are algebraic K-theory, invariant theory, and elliptic functions.

BRUCE BERNDT received his A.B. degree from Albion College, Albion, Michigan in 1961 and his Ph.D. from the University of Wisconsin, Madison, in 1966. Since 1977, he has devoted all of his research efforts to proving the hitherto unproven results in Ramanujan's notebooks. His book, *Ramanujan's Notebooks*, Part I (Springer-Verlag, 1985), is the first of either three or four volumes to be published on this project.

Virtue and sense, with female-softness join'd
(All that subdues and captivates mankind!)
In Britain's matchless fair resplendent shine;
They rule Love's empire by a right divine:
Justly their charms the astonished World admires,
Whom Royal Charlotte's bright example fires.

1. Introduction. The arithmetic-geometric mean was first discovered by Lagrange and rediscovered by Gauss a few years later while he was a teenager. However, Gauss's major contributions, including an elegant integral representation, were made about 7–9 years later. The first purpose of this article is, then, to explain the arithmetic-geometric mean and to describe some of its major properties, many of which are due to Gauss.

*Research partially supported by the Vaughn Foundation.

Because of its rapid convergence, the arithmetic-geometric mean has been significantly employed in the past decade in fast machine computation. A second purpose of this article is thus to delineate its role in the computation of π. We emphasize that the arithmetic-geometric mean has much broader applications, e.g., to the calculation of elementary functions such as $\log x$, e^x, $\sin x$, and $\cos x$. The interested reader should further consult the several references cited here, especially Brent's paper **[14]** and the Borweins' book **[13]**.

The determination of the arithmetic-geometric mean is intimately related to the calculation of the perimeter of an ellipse. Since the days of Kepler and Euler, several approximate formulas have been devised to calculate the perimeter. The primary motivation in deriving such approximations was evidently the desire to accurately calculate the elliptical orbits of planets. A third purpose of this article is thus to describe the connections between the arithmetic-geometric mean and the perimeter of an ellipse, and to survey many of the approximate formulas that have been given in the literature. The most accurate of these is due to Ramanujan, who also found some extraordinarily unusual and exotic approximations to elliptical perimeters. The latter results are found in his notebooks and have never been published, and so we shall pay particular attention to these approximations.

Also contributing to this circle of ideas is the English mathematician John Landen. In the study of both the arithmetic-geometric mean and the determination of elliptical perimeters, there arises his most important mathematical contribution, which is now called Landen's transformation. Many very important and seemingly unrelated guises of Landen's transformation exist in the literature. Thus, a fourth purpose of this article is to delineate several formulations of Landen's transformation as well as to provide a short biography of this undeservedly, rather obscure, mathematician.

For several years, Landen published almost exclusively in the *Ladies Diary*. This is, historically, the first regularly published periodical to contain a section devoted to the posing of mathematical problems and their solutions. Because an important feature of the MONTHLY has its roots in the *Ladies Diary*, it seems then dually appropriate in this paper to provide a brief description of the *Ladies Diary*.

2. Gauss and the arithmetic-geometric mean. As we previously alluded, the arithmetic-geometric mean was first set forth in a memoir of Lagrange **[30]** published in 1784–85. However, in a letter, dated April 16, 1816, to a friend, H. C. Schumacher, Gauss confided that he independently discovered the arithmetic-geometric mean in 1791 at the age of 14. At about the age of 22 or 23, Gauss wrote a long paper **[23]** describing his many discoveries on the arithmetic-geometric mean. However, this work, like many others by Gauss, was not published until after his death. Gauss's fundamental paper thus did not appear until 1866 when E. Schering, the editor of Gauss's complete works, published the paper as part of Gauss's *Nachlass*. Gauss obviously attached considerable importance to his findings on the arithmetic-geometric mean, for several of the entries in his diary, in particular, from the years 1799 to 1800, pertain to the arithmetic-geometric mean. Some of these entries are quite vague, and we may still not know everything that Gauss discovered about the arithmetic-geometric mean. (For an English translation of Gauss's diary together with commentary, see a paper by J. J. Gray **[24]**.)

By now, the reader is anxious to learn about the arithmetic-geometric mean and what the young Gauss discovered.

Let a and b denote positive numbers with $a > b$. Construct a sequence of arithmetic means and a sequence of geometric means as follows:

$$\begin{array}{ll} a_1 = \frac{1}{2}(a+b), & b_1 = \sqrt{ab}, \\ a_2 = \frac{1}{2}(a_1+b_1), & b_2 = \sqrt{a_1 b_1}, \\ \vdots & \vdots \\ a_{n+1} = \frac{1}{2}(a_n+b_n), & b_{n+1} = \sqrt{a_n b_n}, \\ \vdots & \vdots \end{array}$$

Gauss [23] gives four numerical examples, of which we reproduce one. Let $a = 1$ and $b = 0.8$. Then

$$\begin{array}{ll} a_1 = 0.9, & b_1 = 0.894427190999915878564, \\ a_2 = 0.897213595499957939282, & b_2 = 0.897209268732734, \\ a_3 = 0.897211432116346, & b_3 = 0.897211432113738, \\ a_4 = 0.897211432115042, & b_4 = 0.897211432115042. \end{array}$$

(Obviously, Gauss did not shirk from numerical calculations.) It appears from this example that $\{a_n\}$ and $\{b_n\}$ converge to the same limit, and that furthermore this convergence is very rapid. This we now demonstrate.

Observe that

$$b < b_1 < a_1 < a,$$
$$b < b_1 < b_2 < a_2 < a_1 < a,$$
$$b < b_1 < b_2 < b_3 < a_3 < a_2 < a_1 < a,$$

etc. Thus, $\{b_n\}$ is increasing and bounded, and $\{a_n\}$ is decreasing and bounded. Each sequence therefore converges. Elementary algebraic manipulation now shows that

$$\frac{a_1 - b_1}{a - b} = \frac{a-b}{4(a_1+b_1)} = \frac{a-b}{2(a+b)+4b_1} < \frac{1}{2}.$$

Iterating this procedure, we deduce that

$$a_n - b_n < \left(\frac{1}{2}\right)^n (a-b), \qquad n \geqslant 1,$$

which tends to 0 as n tends to ∞. Thus, a_n and b_n converge to the same limit, which we denote by $M(a,b)$. By definition, $M(a,b)$ *is the arithmetic-geometric mean of a and b.*

To provide a more quantitative measure of the rapidity of convergence, first define

$$c_n = \sqrt{a_n^2 - b_n^2}, \qquad n \geqslant 0, \tag{1}$$

where $a_0 = a$ and $b_0 = b$. Observe that

$$c_{n+1} = \frac{1}{2}(a_n - b_n) = \frac{c_n^2}{4a_{n+1}} \leqslant \frac{c_n^2}{4M(a,b)}.$$

Thus, c_n tends to 0 quadratically, or the convergence is of the second order. More generally, suppose that $\{\alpha_n\}$ converges to L and assume that there exist constants $C > 0$ and $m \geqslant 1$ such that

$$|\alpha_{n+1} - L| \leqslant C|\alpha_n - L|^m, \qquad n \geqslant 1.$$

Then we say that the convergence is of the mth order.

Perhaps the most significant theorem in Gauss's paper [23] is the following representation for M for which we provide Gauss's ingenious proof.

THEOREM 1. *Let* $|x| < 1$, *and define*

$$K(x) = \int_0^{\pi/2} (1 - x^2\sin^2\varphi)^{-1/2}\,d\varphi. \tag{2}$$

Then

$$M(1+x, 1-x) = \frac{\pi}{2K(x)}.$$

The integral $K(x)$ is called the complete elliptic integral of the first kind. Observe that in the definition of $K(x)$, $\sin^2\varphi$ may be replaced by $\cos^2\varphi$.

Before proving Theorem 1, we give a reformulation of it. Define

$$I(a,b) = \int_0^{\pi/2} (a^2\cos^2\varphi + b^2\sin^2\varphi)^{-1/2}\,d\varphi. \tag{3}$$

It is easy to see that

$$I(a,b) = \frac{1}{a}K(x),$$

where

$$x = \frac{1}{a}\sqrt{a^2 - b^2}.$$

Since

$$M(a,b) = M(a_1, b_1) \quad \text{and} \quad M(ca, cb) = cM(a,b), \tag{4}$$

for any constant c, it follows that, with x as above,

$$M(1+x, 1-x) = \frac{1}{a}M(a,b).$$

The following reformulation of Theorem 1 is now immediate.

THEOREM 1′. *Let* $a > b > 0$. *Then*

$$M(a,b) = \frac{\pi}{2I(a,b)}.$$

Proof. Clearly, $M(1+x,1-x)$ is an even function of x. Gauss then *assumes* that

$$\frac{1}{M(1+x,1-x)}=\sum_{k=0}^{\infty}A_k x^{2k}. \tag{5}$$

Now make the substitution $x=2t/(1+t^2)$. From (4), it follows that

$$M(1+x,1-x)=\frac{1}{1+t^2}M\big((1+t)^2,(1-t)^2\big)=\frac{1}{1+t^2}M(1+t^2,1-t^2).$$

Substituting in (5), we find that

$$(1+t^2)\sum_{k=0}^{\infty}A_k t^{4k}=\sum_{k=0}^{\infty}A_k\left(\frac{2t}{1+t^2}\right)^{2k}.$$

Clearly, $A_0=1$. Expanding $(1+t^2)^{-2k-1}$, $k\geqslant 0$, in a binomial series and equating coefficients of like powers of t on both sides, we eventually find that

$$\begin{aligned}\frac{1}{M(1+x,1-x)}&=1+\left(\frac{1}{2}\right)^2x^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2x^4+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2x^6+\cdots\\&=\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k^2}{(k!)^2}x^{2k}.\end{aligned} \tag{6}$$

Here we have introduced the notation

$$(\alpha)_k=\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1). \tag{7}$$

Complete details for the derivation of (6) may be found in Gauss's paper [**23**, pp. 367–369].

We now must identify the series in (6) with $K(x)$. Expanding the integrand of $K(x)$ in a binomial series and integrating termwise, we find that

$$\begin{aligned}K(x)&=\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k}{k!}x^{2k}\int_0^{\pi/2}\sin^{2k}\varphi\,d\varphi\\&=\frac{\pi}{2}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k^2}{(k!)^2}x^{2k}.\end{aligned} \tag{8}$$

Combining (6) and (8), we complete the proof of Gauss's theorem.

Another short, elegant proof of Theorem 1 has been given by Newman [**39**] and is sketched by J. M. and P. B. Borwein [**10**].

For a very readable, excellent account of Gauss's many contributions to the arithmetic-geometric mean, see Cox's paper [**18**]. We shall continue the discussion of some of Gauss's discoveries in Section 5.

3. Landen and the Ladies Diary. We next sketch another proof of Theorem 1 (or Theorem 1′) which is essentially due to the eighteenth-century English mathematician John Landen.

Second proof. Although the basic idea is due to Landen, the iterative procedure that we shall describe is apparently due to Legendre [**33**, pp. 79–83] some years later.

For brevity, set

$$x_n = c_n/a_n, \qquad n \geqslant 0, \tag{9}$$

where c_n is defined by (1). In the complete elliptic integral of the first kind (2), make the substitution

$$\tan\varphi_1 = \frac{\sin(2\varphi)}{x_1 + \cos(2\varphi)}. \tag{10}$$

This is called Landen's transformation. After a considerable amount of work, we find that

$$K(x) = (1 + x_1)K(x_1).$$

Upon n iterations, we deduce that

$$K(x) = (1 + x_1)(1 + x_2)\cdots(1 + x_n)K(x_n). \tag{11}$$

Since, by (1) and (9), $1 + x_k = a_{k-1}/a_k$, $k \geqslant 1$, we see that (11) reduces to

$$K(x) = \frac{a}{a_n}K(x_n).$$

We now let n tend to ∞. Since a_n tends to $M(a, b)$ and x_n tends to 0, we conclude that

$$K(x) = \frac{a}{M(a,b)}K(0) = \frac{a\pi}{2M(a,b)}.$$

Landen's transformation (10) was introduced by him in a paper [**31**] published in 1771 and in more developed form in his most famous paper [**32**] published in 1775. There exist several versions of Landen's transformation. Often Landen's transformation is expressed as an equality between two differentials in the theory of elliptic functions [**17**], [**37**]. The importance of Landen's transformation is conveyed by Mittag-Leffler who, in his very perceptive survey [**37**, p. 291] on the theory of elliptic functions, remarks, "Euler's addition theorem and the transformation theorem of Landen and Lagrange were the two fundamental ideas of which the theory of elliptic functions was in possession when this theory was brought up for renewed consideration by Legendre in 1786."

In Section 4, we shall prove the following theorem, which is often called Landen's transformation for complete elliptic integrals of the first kind.

THEOREM 2. *If* $0 \leqslant x < 1$, *then*

$$K\left(\frac{2\sqrt{x}}{1+x}\right) = (1 + x)K(x).$$

In fact, Theorem 2 is the special case $\alpha = \pi$, $\beta = \pi/2$ of the following more general formula. If $x\sin\alpha = \sin(2\beta - \alpha)$, then

$$(1 + x)\int_0^{\alpha}(1 - x^2\sin^2\varphi)^{-1/2}\,d\varphi = 2\int_0^{\beta}\left(1 - \frac{4x}{(1+x)^2}\sin^2\varphi\right)^{-1/2} d\varphi,$$

which is known as Landen's transformation for incomplete elliptic integrals of the first kind.

To describe another form of Landen's transformation, we introduce Gauss's ordinary hypergeometric series

$$F(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k, \qquad |x| < 1, \tag{12}$$

where a, b, and c denote arbitrary complex numbers and $(\alpha)_k$ is defined by (7). Then

$$F\left(a, b; 2b; \frac{4x}{(1+x)^2}\right) = (1+x)^{2a} F\left(a, a-b+\frac{1}{2}; b+\frac{1}{2}; x^2\right) \tag{13}$$

is Landen's transformation for hypergeometric series. Theorems 1 and 2 imply the special case

$$F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{4x}{(1+x)^2}\right) = (1+x) F\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right).$$

Thus, a seemingly innocent "change of variable" (10) has many important ramifications. Indeed, Landen himself evidently never realized the importance of his idea.

Since Landen undoubtedly is not known to most readers, it seems appropriate here to give a brief biography. He was born in 1719. According to the Encyclopedia Britannica [**20**], "He lived a very retired life, and saw little or nothing of society; when he did mingle in it, his dogmatism and pugnacity caused him to be generally shunned." In 1762, he was appointed as the land-agent to the Earl Fitzwilliam, a post he held until two years before his death in 1790.

As a mathematician, Landen was primarily an analyst and geometer. Most of his important works were published in the latter part of his career. These include the aforementioned papers and *Mathematical Memoirs*, published in 1780 and 1789. For several years, Landen contributed many problems and solutions to the *Ladies Diary*. From 1743–1749, he posed a total of eleven problems and published thirteen solutions to problems. However, Leybourn [**34**] has disclosed that contributors to the *Ladies Diary* frequently employed aliases. In particular, Landen used the pseudonyms Sir Stately Stiff, Peter Walton, Waltoniensis, C. Bumpkin, and Peter Puzzlem, who, collectively, proposed ten problems and answered seventeen. Leybourn [**34**] has compiled in four volumes the problems and solutions from the *Ladies Diary* from 1704–1816. Especially valuable are his indices of subject classifications and contributors. (The problems and solutions from the years 1704–1760 had been previously collected by others in one volume in 1774 [**50**].)

First published in 1704, the annual *Ladies Diary* evidently was very popular in England with a yearly circulation of several thousand. The *Ladies Diary* is "designed principally for the amusement and instruction of the fair sex." It contains "new improvements in arts and sciences, and many entertaining particulars... for the use and diversion of the fair sex." The cover is graced by a poem dedicated to the reigning queen and which normally changed little from year to year. Our paper begins with the poem from 1776 paying eloquent homage to the beloved of King George III. Among other things, the *Ladies Diary* contains a "chronology of remarkable events," birth dates of the royal family, enigmas, and answers to enigmas from the previous year. The enigmas as well as the answers were normally set to verse.

The largest portion of the *Ladies Diary* is devoted to the solutions of mathematical problems posed in the previous issue. Despite the name of the journal, very few contributors were women. Leybourn's [34] index lists a total of 913 contributors of which 32 were women. Because proposers and solvers did occasionally employ pen names such as Plus Minus, Mathematicus, Amicus, Archimedes, Diophantoides, and the aforementioned aliases for Landen, it is possible that the number of female contributors is slightly higher. In 1747, Landen gave a solution to a problem which was "designed to improve gunnery of which there are several things wanting." Does not this have a familiar ring today? Geometrical problems were popular, and rigor was lax at times. Here is an example from 1783. Let

$$a = \sum_{k=0}^{\infty} \frac{1}{\sqrt{2k+1}} \quad \text{and} \quad b = \sum_{k=1}^{\infty} \frac{1}{\sqrt{2k}}.$$

Show that $a/b = \sqrt{2} - 1$. In 1784, Joseph French provided the following "elegant" solution. We see that

$$\sqrt{2} \sum_{k=1}^{\infty} \frac{1}{\sqrt{2k}} = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = a + b.$$

Thus, $b(\sqrt{2} - 1) = a$, and the result follows.

Those readers wishing to learn more about Landen's work should consult Watson's very delightful article, "The Marquis and the Land-agent" [52]. Readers desiring more knowledge of the mathematical content of the *Ladies Diary* should definitely consult Leybourn's compendium [34]. (Only a few libraries in the U.S. possess copies of the *Ladies Diary*. The University of Illinois Library has a fairly complete collection, although there are several gaps prior to 1774. T. Perl [43] has written a detailed description of the *Ladies Diary* with an emphasis on the contributions by women and an analysis of both the positive and negative sociological factors on womens' mathematical education during the years of the *Diary*. For additional historical information about the *Ladies Diary* and other obscure English journals containing mathematics, see Archibald's paper [2].)

4. Ivory and Landen's transformation. In 1796, J. Ivory [25] published a new formula for the perimeter of an ellipse. A very similar proof establishes Theorem 2, a version of Landen's transformation discussed in the previous section.

Before proving Theorem 2, we note that it implies a new version of Theorem 1.

THEOREM 1″. *If $x > 0$, then*

$$M(1+x, 1-x) = \frac{\pi(1+x)}{2K\left(\dfrac{2\sqrt{x}}{1+x}\right)}.$$

Theorem 1′ also follows from Theorem 1″; put $x = (a-b)/(a+b)$ and utilize (4).

Proof of Theorem 2. Using the definition (2) of K, employing the binomial series, and inverting the order of summation and integration below, we find that

$$\begin{aligned}
K\left(\frac{2\sqrt{x}}{1+x}\right) &= \frac{1}{2}\int_0^{\pi}\left(1-\frac{4x}{(1+x)^2}\sin^2\varphi\right)^{-1/2} d\varphi \\
&= \frac{1}{2}\int_0^{\pi}\left(1-\frac{2x}{(1+x)^2}(1-\cos(2\varphi))\right)^{-1/2} d\varphi \\
&= \frac{1}{2}(1+x)\int_0^{\pi}\left(1+x^2+2x\cos(2\varphi)\right)^{-1/2} d\varphi \\
&= \frac{1}{2}(1+x)\int_0^{\pi}\left(1+xe^{2i\varphi}\right)^{-1/2}\left(1+xe^{-2i\varphi}\right)^{-1/2} d\varphi \\
&= \frac{1}{2}(1+x)\sum_{m=0}^{\infty}\frac{\left(\frac{1}{2}\right)_m(-x)^m}{m!}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n(-x)^n}{n!}\int_0^{\pi} e^{2i(m-n)\varphi}\, d\varphi \\
&= \frac{\pi}{2}(1+x)\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^2 x^{2n}}{(n!)^2} \\
&= (1+x)K(x),
\end{aligned}$$

by (8). This concludes the proof.

Ivory's paper **[25]**, establishing an analogue of Theorem 2, possesses an unusual feature in that it begins with the "cover letter" that Ivory sent to the editor John Playfair when he submitted his paper! In this letter, Ivory informs Playfair about what led him to his discovery. Evidently then, the editor deemed it fair play to publish Ivory's letter as a preamble to his paper. The letter reads as follows.

Dear Sir,
Having, as you know, bestowed a good deal of time and attention on the study of that part of physical astronomy which relates to the mutual disturbances of the planets, I have, naturally, been led to consider the various methods of resolving the formula $(a^2 + b^2 - 2ab\cos\varphi)^n$ into infinite series of the form $A + B\cos\varphi + C\cos 2\varphi + \&c.$ In the course of these investigations, a series for the rectification of the ellipsis occurred to me, remarkable for its simplicity, as well as its rapid convergency. As I believe it to be new, I send it to you, inclosed, together with some remarks on the evolution of the formula just mentioned, which if you think proper, you may submit to the consideration of the Royal Society.

I am, Dear Sir,
Yours, & *c*.
James Ivory

5. Calculation of π. First, we define the complete elliptic integral of the second kind,

$$E(x) := \int_0^{\pi/2}\left(1-x^2\sin^2\varphi\right)^{1/2} d\varphi,$$

where $|x| < 1$. Two formulas relating the elliptic integrals $E(x)$ and $K(x)$ are the basis for one of the currently most efficient methods to calculate π. The first is due to Legendre [33, p. 61]. We give below a simple proof that appears not to have been, heretofore, given.

THEOREM 3. *Let* $x' = \sqrt{1-x^2}$, *where* $0 < x < 1$. *Then*

$$K(x)E(x') + K(x')E(x) - K(x)K(x') = \frac{\pi}{2}. \tag{14}$$

Proof. Let $c = x^2$ and $c' = 1 - c$. A straightforward calculation gives

$$\begin{aligned}\frac{d}{dc}(E-K) &= -\frac{d}{dc}\int_0^{\pi/2}\frac{c\sin^2\varphi}{(1-c\sin^2\varphi)^{1/2}}\,d\varphi\\ &= \frac{E}{2c} - \frac{1}{2c}\int_0^{\pi/2}\frac{d\varphi}{(1-c\sin^2\varphi)^{3/2}}.\end{aligned}$$

Since

$$\frac{d}{d\varphi}\left(\frac{\sin\varphi\cos\varphi}{(1-c\sin^2\varphi)^{1/2}}\right) = \frac{1}{c}(1-c\sin^2\varphi)^{1/2} - \frac{c'}{c}(1-c\sin^2\varphi)^{-3/2},$$

we deduce that

$$\begin{aligned}\frac{d}{dc}(E-K) &= \frac{E}{2c} - \frac{E}{2cc'} + \frac{1}{2c'}\int_0^{\pi/2}\frac{d}{d\varphi}\left(\frac{\sin\varphi\cos\varphi}{(1-c\sin^2\varphi)^{1/2}}\right)d\varphi\\ &= \frac{E}{2c}\left(1-\frac{1}{c'}\right) = -\frac{E}{2c'}.\end{aligned} \tag{15}$$

For brevity, put $K' = K(c')$ and $E' = E(c')$. Since $c' = 1 - c$, it follows that

$$\frac{d}{dc}(E'-K') = \frac{E'}{2c}. \tag{16}$$

Lastly, easy calculations yield

$$\frac{dE}{dc} = \frac{E-K}{2c} \quad \text{and} \quad \frac{dE'}{dc} = -\frac{E'-K'}{2c'}. \tag{17}$$

If L denotes the left side of (14), we may write L in the form

$$L = EE' - (E-K)(E'-K').$$

Employing (15)–(17), we find that

$$\frac{dL}{dc} = \frac{(E-K)E'}{2c} - \frac{E(E'-K')}{2c'} + \frac{E(E'-K')}{2c'} - \frac{(E-K)E'}{2c} = 0.$$

Hence, L is a constant, and we will find its value by letting c approach 0.
First,

$$E - K = -c\int_0^{\pi/2}\frac{\sin^2\varphi}{(1-c\sin^2\varphi)^{1/2}}\,d\varphi = O(c)$$

as c tends to 0. Next,

$$K' = \int_0^{\pi/2} (1 - c'\sin^2\varphi)^{-1/2}\, d\varphi \leqslant \int_0^{\pi/2} (1 - c')^{-1/2}\, d\varphi$$
$$= O(c^{-1/2}),$$

as c tends to 0. Thus,

$$\lim_{c\to 0} L = \lim_{c \to 0} \{(E - K)K' + E'K\}$$
$$= \lim_{c\to 0} \left\{ O(c^{1/2}) + 1 \cdot \frac{\pi}{2} \right\} = \frac{\pi}{2},$$

and the proof is complete.

The second key formula, given in Theorem 4 below, can be proved via an iterative process involving Landen's transformation. We forego a proof here; a proof may be found, for example, in King's book [**29**, pp. 7, 8].

THEOREM 4. *Let, for* $a > b > 0$,

$$J(a, b) = \int_0^{\pi/2} (a^2\cos^2\varphi + b^2\sin^2\varphi)^{1/2}\, d\varphi, \tag{18}$$

and recall that c_n *is defined by* (1). *Then*

$$J(a, b) = \left(a^2 - \frac{1}{2}\sum_{n=0}^{\infty} 2^n c_n^2\right) I(a, b),$$

where $I(a, b)$ *is defined by* (3).

Note that

$$J(a, b) = aE(x),$$

where $x = (1/a)\sqrt{a^2 - b^2}$.

Theorems 3 and 4 now lead to a formula for π which is highly suitable for computation.

THEOREM 5. *If* c_n *is defined by* (1), *then*

$$\pi = \frac{4M^2(1, 1/\sqrt{2})}{1 - \sum_{n=1}^{\infty} 2^{n+1}c_n^2}.$$

Proof. Letting $x = x' = 1/\sqrt{2}$ in Theorem 3, we find that

$$2K\left(\frac{1}{\sqrt{2}}\right)E\left(\frac{1}{\sqrt{2}}\right) - K^2\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{2}. \tag{19}$$

Setting $a = 1$ and $b = 1/\sqrt{2}$ in Theorem 4, we see that

$$E\left(\frac{1}{\sqrt{2}}\right) = \left(1 - \frac{1}{2}\sum_{n=0}^{\infty} 2^n c_n^2\right) K\left(\frac{1}{\sqrt{2}}\right), \tag{20}$$

since $I(1,\sqrt{2}) = K(1/\sqrt{2})$ and $J(1,1/\sqrt{2}) = E(1/\sqrt{2})$. Lastly, by Theorem 1′,

$$M(1,1/\sqrt{2}) = \frac{\pi}{2K(1/\sqrt{2})}. \tag{21}$$

Substituting (20) into (19), employing (21), noting that $c_0^2 = 1/2$, and solving for π, we complete the proof.

According to King [**29**, pp. 8, 9, 12], an equivalent form of Theorem 5 was established by Gauss. Observe that in the proof of Theorem 5, we used only the special case $x = x' = 1/\sqrt{2}$ of Legendre's identity, Theorem 3. We would like to show now that this special case is equivalent to the formula

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} \int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} = \frac{\pi}{4}, \tag{22}$$

first proved by Euler [**22**] in 1782. (Watson [**52**, p. 12] claimed that an equivalent formulation of (22) was earlier established by both Landen and Wallis, but we have been unable to verify this.) The former integral in (22) is one quarter of the arc length of the lemniscate given by $r^2 = \cos(2\varphi)$, $0 \leqslant \varphi \leqslant 2\pi$. The latter integral in (22) is intimately connected with the classical elastic curve. For a further elaboration of the connections of these two curves with the arithmetic-geometric mean, see Cox's paper [**18**].

In order to prove (22), make the substitution $x = \cos\varphi$. Then straightforward calculations yield

$$K\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}\int_0^1 \frac{dx}{\sqrt{1-x^4}}$$

and

$$2E\left(\frac{1}{\sqrt{2}}\right) - K\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}}.$$

It is now easy to see that Legendre's relation Theorem 3 in the case $x = x' = 1/\sqrt{2}$ implies (22).

In 1976, Salamin [**47**] rederived the forgotten Theorem 5, from which he established a rapidly convergent algorithm for the computation of π. Recall that in Section 2 we demonstrated how rapidly the arithmetic-geometric mean converges and thus how fast c_n tends to 0. Tamura and Kanada have used this algorithm to compute π to 2^{24} (over 16 million) decimal places. An announcement about their calculation of π to 2^{23} decimal places was made in *Scientific American* [**48**]. Their paper [**27**] describes their calculation to 10,013,395 decimal places. More recently, D. H. Bailey [**3**] has used a quartically convergent algorithm to calculate π to 29,360,000 digits.

Newman [**40**] has obtained a quadratic algorithm for the computation of π that is somewhat simpler than Salamin's. His proof is quite elementary and avoids Legendre's identity. It should be remarked that Newman's estimates of some integrals are not quite correct. However, the final result (middle of p. 209) is correct.

In 1977, Brent [**14**] observed that the arithmetic-geometric mean could be implemented to calculate elementary functions as well. Let us briefly indicate how to calculate $\log 2$. From Whittaker and Watson's text [**53**, p. 522], as x tends to $1-$,

$$K(x) \sim \log \frac{4}{\sqrt{1-x^2}}.$$

From Theorem 1′,

$$K(x) = \frac{\pi}{2M\left(1, \sqrt{1-x^2}\right)},$$

and so

$$\log \frac{4}{\sqrt{1-x^2}} \approx \frac{\pi}{2M\left(1, \sqrt{1-x^2}\right)}.$$

Taking $\sqrt{1-x^2} = 4 \cdot 2^{-n}$, we find that

$$\log 2 \approx \frac{\pi}{2nM\left(1, 2^{2-n}\right)},$$

for large n.

Further improvements in both the calculation of π and elementary functions have been made by J. M. and P. B. Borwein [**8**], [**9**], [**10**], [**11**], [**12**], [**13**]. In particular, in [**8**], [**11**], and [**12**], they have utilized elliptic integrals and modular equations to obtain algorithms of higher order convergence to approximate π. The survey article [**10**] by the Borwein brothers is to be especially recommended. Carlson [**16**] has written an earlier survey on algorithms dependent on the arithmetic-geometric mean and variants thereof.

Postscript to π. The challenge of approximating and calculating π has been with us for over 4000 years. By 1844, π was known to 200 decimal places. This stupendous feat was accomplished by a calculating prodigy named Johann Dase in less than two months. On Gauss's recommendation, the Hamburg Academy of Sciences hired Dase to compute the factors of all integers between 7,000,000 and 10,000,000. Thus, our ideas have come to a full circle. As Beckmann [**5**, p. 104] remarks, "It would thus appear that Carl Friedrich Gauss, who holds so many firsts in all branches of mathematics, was also the first to introduce payment for computer time." The computer time now for 29 million digits (28 hours) is considerably less than the computer time for 200 digits by Gauss's computer, Dase.

6. Approximations for the perimeter L of an ellipse. If an ellipse is given by the parametric equations $x = a\cos\varphi$ and $y = b\sin\varphi$, $0 \leqslant \varphi \leqslant 2\pi$, then from elementary calculus,

$$\begin{aligned} L = L(a,b) &= \int_0^{2\pi} \left(a^2\sin^2\varphi + b^2\cos^2\varphi\right)^{1/2} d\varphi \\ &= 4J(b,a), \end{aligned} \tag{23}$$

where $J(b, a)$ is defined by (18). Thus, we see immediately from Theorems 1′ and 4 that elliptical perimeters and arithmetic-geometric means are inextricably intertwined. Ivory's letter and our concomitant comments also unmistakenly pointed to this union.

Before discussing approximations for L, we offer two exact formulas. The former is due to MacLaurin [36] in 1742, and the latter was initially found by Ivory [25] in 1796, although it is implicit in the earlier work of Landen.

THEOREM 6. *Let* $x = a\cos\varphi$ *and* $y = b\sin\varphi$, $0 \leqslant \varphi \leqslant 2\pi$. *Let* $e = (1/a)\sqrt{a^2-b^2}$, *the eccentricity of the ellipse. Then if F is defined by* (12),

$$L(a,b) = 2\pi a F\left(\frac{1}{2}, -\frac{1}{2}; 1; e^2\right) \tag{24}$$

$$= \pi(a+b)F\left(-\frac{1}{2}, -\frac{1}{2}; 1; \lambda^2\right), \tag{25}$$

where

$$\lambda = \frac{a-b}{a+b}.$$

Proof. The proofs are very similar to those in Sections 2 and 4. First, using (23), expanding the integrand in a binomial series, and integrating termwise, we deduce that

$$\begin{aligned} L(a,b) &= 4a\int_0^{\pi/2}(1-e^2\cos^2\varphi)^{1/2}\,d\varphi \\ &= 4a\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_n}{n!}e^{2n}\int_0^{\pi/2}\cos^{2n}\varphi\,d\varphi \qquad (26) \\ &= 2\pi a F\left(\frac{1}{2}, -\frac{1}{2}; 1; e^2\right). \end{aligned}$$

Thus, (24) is established.

We indicate two proofs of (25). First, in Landen's transformation (13) of hypergeometric series, set $a = -1/2$, $b = 1/2$, and $x = \lambda$. We immediately find that

$$F\left(-\frac{1}{2}, \frac{1}{2}; 1; e^2\right) = \frac{a+b}{2a}F\left(-\frac{1}{2}, -\frac{1}{2}; 1; \lambda^2\right).$$

By this formula and (24), formula (25) is demonstrated.

The second proof that we mention is that of Ivory [25]. Using (26), proceed exactly in the same fashion as in the proof of Theorem 2 in Section 4.

In fact, there exists a third early formula for $L(a,b)$. In 1773, Euler [21] proved that

$$L(a,b) = \pi\sqrt{2(a^2+b^2)}\,F\left(-\frac{1}{4}, \frac{1}{4}; 1; \left(\frac{a^2-b^2}{a^2+b^2}\right)^2\right).$$

Although Euler proceeded differently, we mention that his formula may be derived from MacLaurin's via a certain quadratic transformation for hypergeometric series that is different from Landen's. Euler's formula also trivially leads to an approximation for $L(a,b)$ given in our table below.

The problem of determining $L(a, b)$ is not as venerable as that for determining π. However, some have argued (not very convincingly) that the problem goes back to the time of King Solomon, who hired a craftsman Huram to make a tank. According to 1 Kings 7 : 23, "Huram made a round tank of bronze 5 cubits deep, 10 cubits in diameter, and 30 cubits in circumference." The implication is clear that the ancient Hebrews regarded π as being equal to 3. It has been suggested, perhaps by someone who believes that "God makes no mistakes," that "round" and "depth" are to be interpreted loosely, and that the tank really was elliptical in shape, with the major axis being 10 cubits and the minor axis being about 9.53 cubits in length.

As might be expected. the primary impetus in finding methods for calculating elliptical perimeters arises from astronomy. In 1609, Kepler [**28**] offered perhaps the first legitimate approximations

$$L \approx \pi(a+b) \quad \text{and} \quad L \approx 2\pi\sqrt{ab},$$

although, as we shall see, his arguments were not very rigorous and $2\pi\sqrt{ab}$ was intended to be only a *lower bound* for L. Kepler [**28**, p. 307] first remarks that the ellipse with semiaxes a and b and the circle with radius $\sqrt{ab}$ have the same areas. Since the circle has the smaller circumference,

$$L \geqslant 2\pi\sqrt{ab}.$$

He [**28**, p. 368] furthermore remarks that $(1/2)(a+b) \geqslant \sqrt{ab}$, and so concludes that

$$L \approx 2\pi\frac{1}{2}(a+b).$$

Kepler appears to be using the dubious principle that quantities larger than the same number must be about equal.

Approximations of several types, depending upon the relative sizes of a and b, exist in the literature. In this section, we concentrate on estimates that are best for a close to b. Thus, we shall write all of our approximations in terms of $\lambda = (a-b)/(a+b)$ and compare them with the expansion (25). For example, Kepler's second approximation can be written in the form

$$L \approx \pi(a+b)(1-\lambda^2)^{1/2}.$$

We now show how the formula

$$L(a,b) = 4J(a,b) = \frac{2\pi}{M(a,b)}\left(a^2 - \frac{1}{2}\sum_{n=0}^{\infty} 2^n c_n^2\right), \tag{27}$$

arising from Theorems 1′ and 4, can be used to find approximations to the perimeter of an ellipse. Replacing $M(a, b)$ by a_2 and neglecting the terms with $n \geqslant 2$, we find that

$$L(a,b) \approx \frac{2\pi}{a_2}\left(a^2 - \frac{c_0^2}{2} - c_1^2\right) = \frac{2\pi a_1^2}{a_2} = 2\pi\left(\frac{a+b}{\sqrt{a}+\sqrt{b}}\right)^2.$$

This formula was first obtained by Ekwall [**19**] in 1973 as a consequence of a formula by Sipos from 1792 [**54**].

If we replace $M(a, b)$ by a_3 in (27) and neglect all terms with $n \geqslant 3$, we find, after some calculation, that

$$L(a,b) \approx 2\pi \frac{2(a+b)^2 - (\sqrt{a} - \sqrt{b})^4}{(\sqrt{a} + \sqrt{b})^2 + 2\sqrt{2}\sqrt{a+b}\sqrt[4]{ab}}.$$

This formula is complicated enough to dissuade us from calculating further approximations by this method.

We now provide a table of approximations for $L(a, b)$ that have been given in the literature. At the left, we list the discoverer (or source) and year of discovery (if known). The approximation $A(\lambda)$ for $L(a, b)/\pi(a + b)$ is given in the second column in two forms. In the last column, the first nonzero term in the power series for

$$A(\lambda) - \frac{L(a,b)}{\pi(a+b)} = A(\lambda) - F\left(-\frac{1}{2}, -\frac{1}{2}; 1; \lambda^2\right)$$

is offered so that the accuracy of the approximating formula can be discerned. For convenience, we note that

$$F\left(-\frac{1}{2}, -\frac{1}{2}; 1; \lambda^2\right) = 1 + \frac{1}{4}\lambda^2 + \frac{1}{4^3}\lambda^4 + \frac{1}{4^4}\lambda^6 + \frac{25}{4^7}\lambda^8 + \frac{49}{4^8}\lambda^{10} + \cdots.$$

Kepler [**28**], 1609	$\dfrac{2\sqrt{ab}}{a+b} = (1-\lambda^2)^{1/2}$	$-\dfrac{3}{4}\lambda^2$
Euler [**21**], 1773	$\dfrac{\sqrt{2(a^2+b^2)}}{a+b} = (1+\lambda^2)^{1/2}$	$\dfrac{1}{4}\lambda^2$
Sipos [**54**], 1792 Ekwall [**19**], 1973	$\dfrac{2(a+b)}{(\sqrt{a}+\sqrt{b})^2} = \dfrac{2}{1+\sqrt{1-\lambda^2}}$	$\dfrac{7}{64}\lambda^4$
Peano [**42**], 1889	$\dfrac{3}{2} - \dfrac{\sqrt{ab}}{a+b} = \dfrac{3}{2} - \dfrac{1}{2}(1-\lambda^2)^{1/2}$	$\dfrac{3}{64}\lambda^4$
Muir [**38**], 1883	$\dfrac{2}{a+b}\left(\dfrac{a^{3/2}+b^{3/2}}{2}\right)^{2/3}$ $= \dfrac{1}{2^{2/3}}\{(1+\lambda)^{3/2} + (1-\lambda)^{3/2}\}^{2/3}$	$-\dfrac{1}{64}\lambda^4$
Lindner [**35**, p. 439], 1904–1920 Nyvoll [**41**], 1978	$\left\{1 + \dfrac{1}{8}\left(\dfrac{a-b}{a+b}\right)^2\right\}^2$ $= \left(1 + \dfrac{1}{8}\lambda^2\right)^2$	$-\dfrac{1}{2^8}\lambda^6$
Selmer [**49**], 1975	$1 + \dfrac{4(a-b)^2}{(5a+3b)(3a+5b)}$ $= 1 + \dfrac{1}{4}\lambda^2 \dfrac{1}{1 - \dfrac{1}{16}\lambda^2}$	$-\dfrac{3}{2^{10}}\lambda^6$

Ramanujan [44], [45], 1914 Fergestad [49], 1951	$\dfrac{3-\sqrt{(a+3b)(3a+b)}}{a+b}$ $= 3-\sqrt{4-\lambda^2}$	$-\dfrac{1}{2^9}\lambda^6$
Almkvist [1], 1978	$2\dfrac{2(a+b)^2-(\sqrt{a}-\sqrt{b})^4}{(a+b)\left\{(\sqrt{a}+\sqrt{b})^2+2\sqrt{2}\sqrt{a+b}\sqrt[4]{ab}\right\}}$ $=2\dfrac{\left(1+\sqrt{1-\lambda^2}\right)^2+\lambda^2\sqrt{1-\lambda^2}}{\left(1+\sqrt{1-\lambda^2}\right)\left(1+\sqrt[4]{1-\lambda^2}\right)^2}$	$\dfrac{15}{2^{14}}\lambda^8$
Bronshtein and Semendyayev [15], 1964 Selmer [49], 1975	$\dfrac{1}{16}\dfrac{64(a+b)^4-3(a-b)^4}{(a+b)^2(3a+b)(a+3b)}$ $=\dfrac{64-3\lambda^4}{64-16\lambda^2}$	$-\dfrac{9}{2^{14}}\lambda^8$
Selmer [49], 1975	$\dfrac{1}{8}\left\{12+\left(\dfrac{a-b}{a+b}\right)^2-\dfrac{2\sqrt{2(a^2+6ab+b^2)}}{a+b}\right\}$ $=\dfrac{3}{2}+\dfrac{1}{8}\lambda^2-\dfrac{1}{2}\sqrt{1-\dfrac{1}{2}\lambda^2}$	$-\dfrac{5}{2^{14}}\lambda^8$
Jacobsen and Waadeland [26], 1985	$\dfrac{256-48\lambda^2-21\lambda^4}{256-112\lambda^2+3\lambda^4}$	$-\dfrac{33}{2^{18}}\lambda^{10}$
Ramanujan [44], [45], 1914	$1+\dfrac{3\lambda^2}{10+\sqrt{4-3\lambda^2}}$	$-\dfrac{3}{2^{17}}\lambda^{10}$

The two approximations by India's great mathematician, S. Ramanujan, were first stated by him in his notebooks [46, p. 217], and then later at the end of his paper [44], [45, p. 39], where he says that they were discovered empirically. Ramanujan [44], [45] also provides error approximations, but they are in a form different from that given here. Since

$$\lambda = \frac{a-b}{a+b} = \frac{1-\sqrt{1-e^2}}{1+\sqrt{1-e^2}} \approx \frac{e^2}{4},$$

we find that, for the first approximation,

$$\pi(a+b)\frac{\lambda^6}{2^9} \approx \pi a\left(1+\sqrt{1-e^2}\right)\frac{\left(e^2/4\right)^6}{2^9} < 2\pi a\frac{e^{12}}{2^{21}} = \pi a\frac{e^{12}}{2^{20}},$$

which is the approximate error given by Ramanujan. Similarly, for the second approximation, Ramanujan states that the error is approximately equal to

$$3\pi a\frac{e^{20}}{2^{36}},$$

which is in agreement with our claim. The exactness of Ramanujan's second formula for eccentricities that are not too large is very good. For example, for the orbit of

Mercury ($e = 0.206$), the absolute error is about 1.5×10^{-13} meters. Note that if we set $b = 0$ in Ramanujan's second formula, we find that $\pi \approx 22/7$.

Fergestad [49] rediscovered Ramanujan's first formula several years later.

Despite Ramanujan's remark on the discovery of these two formulas, Jacobsen and Waadeland [26] have offered a very plausible explanation of Ramanujan's approximations. We confine our attention to the latter approximation, since the arguments are similar. Write

$$F\left(-\frac{1}{2}, -\frac{1}{2}; 1; \lambda^2\right) = 1 + \frac{\lambda^2}{4(1+w)}. \tag{28}$$

Then it can be shown that w has the continued fraction expansion

$$w = \frac{1}{3}\left\{\frac{-\frac{3}{16}\lambda^2}{1} + \frac{-\frac{3}{16}\lambda^2}{1} + \frac{-\frac{3}{16}\lambda^2}{1} + \frac{-\frac{11}{48}\lambda^2}{1} + \cdots\right\}.$$

If each numerator above is replaced by $-3\lambda^2/16$, then we obtain the approximation

$$w \approx \frac{1}{12}\left(-2 + \sqrt{4 - 3\lambda^2}\right).$$

Substituting this approximation in (28) and then using (25), we are immediately led to the estimate

$$\frac{L(a,b)}{\pi(a+b)} \approx 1 + \frac{3\lambda^2}{10 + \sqrt{4 - 3\lambda^2}}.$$

Since Ramanujan's facility in representing analytic functions by continued fractions is unmatched in mathematical history, it seems likely that Ramanujan discovered his approximations in this manner.

In the next section, we examine some approximations for $L(a,b)$ of a different type given by Ramanujan in his notebooks [46].

7. Further approximations given by Ramanujan. In his notebooks [46], Ramanujan offers some very unusual formulas, expressed in sexagesimal notation, for $L(a,b)$. The first is related to his approximation $3 - \sqrt{4 - \lambda^2}$ given in Section 6.

THEOREM 7. *Put*

$$L(a,b) = \pi(a+b)\left(1 + 4\sin^2\frac{1}{2}\theta\right), \quad 0 \leqslant \theta \leqslant \pi/4, \tag{29}$$

and

$$\sin\theta = \lambda \sin\alpha, \qquad \lambda = \frac{a-b}{a+b}. \tag{30}$$

Then, when the eccentricity $e = 1$, $\alpha = 30°18'6''$, *and as* e *tends to* 0, α *tends monotonically to* $30°$.

It is not clear how Ramanujan was led to this very unusual theorem. The variance of α over such a small interval is curious.

Proof. We shall prove Theorem 7 except for the conclusion about monotonicity. However, we shall show that $\alpha \geqslant \pi/6$ always.

For brevity, we write (25) in the form

$$\frac{L(a,b)}{\pi(a+b)} = \sum_{n=0}^{\infty} \alpha_n \lambda^{2n}, \qquad |\lambda| < 1. \tag{31}$$

It then follows from (29) and (30) that

$$3 - 2\sqrt{1-\lambda^2\sin^2\alpha} = 1 + 4\sin^2\frac{1}{2}\theta = \sum_{n=0}^{\infty} \alpha_n \lambda^{2n}, \qquad |\lambda| < 1. \tag{32}$$

Next set

$$3 - \sqrt{4-\lambda^2} = \sum_{n=0}^{\infty} \beta_n \lambda^{2n}, \qquad |\lambda| < 2. \tag{33}$$

As implied in Section 6, $\alpha_n = \beta_n$, $n = 0,1,2$. We shall further show that, for $n \geqslant 3$,

$$\beta_n \leqslant \alpha_n / 2^{n-2}. \tag{34}$$

From the definitions (31) and (33), respectively, short calculations show that, for $n \geqslant 1$,

$$\frac{\alpha_{n+1}}{\alpha_n} = \frac{(2n-1)^2}{(2n+2)^2} \quad \text{and} \quad \frac{\beta_{n+1}}{\beta_n} = \frac{2n-1}{8(n+1)}.$$

Thus,

$$\frac{\beta_{n+1}}{\beta_n} \bigg/ \frac{\alpha_{n+1}}{\alpha_n} = \frac{n+1}{2(2n-1)} \leqslant \frac{1}{2},$$

if $n \geqslant 2$. Proceeding by induction, we deduce that

$$\frac{\beta_{n+1}}{\alpha_{n+1}} \leqslant \frac{1}{2}\frac{\beta_n}{\alpha_n} \leqslant \frac{1}{2^{n-1}},$$

for $n \geqslant 2$, and the proof of (34) is complete.

From (32) and (34), it follows that

$$3 - \sqrt{4-\lambda^2} \leqslant 3 - 2\sqrt{1-\lambda^2\sin^2\alpha}.$$

Solving this inequality, we find that $\sin^2\alpha \geqslant 1/4$, or $\alpha \geqslant \pi/6$.

Second, we calculate α when $e = 1$. Thus, $\lambda = 1$ and $\theta = \alpha$. Therefore, from (25) and (32),

$$1 + 4\sin^2\frac{1}{2}\alpha = F\left(-\frac{1}{2}, -\frac{1}{2}; 1; 1\right) = \frac{4}{\pi}. \tag{35}$$

This evaluation follows from a general theorem of Gauss on the evaluation of hypergeometric series at the argument 1 [**4**, p. 2]. Moreover, this particular series is found in Gauss's diary under the date June, 1798 [**24**]. Thus,

$$\sin^2\frac{1}{2}\alpha = \frac{1}{\pi} - \frac{1}{4} = 0.0683098861.$$

It follows that $\alpha = 30°18'6''$.

Third, we calculate α when $e = 0$. From (30) and (32),

$$\lim_{\lambda \to 0} \sin^2\alpha = \lim_{\lambda \to 0} \frac{\sin^2\theta}{\lambda^2} = \lim_{\lambda \to 0} \frac{4\sin^2\frac{1}{2}\theta}{\lambda^2}$$

$$= \lim_{\lambda \to 0} \lambda^{-2} \sum_{n=1}^{\infty} \alpha_n \lambda^{2n} = \alpha_1 = \frac{1}{4}.$$

Thus, α tends to $\pi/6$ as e tends to 0.

Ramanujan [**46**, p. 224] offers another theorem, which we do not state, like Theorem 7 but which appears to be motivated by his second approximation for $L(a, b)$.

Ramanujan [**46**, p. 224] states two additional formulas each of which combines two approximations, one for e near 0 and the other for e close to 1. Again, we give just one of the pair. A complete proof of Theorem 8 below would be too lengthy for this paper, and so we shall just sketch the main ideas of the proof. Complete details may be found in [**7**].

THEOREM 8. *Set*

$$L(a, b) = \pi(a + b)\frac{\tan\theta}{\theta}, \qquad 0 \leqslant \theta < \pi/2, \tag{36}$$

and

$$\tan\theta = \lambda\cos\alpha, \qquad \lambda = \frac{a - b}{a + b}. \tag{37}$$

Then as e increases from 0 *to* 1, α *decreases from* $\pi/6$ *to* 0. *Furthermore*, α *is approximately given by*

$$\frac{2\sqrt{ab}}{a + b}\left\{30^\circ + 6^\circ 18' 49'' \frac{(\sqrt{a} - \sqrt{b})^2}{a + b} - 1^\circ 10' 55''\left(\frac{a - b}{a + b}\right)^2\right\}. \tag{38}$$

Proof. If $e = 0$, then $\lambda = 0$ and $\theta = 0$. The argument is very similar to that in the proof of Theorem 7, and we find that

$$\lim_{\lambda \to 0} \cos^2\alpha = 3\alpha_1 = 3/4.$$

Thus, $\alpha = \pi/6$ when $\lambda = 0 = e$.

We next determine α when $e = 1$. Thus, $\lambda = 1$ and $\tan\theta = \cos\alpha$ by (37). From (25), (36), and (35),

$$\left.\frac{\tan\theta}{\theta}\right|_{\lambda=1} = F\left(-\frac{1}{2}, -\frac{1}{2}; 1; 1\right) = \frac{4}{\pi}.$$

Thus, $\theta = \pi/4$ and $\alpha = 0$.

It appears to be extremely difficult to show that as λ goes from 0 to 1, α *monotonically* decreases from $\pi/6$ to 0. It can be shown [**7**], however, that $0 \leqslant \alpha \leqslant \pi/6$, always. A proof depends upon a continued fraction for $\tan^{-1}x$.

The proof of (38) is very difficult, and we provide only a brief sketch. We observe (again) that

$$\sqrt{1-\lambda^2} = \frac{2\sqrt{ab}}{a+b},$$

and so

$$\sqrt{1-\lambda^2} - (1-\lambda^2) = \frac{2\sqrt{ab}}{(a+b)^2}(\sqrt{a}-\sqrt{b})^2.$$

Thus, Ramanujan is attempting to find an approximation to α of the form

$$\sqrt{1-\lambda^2}\left(A + B\left\{1-\sqrt{1-\lambda^2}\right\} + C\lambda^2\right), \tag{39}$$

which will be a good approximation both when λ is close to 0 and when λ is near 1. Our task is then to determine A, B, and C.

With a considerable amount of effort, it can be shown that [7]

$$\alpha = \frac{\pi}{6} - \frac{21\sqrt{3}}{160}\lambda^2 + O(\lambda^4) \tag{40}$$

in a neighborhood of $\lambda = 0$. The proper expansion near $\lambda = 1$ is even more difficult to obtain because $F(-\frac{1}{2}, -\frac{1}{2}; 1; \lambda^2)$ is not analytic at $\lambda = 1$. However, there does exist an asymptotic expansion for $F(-\frac{1}{2}, -\frac{1}{2}; 1; \lambda^2)$ as λ tends to $1-$, and employing this, we can show that [7]

$$\alpha = \sqrt{\frac{4-\pi}{2\pi-4}}\sqrt{1-\lambda^2} + o\left(\sqrt{1-\lambda^2}\right), \tag{41}$$

as λ tends to $1-$.

Having omitted the hard analysis, we now determine A, B, and C from (40) and (41) with little difficulty. When λ tends to 0, (39) tends to A. Thus, $A = \pi/6$, by (40). Next, examine $(\alpha - \pi/6)/\lambda^2$ as λ tends to 0. From (39) and (40), we find that

$$-\frac{\pi}{12} + \frac{1}{2}B + C = -\frac{21\sqrt{3}}{160}.$$

Now check $\alpha/\sqrt{1-\lambda^2}$ as λ tends to $1-$. From (39) and (41), we see that

$$\frac{\pi}{6} + B + C = \sqrt{\frac{4-\pi}{2\pi-4}}.$$

Simultaneously solving these last two equalities, we conclude that

$$B = 2\sqrt{\frac{4-\pi}{2\pi-4}} + \frac{21\sqrt{3}}{80} - \frac{\pi}{2} = 0.1101935$$

and

$$C = \frac{\pi}{3} - \sqrt{\frac{4-\pi}{2\pi-4}} - \frac{21\sqrt{3}}{80} = -0.0206291.$$

Converting A, B, and C to the sexagesimal system and substituting in (39), we complete the proof.

Although Ramanujan is well known for his approximations and asymptotic formulas in number theory, he has not been adequately recognized for his deep contributions to approximations and asymptotic series in analysis, because the vast majority of his results in the latter field have been hidden in his notebooks. These notebooks were begun in about 1903, when he was 15 or 16, and are a compilation of his mathematical discoveries without proofs. The last entries were made in 1914, when he sailed to England at the urging of G. H. Hardy. Although the editing of Ramanujan's notebooks was strongly advocated by Hardy and others immediately after Ramanujan's death in 1920, it is only recently that this has come to fruition **[6]**.

We have not attempted to give complete proofs of some of the theorems that we have described, but we hope that the principal ideas have been made clear. We have seen that a chain of related ideas stretches back over a period exceeding two centuries and provides impetus to contemporary mathematics. Ideas and topics that appear disparate are found to have common roots and merge together. For further elaboration of these ideas, readers should consult the works cited, especially Cox's paper **[18]**, the papers and book of J. M. and P. B. Borwein **[8]**–**[13]**, a paper by Almkvist **[1]** written in Swedish, and Berndt's forthcoming book **[7]**.

We are most grateful to Birger Ekwall for providing some very useful references.

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