PLATEAU'S PROBLEMS AND THEIR MODERN RAMIFICATIONS

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Today's special session within the National Meeting of the American Chemical Society is held in honor of Joseph-Antoine-Ferdinand Plateau whose magnum opus, *Statique expérimentale et théorique des liquides soumis aux seules forces moléculaires*, was published a century ago, in 1873. This book, in the main unifying the contents of a series of eleven long scientific papers which appeared during the period 1843–1869, may be described as the condensation of Plateau's experimental observations and measurements regarding the phenomena caused by capillary forces. It contains a complete historical account and a summary of theoretical insights as well. The book's impact on its contemporaries and on subsequent generations cannot be exaggerated. His reports were studied in many quarters. His experiments were repeated by many scientists, among them Faraday, Brewster and Boys. In present days Plateau's work has become rather lost in obscurity. Surface chemists who, with a few notable exceptions, now often shun the use of mathematical approaches, are generally content with a brief reference to Plateau in their text books, and it is really in mathematics where his name lives on.

1. Liquids in contact with their own vapors or air possess a surface tension which, unless opposed by external forces—for example, gravitation, centrifugal forces, the influence of an electric field, etc.—, causes the interface to assume the configuration of minimum area. The laws which govern the behavior of surfaces separating one medium from another clearly belong in the domain of physicists and chemists. Once these laws are conceived and formulated, however, it is not hard to understand why the investigation of the shape and the stability of such interfaces, which are, after all, geometrical objects subject to variational principles, differential equations and various boundary conditions, has over the years stimulated mathematical activities of considerable consequences. Some of the most beautiful purely mathematical developments and, more than incidentally, the mobilization of results from a variety of mathematical fields can be traced back to the study of capillary phenomena and also to the influence of Plateau. The theory of minimal surfaces, the study of surfaces of prescribed mean curvature, as well as the whole complex of questions which mathematicians describe as Plateau's problem, are prominent, but by far not the only examples. Mathematics, in turn, has served its users well although even now substantial problems remain unsolved, posing a challenge which guarantees further fruitful interaction. Without doubt, here is a case where a mathematician need not be ashamed of a lack of "relevance", a term menacingly applied by

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JOSEPH-ANTOINE-FERDINAND PLATEAU

the (often self-appointed and often misunderstanding) custodians of today's society. It is well known, and helpful beyond the realm of frivolity, that portions of the surfaces under consideration, at least as long as these portions remain in stable equilibrium, can often be experimentally realized with soap films or soap bubbles bounded by suitable fixed, or movable, frames. From time immemorial such experiments have been recorded in many sources and have even been depicted in paintings (*e.g.*, by Murillo, v. Oost, Chardin, Hamilton, Spitzweg, Manet). Leonardo da Vinci already knew about capillary phenomena; subsequent attempts aimed at their explanation and at their utilization have occupied great minds ever since. Capillary action is at work in the processes of wetting, displacing, dyeing, coalescing, flotation, emulsification—all of utmost significance in nature and technology. It seems unnecessary to elaborate further on the importance of our subject in a city like Dallas which lies in the heartland of the oil industry, whose research efforts must include the mastery of surface forces. A decrease of the surface tension between water and oil by several orders of magnitude, which is now possible through the injection of newly discovered chemicals, will in fact extend the useful life of the Texas oil fields by several decades.

The organizers of this session, my colleague Dr. Scriven from the University of Minnesota and Dr. Melrose from the Mobile Research and Development Corporation, both distinguished experts in the field and enthusiastic connoisseurs of Plateau, have asked me, a mathematician by profession, to speak about Plateau, to elucidate the influence of his work on mathematical thought and to pursue, in anecdotal form or in mathematical guise, whatever would seem most suitable, a few of the problems raised to their modern habitat in mathematics. Appreciating a mathematician's shortcomings, particularly when communication with scientists in other disciplines is at stake, and charged with—here I quote—"being as intriguing and as off-beat as possible," I am aware of the difficulties of my task. Quite recently I have completed a voluminous monograph on the theory of minimal surfaces which will soon appear as number 199 in the "yellow series" of the Springer Publishing Company. I have thus had many occasions to look into Plateau's treatise. Only in the last weeks, however, as in preparation for this lecture I went through the book again more systematically, did I come to fully appreciate the wealth of the material treated and the complexity of the experiments which Plateau and his disciples conducted with remarkable skill and perception. This experience in conjunction with the recognition of the abundance of problems initiated and of the intricacies inherent to mathematical techniques, which in analysis unfortunately often tend to distract from the essence of a problem and thus mar its conceptual simplicity, made me uncertain to the point that I would have preferred not even to attempt to do justice to our hero and to relate instead some of the interesting particulars of his rather colorful life. Realizing the inappropriateness of such a course, but mindful of the limited time available, I decided to proceed as follows. After briefly reminiscing about Plateau's life and about his general activities, I shall select a few specific

topics from his writings, five or six at most, whose substance will then be explained and whose fate will be discussed. Each case will be concluded with the statement of open problems. It must be kept in mind that the problems chosen are nothing but a small sample. An observation which to me came as a surprise may already be mentioned here. Originally it had been my expectation that my main objective should consist first of my pointing to the forces of mathematical generalization and then of the demonstration that, with suitable safeguards, generalization may lead better than anything to the heart of a mathematical idea and, at the same time, to the definitive solution of a problem. This, of course, is one of the strengths and virtues of the mathematical process, and we shall see it here at work in the molding of the concept of a surface. It turned out, however, that another, equally important, part of Plateau's discussions leads to questions, among them hard problems of geometry and analysis (often referred to as problems of hard analysis), whose formulation today is as classical as it was a hundred years ago and for whose solution new insights are mandatory while a generalization of concepts appears altogether useless. References to the literature will be suppressed*, and names will be mentioned only sparingly. Although our subject, more than other mathematical fields, invites, and greatly gains by, experimental illustrations, I shall be forced to keep such illustrations to a minimum. The few models which I have brought along, pretty as they are, can be looked at outside of my lecture.

2. Plateau was born in Brussels in 1801 and died in 1883, an 82-year-old man. From his father he had inherited artistic skills. Already in elementary school he displayed a lively interest in physical experiments and in mechanical contrivances as well as a love of nature. The boy's chasing of butterflies led to a fine collection of these insects later in life. The misfortune of his losing in quick succession first his mother and then his father and becoming an orphan at age fourteen, made Joseph severely ill. A stay in the country, in a little village near Waterloo, appeared to be

*) The reader interested in references may consult the following sources:

On capillarity and stability: H. Minkowski: Kapillarität. Enc. d. math. Wiss. 5.1.9; Teubner, 1903–21. — N. K. Adam: The physics and chemistry of surfaces. 3rd ed., Oxford Univ. Press, 1941. — K. L. Wolf: Physik und Chemie der Grenzflächen. Springer, Vol. 1, 1957; Vol. 2, 1959. — Further also J. C. Maxwell's article "Capillary action" in the 11th edition (Vol. 5, pp. 256–275) and A. W. Porter's article "Surface tension" in recent editions of the Encyclopaedia Britannica.

On minimal surfaces: R. Courant: Dirichlet's principle, conformal mapping, and minimal surfaces. Interscience, 1950. — J. C. C. Nitsche: Vorlesungen über Minimalflächen. Springer, 1974.

On geometric measure theory: F. J. Almgren: Plateau's problem; an inviation to varifold geometry. Benjamin, 1966. — H. Federer: Geometric measure theory. Springer, 1969. — W. K. Allard: On the first variation of a varifold. Ann. of Math. (2) **95** (1972), 417–491.

On the calculus of variations and Morse theory: O. Bolza: Vorlesungen über Variationsrechnung. Teubner, 1909. — M. Morse and C. Tomkins: The existence of minimal surfaces of general critical types. Ann. of Math. (2) **40** (1939) 443–472; (2) **42** (1941), 331. — M. Shiffman: The Plateau problem for non-relative minima. Ann. of Math. (2) **40** (1939), 834–854. — C. B. Morrey: Multiple integrals in the calculus of variations. Springer, 1966.

beneficial for him in the eyes of his uncle, a lawyer, who became his guardian. By an unforeseen coincidence the journey of Plateau and his two younger sisters fell on the eve of the battle of Waterloo. Together with other villagers they hurried to hide in the woods around Soignies. Young Plateau remained oblivious to the fearful events of those days and nights. While bombardments rumbled all around he enjoyed himself eating country-fried potatoes and catching butterflies. Recovery followed. Upon his return to Brussels his schooling continued with best results. Again there are reports about his experimental skills and interests. In the Atheneum, Plateau impressed Quetelet, a renowned scientist and later secretary of the Royal Academy of Brussels over four decades (1834–1874), and became one of Quetelet's protégés for life. In 1822 Plateau entered the University at Liège. His uncle, full of admiration for the profession of the devoted barrister who defends widows and orphans, would have liked to see him become a lawyer but eventually agreed to let him study natural sciences and mathematics and to cultivate the art of observation. Material needs, among them Plateau's responsibility for his sister, Joséphine, led him to accept a teaching position first in Liège, where he received his doctor's diploma in 1829, and later in Brussels.

Plateau's first scientific interests concerned physiological optics, particularly the study of the sensations produced in the human eye by fixed or moving light sources, a subject in which he made substantial discoveries and which, as a matter of record, led him to the invention of various optical toys. A special experiment, without doubt related to these interests, caused him once (in 1829) to view the sun with his naked eyes for longer than 25 seconds. This proved to be a fateful incident. Painful treatments could not prevent his irrevocable blindness in 1843, which his biographer considers to be a direct consequence of the 1829 experiment. (An ophthalmologist has advised me, however, that the exposure to the sun alone could not have been the cause of a total loss of sight so many years later.) In the meantime Plateau had gotten married. In this connection the following little incident is reported. Visiting Paris on his honeymoon, he seized the opportunity to look up a few of the French professors with whom he had cultivated scientific contacts. As he stayed away from his hotel far longer than anticipated, his wife got extremely worried. Finally, upon his return he admitted to having forgotten that he was married. His loss of sight, of course, opened depressing perspectives. Fortunately, in 1844 Plateau was named ordinary (full) professor, and a royal edict soon to follow relieved him of his teaching duties and of all material worries. Having thus become what today would be called a research professor he could devote all his efforts to his work which now had begun to concentrate on a systematic study of capillary phenomena, an endeavor which culminated in his celebrated book. Plateau's accomplishments and his perseverance in performing delicate experiments under so adverse circumstances deserve our admiration. To be sure, he enjoyed the assistance of members of his family and of dedicated pupils, substantial scholars and later professors themselves, among them Lamarle and Van der Mensbrugghe. (The latter became his

son-in-law in 1871.) Whether the relationship between Plateau and his selfless younger helpers was always untroubled, whether his collaborators ever pursued their joint investigations while gritting their teeth or feeling impeded in their independence and originality, these are questions worth asking.

Much has been passed on concerning Plateau's later years, his family life, and his working and dictating habits. Illuminating as these things might be, they would lead us too far. Thus, before proceeding to the next part of my lecture let me merely state in a summary fashion that Plateau's total scientific activities did cover rather diverse areas, not all equal in importance, and let me mention two instances. According to a popular superstition of the time, the prophet Mohammed's tomb was said to be held in mid-air by the action of strong magnets. Plateau took on this myth and conclusively proved the impossibility of a stable equilibrium for the tomb in any arrangement. For his argument it is essential that the distance between elements appears with exponent -2 in the laws of electromagnetism. Were one to substitute another exponent in these laws, the situation would be different. Experiments were conducted to illustrate this.

The following observation belongs to the realm of mathematical recreation. Given any odd number q, not divisible by 5, say $q = 7$, and any integer d between 1 and 9, say $d = 9$. Then a digital number formed by a suitable repetition of d, that is, a number of the form $ddd \cdots d$, is divisible by q. In our case we find that $999999 = 7 \cdot 142857$. In other words: For a correctly chosen natural number m, here $m = 6$, the number $10^m - 1$ is divisible by q.

3. The normal pressure on an interface, due to the existence of surface tension is, per unit area, equal to

$$p = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \quad \sigma = \text{surface (or interfacial) tension.}$$

Thus the conditions of equilibrium for the boundary surface of a liquid which is free from the influence of gravity are expressed by the equation

$$H \equiv \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \text{const}.$$

Here R_1 and R_2 are the principal curvature radii of the surface and the quantity H is called its mean curvature. If the surface has a representation of the form $z = f(x, y)$, then we have

$$H = \frac{(1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy}}{2[1 + f_x^2 + f_y^2]^{3/2}}.$$

In Figure 1 various types of surface curvature are illustrated.

Plateau's investigations of the effects of capillary forces may well be viewed as grand variations on the theme $H = \text{const}$, which, in the framework of analysis,

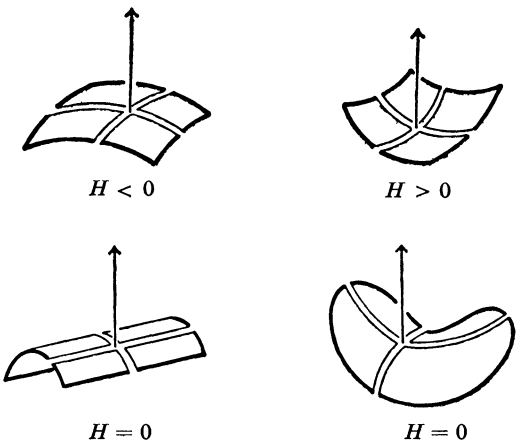


FIG. 1

turns out to be a non-linear second order elliptic partial differential equation, often called the Laplace-Young equation by surface chemists. Taking up a suggestion made by Segner in 1751, Plateau got rid of the effects of gravity in the following way. He formed a mixture of alcohol and water of precisely the same density as olive oil and then introduced a quantity of oil into the mixture. Under the action of surface tension alone the portion of oil assumes the form of a sphere, never any other form. This fact seems to substantiate the mathematical theorem that a closed surface of constant mean curvature must be a sphere. Plateau was well aware of the difficulties connected with this proposition. Only much later, in 1900, Liebmann proved the theorem that every *convex* surface of constant mean curvature is a sphere. A more general earlier attempt of Jellet, who already in 1853 considered starshaped surfaces, seems to have been overlooked by the mathematicians. Neither theorem, of course, answers the question about the shape of a general closed surface of constant mean curvature. Aside from the sphere (of genus $g = 0$) could there be such ring-type surfaces (of genus $g = 1$), or pretzel-type surfaces (of genus $g = 2$) etc.? See Figure 2.

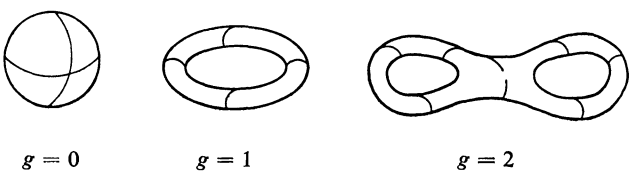
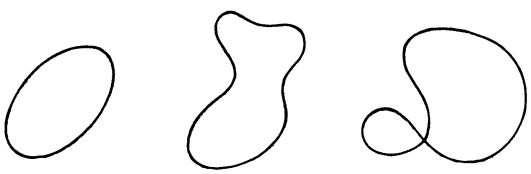


FIG. 2

In physical terms: Are there bodies, different from spheres, in equilibrium under the sole influence of surface forces? Experimental evidence seems to negate this

question. At this point a distinction has to be brought out between physical and mathematical surfaces. A physical surface appears as boundary of a domain, i.e., as the interface separating a quantity of matter from its outside, and thus obviously cannot possess self-intersections. Physical surfaces are also called simple. A mathematical surface, on the other hand, may well intersect itself. The difference is illustrated in Figure 3.



convex surface | physical surface | mathematical surface

FIG. 3

In 1951, H. Hopf proved that a closed (mathematical) surface of genus zero, i.e., a sphere-like surface, having constant mean curvature must in fact be a sphere. A few years later the Russian mathematician A. D. Alexandrov, using ingenious arguments which were often copied in later years, demonstrated that any physical surface of constant mean curvature and of *arbitrary* genus by necessity is a sphere. It takes Alexandrov's proof to answer the physical question. The general question, however, continues to remain open: It is not known today whether there are closed surfaces of constant mean curvature other than the sphere. Here we have one of the outstanding problems of global differential geometry. Most mathematicians do not even try any more to answer it.

Instead of characterizing our surfaces by the constancy of their mean curvature we could have, as we did at the outset of this lecture, described them as surfaces of minimum area. There is indeed a close connection with the isoperimetric problem. In its classical form this problem calls for the determination of a region of prescribed volume whose boundary has minimal surface area. It can be shown that a closed surface has constant mean curvature if, and only if, its area is stationary with respect to volume-preserving variations. For non-simple closed surfaces it is, of course, first necessary to generalize the notion of volume. What has to be done can best be explained on the example of a closed curve in the plane and the area enclosed by it. For an oriented closed curve $\mathscr{C}$ we define the order of a point p not situated on $\mathscr{C}$ as the algebraic number of times $\mathscr{C}$ winds around p. For each point in a connected component R of the complement of $\mathscr{C}$ the order is the same integer, as indicated in Figure 4.

Denoting the various components by R_i, their areas by $|R_i|$ and the corresponding orders by d_i, the area enclosed by $\mathscr{C}$ is now defined as the weighted sum $\sum d_i |R_i|$. The definition of the volume of a closed surface is now evident.

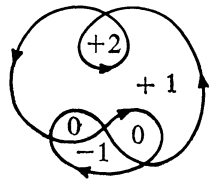
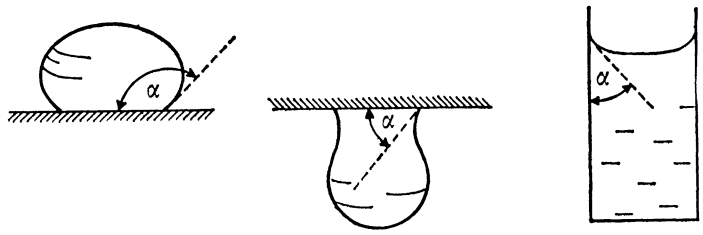


FIG. 4.

4. Leaving the case of equilibrium in the presence of surface forces only, we shall now briefly consider a situation in which external influences, say gravitational forces, are present. Then the mean curvature of the interface under consideration is no longer a constant but rather a (linear) function of the space variables. Typical examples are liquid drops, supported by or suspended from a horizontal plane, liquids in tubes and containers, bubbles, menisci, etc. See Figure 5. The liquid contacts the supporting



sessile drop pendant drop liquid in tubes

FIG. 5

boundary at a certain contact angle α. This angle varies from almost zero for liquids that "wet" the solid boundary to about 140° for mercury on glass. For fats on wool as an example we have $\alpha = 0$, incidentally, a problem for stain removers. Under ideal circumstances the contact angle, however elusive its actual experimental determination may be, can be considered as a material constant.

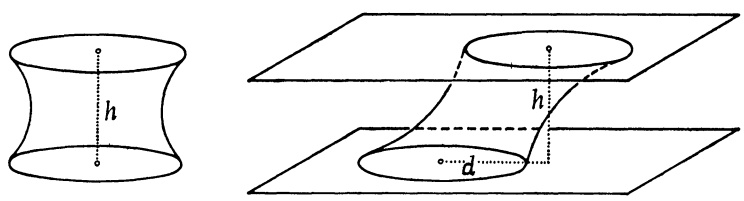
Let us look at the sessile drop. What is its shape? In a coordinate system with vertical z-axis (parallel to the direction of gravitational forces) the boundary of the drop satisfies the differential equation

$$H = az + b$$

in which a and b are certain constants. The solution of this harmless looking elliptic partial differential equation poses formidable difficulties, particularly if one, as he must, tries to satisfy the correct boundary conditions. Were we allowed to assume the drop to be rotationally symmetric the partial differential equation above could be transformed into an ordinary differential equation. Even in order to solve the latter, one has to take refuge in numerical computations. So far, however, it has never been demonstrated that the assumption of rotational symmetry is warranted. Such a

demonstration might be feasible with the help of a symmetrization process. Consider a sessile drop which is not axisymmetric. Replacing each of its horizontal cross sections by a circular slice of equal area in the same height and suitably shifting these slices, a new shape with rotational symmetry could be obtained. Its volume as well as the potential energy of its parts are the same while its surface area has been diminished. For a drop whose total potential (surface and gravitational) energy is already an absolute minimum this would not be possible. To complete the argument, however, one would have to convince oneself that the symmetrization could be performed in a way which would not change the contact angle between the drop and its supporting plane. Trying to avoid energy considerations and a discussion of the physical aspects altogether (large drops may be unstable and disintegrate, etc.), I shall formulate the problem in a purely geometrical form as follows: Let S be a (mathematical) surface of the type of the disk (a precise definition will be given later, in section 7) whose mean curvature at each of its points (x, y, z) is a linear function of the vertical coordinate z and whose boundary meets the plane $z = 0$ at a fixed angle $\alpha, 0 \leqq \alpha \leqq \pi$. Prove that S has rotational symmetry. The special case where the surface is assumed to have a non-parametric representation $z = f(x, y)$ and where $0 \leqq \alpha < \pi/2$ has been settled by J. Serrin.)

5. One of the most celebrated of Plateau's activities is of course connected with what is commonly called Plateau's problem. Imagine the following experiment. Take a frame fashioned of one or several thin wires and dip it into a (suitably prepared) soap solution. Upon withdrawal of the frame a soap film spanning the wires may develop. This film is extremely thin so that the influence of gravitation can be neglected and, although it is actually bounded by two surfaces, it presents the image of an ideal surface. Since the latter is generally open, i.e., encloses no volume, both of its sides are subject to the same pressure and its mean curvature must be zero everywhere. Surfaces of vanishing mean curvature are called minimal surfaces. Our experiment shows one way to realize them physically. As anybody who has ever tried his hand at such experiments knows, it is not every time that a soap film develops in the frame. There can be two different reasons for this failure: either a lack of skill on the part of the experimenter, or, more deeply seated, the mathematical fact that no minimal surface bounded by the curves of the frame exists at all. A classical example will illustrate



a b

FIG. 6

our point. Two coaxial unit circles in parallel planes will bound a ring-type minimal surface, in fact the well-known catenoid, as long as they are not too far apart; more precisely, as long as $h \leqq 1.325 \approx 4/3$. See Figure 6a. If we now move these circles away from each other, either vertically or sideways, there will arrive a moment when the minimal surface tears. It has become unstable. From this moment on our circles are not capable any more of bounding a minimal surface of the type of the circular annulus. That the catenoid disintegrates when the circles are moved apart vertically, a case of rotational symmetry in which the pertinent partial differential equation reduces to a much more amenable ordinary differential equation, has been known for more than a century. The limit value, $h_{\text{lim}} \approx 4/3$, had been determined experimentally by Plateau, and its exact value was computed by Lindelöf. The intuitively obvious fact that the minimal surface tears, however, the circles — or, for this matter, any two boundary contours — are moved apart has only recently been stated and studied by the speaker. For the situation illustrated in Figure 6b a detailed numerical table has been computed relating the limit values of h to the lateral distance d. A graph of this relationship is given in Figure 7. Note that the circles cannot bound a

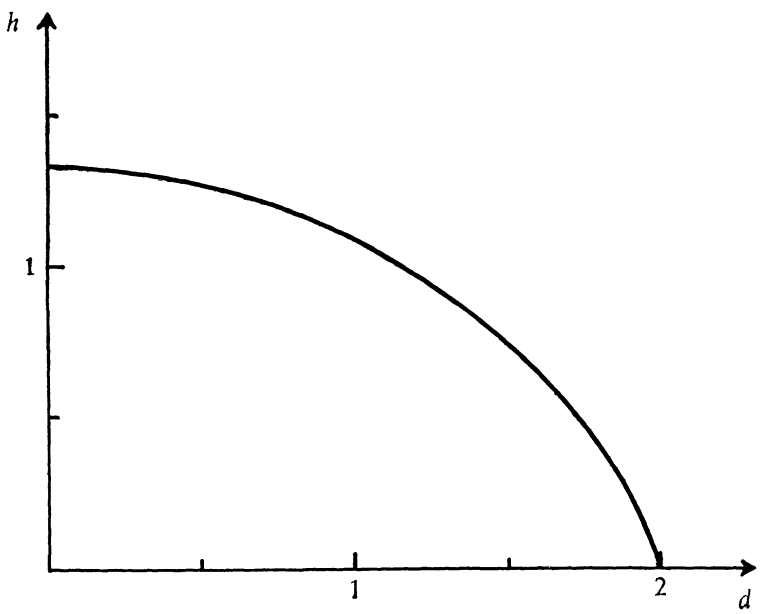


FIG. 7

ring-type minimal surface, no matter how close they are, if their projections do not overlap.

Having been alerted to the disquieting fact that boundary value problems for minimal surfaces may have no solutions at all, let us now mention a favorable case.

A single contour — mathematically speaking, a simple closed curve (or Jordan curve), no matter how bizarre, *always bounds a disk-type minimal surface.* Plateau arrived at this conclusion which he expressed in precise words, on the basis of his elaborate experiments. What he had formulated, of course, was in effect a mathematical statement: that a certain geometric boundary value problem always possesses a solution.

The precise mathematical formulation of "Plateau's problem" is as follows. Given a Jordan curve $\Gamma = \{\boldsymbol{x} = \boldsymbol{y}(\theta);\ 0 \leqq \theta \leqq 2\pi\}$ in Euclidean 3-space. To determine a vector $\boldsymbol{x}(u, v)$ defined in the closure $\bar{P}$ of the unit disk $P = \{u, v;\ u^2 + v^2 < 1\}$ such that

(i) $\boldsymbol{x}(u, v) \in C^2(P) \cap C^0(\bar{P})$;

(ii) $\Delta \boldsymbol{x} = 0,\ \boldsymbol{x}_u^2 = \boldsymbol{x}_v^2,\ \boldsymbol{x}_u \boldsymbol{x}_v = 0$ in P;

(iii) $\boldsymbol{x}(u, v)$ maps the boundary ∂P onto Γ topologically.

Any surface $\{\boldsymbol{x} = \boldsymbol{x}(u,v);\ (u, v) \in \bar{P}\}$ whose position vector satisfies (i), (ii), (iii) is called a solution of Plateau's problem.

If one considers the efforts (and failures) of the leading geometers in providing a rigorous existence proof, efforts which bore fruits only in 1930 through the pioneering work of Douglas and of Radó, one will appreciate the importance of Plateau's observation. It must be mentioned that for a special situation the problem had already been formulated by Lagrange in 1762 (the "birth date" of the theory of minimal surfaces) and that it had been brought again to the attention of the mathematicians by Gergonne in 1816. The only case completely settled for a long time was that of the minimal surface through the sides of a skew quadrilateral whose shape is depicted in Figure 8 and whose equations were derived by H. A. Schwarz in the explicit form

$$x = \operatorname{Re} \int^w (1 - w^2) R(w) dw \qquad w = u + iv$$

$$y = \operatorname{Re} \int^w i(1 + w^2) R(w) dw, \qquad R(w) = \frac{1}{\sqrt{1 - 14w^4 + w^8}}$$

$$z = \operatorname{Re} \int^w 2wR(w) dw.$$

Note that the function $R(w)$ has singularities at the points $w = \pm(\sqrt{3} \pm 1)/\sqrt{2}$, $\pm i(\sqrt{3} \pm 1)/\sqrt{2}$. (The detailed study of the three hyperelliptic integrals and their periodicity properties occupies half of the first volume of Schwarz's *Collected Works.*)

The difficulties in the case of more general, even polygonal, boundaries proved insurmountable for another sixty years. It was necessary first to separate the question of mere existence from the goal of actually determining the solution surface, say, with the help of explicit equations, before progress was possible. Precisely this step, the isolation of the pure existence problem from everything else, has often been cited as a mark of modern mathematics. It can be found in many diverse developments. (Hilbert gained first fame through a comparable step, as he solved Gordan's problem in invariant theory in 1888.)

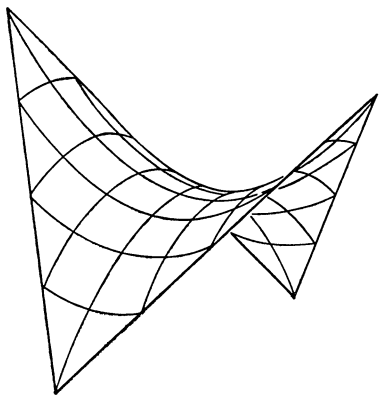


FIG. 8

6. The minimal surfaces which appear as solutions of Plateau's problem possess many special characteristics whose detailed study seemed mandatory. After all, the charge to a mathematician is the same as the charge to the witness in a court of law: To tell the truth, the whole truth and nothing but the truth. Consequently many questions were raised in connection with the attempts to understand all ramifications of Plateau's problem and to fully describe the properties of the solution surfaces, their regularity, their uniqueness or non-uniqueness, their geometrical features and so on. For instance, while the minimal surfaces are analytic in their interior, their precise boundary behavior had been uncertain until five years ago. From the investigations of H. Lewy, the speaker and others we now know that they are, roughly speaking, as regular on their boundary as the Jordan curves which they span. More precisely: If the position vector $y(\theta)$ of the Jordan curve Γ belongs to a certain differentiability class — $C^{m,\alpha}$, or C^{∞}, or C^{ω} — then the vector $x(u,v)$ defining a solution of Plateau's problem is a member of the same differentiability class in the entire closed disk $\bar{P}$.

Another problem, in my opinion currently the most important question connected with the classical Plateau problem, is still open: Can a given contour ever bound infinitely many minimal surfaces, and if not, can one estimate the number of possible solutions in terms of geometric quantities of their boundary?

7. All these questions would lead us too far. Trying rather to concentrate next on one particular aspect in the evolution of Plateau's problem, I shall now make a jump. The existence proofs utilize, in one version or another, the so-called direct methods of the calculus of variations. In order to obtain a surface of least area bounded by a given contour one takes a sequence of comparison surfaces whose areas converge to the minimum value possible. It is then hoped that not only these areas but also the comparison surfaces themselves converge (in some sense) to a limit, hopefully a solution surface. Unfortunately, even in the case of well-behaved comparison surfaces, the most unforeseen things can happen in this limiting process. This fact, among others,

has motivated the mathematicians to reconsider the customary definition of a surface, as well as the objects and quantities related to it (its boundary, its area, etc.). The classical concept of a parameter surface — a definition with which we all (consciously or unconsciously) have become acquainted in college — is often found ill-suited. Such a surface is defined as the mapping into space, effected by a continuous vector, of a domain in the parameter plane. Thus we speak of a surface of the type of the disk if the parameter domain is a disk, a surface of the type of the annulus if the parameter domain is an annulus, etc. Trying to solve Plateau's problem we have as a first step to choose a parameter domain for the comparison surfaces. In doing this we may prejudge the topological type of the solution surface of least area which may well turn out to be different from that of the comparison surfaces. A number of alternate definitions have been suggested designed to avoid this difficulty. All of these definitions are rather abstract. For the purpose of illuminating their crucial ingredient, I shall for a moment digress and turn my attention to a much simpler, yet analogous, concept, that of a function, which has undergone a similar abstraction process.

I am sure we all think we know what a function $y = f(x)$ is: a law which associates with every value of the independent variable x a value of the dependent variable y. We may have heard about objections by logicians and we may be aware of rather unusual "functions," as for instance the δ-function $\delta(x)$ which is zero for all $x \neq 0$ and infinite for $x = 0$. (This "function" has even been differentiated by physicists!) Let me now interject a little anecdote. Assume there are some questions about Mr. X who used to live in Y-city, or Mr. X died, and we want to find out what kind of a man he was. Information is solicited from his former employer, from his neighbors, his former wife, his schoolmates, his girlfriends, his students, the FBI and so forth. Having available all this information we might then be inclined to believe that we know or can determine who and what Mr. X is, or was. For all practical purposes this may in fact be the case. Naturally, I am convinced that there is more to man than this. There may not be more to a function. Instead of these sources of information before (FBI agents, girlfriends, etc.), take now a class of very well-behaved functions $\phi(x)$ — also called test functions (these are infinitely often differentiable functions which vanish outside of a finite interval; we say: functions of class C_0^∞) — and integrate to obtain an expression

$$\int_{-\infty}^{+\infty} f(x)\phi(x)dx \equiv L(\phi).$$

For every test function $\phi(x)$ the integration produces a specific numerical value. In mathematical terms this makes L a functional which is linear since obviously $L(\phi + \psi) = L(\phi) + L(\psi)$. A second important property of L is its continuity (suitably defined) over the space of test functions. It is easy to see that any two reasonable, say, continuous or integrable, functions $f(x)$ and $g(x)$ must be identical if the corresponding functionals are the same, i.e., have the same values for all test functions. Since every unctfion is thus uniquely characterized by the actions upon it of all test functions,

we may actually identify our functions with such functionals. Indeed, forgetting functions altogether, we can now work with these (linear and continuous) functionals regardless of their origin. We then speak of distributions, and our traditional functions simply appear as special distributions, namely, continuous linear functionals which happen to possess a representation utilizing an integral as before. The δ-function is a distribution; it associates to every test function $\phi(x)$ its value $\phi(0)$ at $x = 0$, since formally

$$\int_{-\infty}^{+\infty} \delta(x)\phi(x)dx = \phi(0).$$

In spite of, or rather because of, their generality distributions have many nice properties and are therefore a convenient tool. They can for instance be arbitrarily often differentiated. In view of a formal integration by parts according to which

$$\int_{-\infty}^{+\infty} f'(x)\phi(x)dx = -\int_{-\infty}^{+\infty} f(x)\phi'(x)dx$$

(note that no boundary terms appear, and that $\phi'(x)$ is again a test function), we define the derivative L' of a distribution by the stipulation $L'(\phi) = -L(\phi')$. It is often advantageous to seek the solution to a problem in the framework of distribution theory. Later on it may be possible to prove that the solution actually is a function.

For the purpose of illustrating the last remark let us mention here the simplest example possible. Generalizing the trivial fact that a differentiable function whose derivative vanishes in an interval must be a constant there, we try to characterize all distributions of vanishing derivative L'. In view of the above, this means that $L(\phi') = 0$ for all test functions $\phi(x)$. Denote by $\phi_0(x)$ a fixed test function whose integral is equal to one and consider an arbitrary test function $\phi(x)$. Clearly, the function $\psi(x) = \phi(x) - \lambda\phi_0(x)$, where $\lambda = \int_{-\infty}^{+\infty} \phi(x)dx$, is of class C_0^∞ and $\int_{-\infty}^{+\infty} \psi(x)dx = 0$. Thus, setting $\chi(x) = \int_{-\infty}^{x} \psi(\xi)d\xi$, we see that also $\chi(x)$ is a test function and $\chi'(x) = \psi(x)$. Consequently

$$L(\phi) = L(\lambda\phi_0 + \psi) = \lambda L(\phi_0) + L(\chi') = \lambda L(\phi_0) = \int_{-\infty}^{\infty} c\phi(x)dx.$$

Here $c = L(\phi_0)$. In other words: A distribution L, whose derivative vanishes, is a constant, i.e., L is generated by a constant function.

The preceding remarks about functions will have served to illustrate our point, and we now return to surfaces. Given a regular surface S (rather than a function f as before) we introduce an appropriate class of differential forms Φ (in place of the test functions ϕ before). Each form Φ can be integrated over S leading to a numerical value. Thus we are again encountering a linear functional, and by the same abstraction process we arrive at a class of objects, called currents, which contains our surfaces as special elements but which incorporates more general entities as well. Related to currents are the so-called varifolds. In technical language their definition reads as

follows: "A varifold is a Radon measure on the bundle over an m-dimensional Riemannian manifold M whose fiber at each point p of M is the Grassmann manifold of k-dimensional linear subspaces of the tangent space to M at p." A new powerful branch of mathematics, called geometric measure theory, created by Federer, Fleming, Almgren, Allard, a.o., has scored first successes by attacking Plateau's problem in this general framework.

8. Let us assume that a solution of Plateau's problem has been found in the form of a current or a varifold. One must then set one's sight on obtaining a full description of this structure, primarily on the demonstration that it is really as close to being a classical surface as possible. In the process of this demonstration whose technical intricacies lie beyond the scope of a description here, it turns out that two kinds of points on the solution current must be distinguished, regular points and singular points. In the neighborhood of a regular point the current is manifold-like; that is to say, that part of the current which is contained in a sufficiently small ball about one its regular points is in fact a surface portion in the classical sense. As this regular picture is disturbed near the singular points, any information concerning the size of the "singular set," i.e., the totality of all singular points on the solution structure, becomes of interest. For the simplest case of Plateau's problem (and for two-dimensional surfaces in three-space) the singular set is empty and the solution structure is therefore everywhere regular. Globally it may be quite wild, possessing a number of, even infinitely many, handles, as depicted in Figure 9. Locally, however, it is a

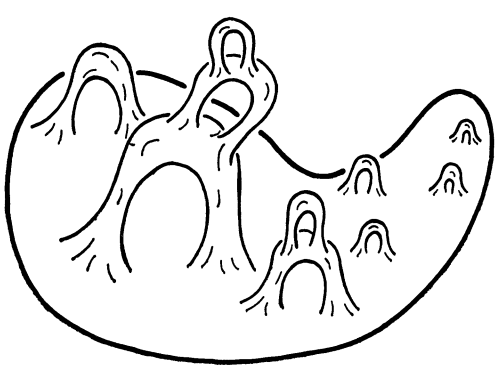


FIG. 9

smooth regular manifold. In more general cases, particularly if we are considering Plateau's problem in its higher dimensional versions, the situation is more complex. It is known that the dimension of the singular set is lower than that of the solution structure. Precisely, how voluminous it can be under the most unfavorable circumstances is an open question today. For structures of codimension one, i.e., for n-dimensional currents in $(n + 1)$-dimensional space a most striking fact was brought out a few years ago through the efforts of a number of mathematicians: If we have

such a current, its singular set will be empty as long as $n \leqq 7$. For $n > 7$ the (so-called Hausdorff) dimension of the singular set cannot be larger than $n - 7$. The appearance of the limit dimension seven is most surprising. Why seven, can be explained by the properties of a certain variational problem, but is still hard to understand intuitively.

More elaborate experiments suggest that the singular set may often possess a rather special shape. For instance, if one dips a frame made up of the edges of a cube in the soap solution one obtains upon withdrawal of the frame a system of thirteen membranes as shown in Figure 10. The singular set is one-dimensional, consisting of

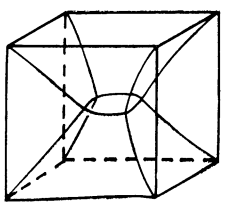


FIG. 10

the branch lines, also called liquid edges, along which several surfaces meet. Plateau's experiments (and the theoretical considerations of Lamarle and others) led him to the following conclusion which has been theoretically substantiated in concrete cases: In a stable configuration it is not possible that more than three membranes come together along a branch line; and if there are three, they meet mutually at equal angles of 120°. As far as the branch lines themselves are concerned, at most four can issue from a point where they then mutually intersect each other at equal angles of 109.47°. Mathematical existence proofs for surface systems similar to those depicted in Figure 10 as well as the investigations of the nature and the regularity of the branch lines are still in their infancy. First progress has been made in the 1972 Princeton dissertation of J. Taylor.

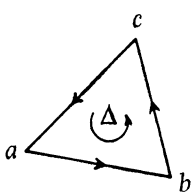
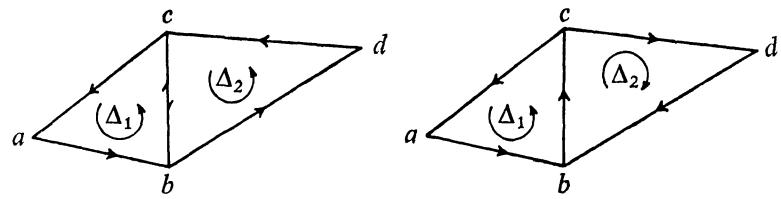


FIG. 11

9. A few words might be in order to explain the methods dealing with such surface systems. In studying manifolds it is a customary procedure of algebraic topology to triangulate, i.e., to consider a manifold as made up of triangles. Each triangle is given an orientation. (In Figure 11 we have employed arrows for this purpose.) An in-

dividual oriented triangle Δ with vertices a, b, c as in Figure 11 is assigned as boundary $\partial\Delta$ the sum $ab + bc + ca$ of oriented segments. Here we use the convention $ab = -ba$, so that $ab + ba = 0$. The boundary of a collection of triangles is obtained by addition, as illustrated in Figure 12, where two choices of orientations are depicted. Depending on the orientations chosen, the boundary of the complex of triangles has different forms. Actually the segment bc, being an interior segment of the



$\partial(\Delta_1 + \Delta_2) = ab + bd + dc + ca$ $\qquad$ $\partial(\Delta_1 + \Delta_2) = ab - bd - dc + ca + 2bc$
$\equiv ab + bd + dc + ca \pmod 2$

FIG. 12

quadrilateral triangulated in Figure 12, should not appear in the boundary expression at all. This can be arranged, even for the "wrong" case, if we agree to reduce every integer coefficient of a segment "modulo two," i.e., if we replace every integer by the remainder it leaves when divided by two. Thus

$$\cdots, -1 \equiv 1 \pmod 2,\ 0 \equiv 0 \pmod 2,\ 1 \equiv 1 \pmod 2,\ 2 \equiv 0 \pmod 2, \cdots$$

and $-bd \equiv bd \pmod 2$, $-dc \equiv dc \pmod 2$, $2bc \equiv 0 \pmod 2$. Then, regardless of the orientations chosen for the individual triangles, the boundaries are the same modulo two.

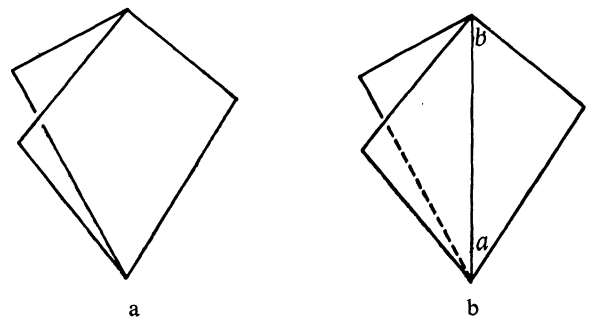


FIG. 13

Let us now consider a frame of three wires as shown in Figure 13a. From soap film experiments we know that this frame spans a system of three surfaces which meet along a branch line connecting the common end points a and b of the arcs as schematically sketched in Figure 13b. In Figure 14 in which the segment ab appears with three specimens (actually to be identified) an orientation is suggested for the triangulation of the system.

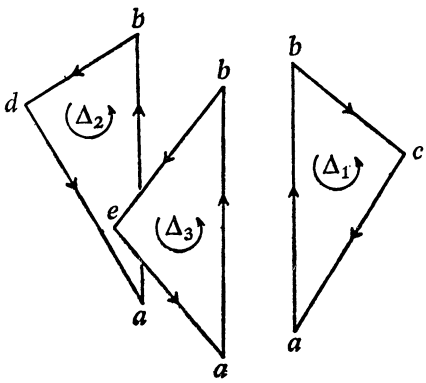


FIG. 14

From

$$\partial\Delta_1 = ab + bc + ca$$
$$\partial\Delta_2 = ab + bd + da$$
$$\partial\Delta_3 = ab + be + ea$$

we find that

$$\partial(\Delta_1 + \Delta_2 + \Delta_3) = 3ab + bc + \cdots + ea.$$

Thus the segment ab appears in the expression for the boundary of the surface system. In our concrete problem, however, ab should be an interior segment of the surface system and the boundary should consist exclusively of the wires of the frame, i.e., of all the segments with the exception of the segment ab. This can be achieved if we this time work modulo three. Since $3 \equiv 0 \pmod 3$, the boundary of our surface system becomes

$$\partial(\Delta_1 + \Delta_2 + \Delta_3) \equiv bc + ca + bd + da + be + ea \pmod 3,$$

which is precisely the desired expression.

These remarks may suffice to indicate how one, working modulo a suitable integer, can adapt the solution procedure to various concrete problems. Whether for a given frame the determination of currents of absolutely least area, but modulo different integers, can lead to different solutions (different also in area), is a question which has not yet been fully answered in all cases.

10. In conclusion of the preceding remarks concerning the modern approaches to the solution of Plateau's problem it must be pointed out that these approaches until now have succeeded only in the determination of those solution structures whose area represents the absolute minimum among the areas of all competing structures

The existence proofs for other solutions, those whose area is a relative minimum only and, in particular, altogether unstable solutions, are more elusive. Here the powerful topological theory, named after its founder Marston Morse, has been applied with success only in the more classical setting of the 1930's. Interesting theorems assuring the existence of unstable minimal surfaces which span a given frame have been proved in 1939 by Morse and Tompkins and by Shiffman. (Peculiarly enough, the subject has not yet been taken up again since then.)

In order to explain these theorems it has to be understood that a surface bounded by the frame or, more precisely, the position vector of such a surface, can be regarded as an element of a certain general function space and that the integral for the area then becomes a functional defined on all the elements of this space. Minimal surfaces are characterized by the property that their area is stationary, i.e., for them the first variation of the area functional vanishes. In this sense minimal surfaces are the critical points for the area functional. It is true that our functional is merely semi-continuous but not continuous, and this fact is the cause of considerable difficulties.

For the purpose of giving a simple illustration, however, let us now instead of position vectors in a function space consider points (x, y) in the Cartesian plane and let us focus our attention on a continuous, better still, a sufficiently often differentiable, function $f(x, y)$ rather than on a (semi-continuous) functional. Stationary, or critical, points for $f(x, y)$ are those points (x, y) in which the first partial derivatives $f_x = \partial f/\partial x$ and $f_y = \partial f/\partial y$ are zero. Critical points describe minima and maxima for $f(x, y)$, but also saddle points and ridges. In Figure 15 a topographical map is reproduced showing

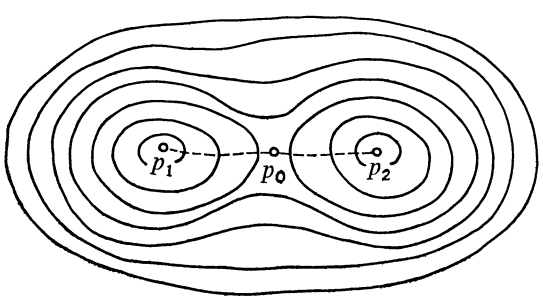


FIG. 15

level lines which may be thought of as the loci of constancy for a function $z = f(x, y)$. We observe that there are valleys, or pits, at p_1 and p_2. In these points our function possesses absolute minima so that there $f_{xx} > 0$, $f_{xx}f_{yy} - f_{xy}^2 > 0$. Searching for further stationary points of $f(x, y)$ let us walk from p_1 to p_2 along a certain path, and let us mark the point of highest elevation which we have to pass on our journey. In the attempt to minimize our efforts we will, of course, choose a path on which the highest elevation is as low as feasible. Such a path is shown as a dotted line in Figure 15. It is intuitively clear, and can also be proved mathematically, that the point p_0 of

highest elevation on *this* path is a stationary point for our function, in the case at hand, a saddle point.

The experience gained from the foregoing example motivates the theorems about minimal surfaces alluded to earlier. A typical statement is the following: Assume that a given contour Γ bounds two minimal surfaces, and that the areas of both represent strict relative minima. Then Γ must still bound a third, generally unstable, minimal surface. The proof for this statement is highly technical.

The degree of sophistication reached by the abstract Morse theory has so far not been fully transferred to the concrete, but difficult, application which Plateau's problem represents. A classification of minimal surfaces according to their critical types as well as an estimate of their number are still out of reach today.

11. The end of my lecture is in sight. My attempts to illustrate by examples the influences of Plateau's work on mathematical developments could at best provide a few glimpses. There are so many mathematical subjects of current interest which should have been, but could not be, discussed:

Minimal surfaces bounded by a flexible, but inextensible, string (the string will assume the shape of a curve of constant space curvature) —

Surfaces of least area which are forced to lie on one side of a fixed obstacle —

Free boundary value problems where the boundary of a solution surface is required to lie on a given manifold —

The intimate relations to the calculus of variations and to the theory of partial differential equations —

The still incomplete existence and non-existence problems for surfaces of prescribed (but variable and not vanishing) mean curvature —

The behavior of interfaces under varying gravity conditions —

Investigations concerning the oscillations of liquid masses —

to mention but a few. I should also have given an account of the powerful numerical methods which were created in connection with the attempts to solve the Young-Laplace equation, but which, far transcending their original goal, have led to new developments in the whole area of ordinary differential equations. By a fine coincidence the basic theoretical and numerical study of Bashforth and Adams regarding the shape of liquid drops was published in 1883, the year Plateau died. With the advent of the computer a classification of all axisymmetric solutions of the Young-Laplace equation has become feasible. Surveying the print-outs Huhs and Scriven as well as Concus and Finn discovered that the (non-parametric) equation of the pendant drop

$$H \equiv \frac{(1 + z_y^2)z_{xx} - 2z_x z_y z_{xy} + (1 + z_x^2)z_{yy}}{2(\sqrt{1 + z_x^2 + z_y^2})^3} = az + b$$

possesses one particular solution $z = z(x, y)$ with an isolated singularity, describing an (unstable) equilibrium configuration, a "pendant spike." Subsequently Finn and Concus have substantiated this interesting fact by a mathematical proof.

12. It seems only fitting, however, that I now conclude this lecture with a few remarks concerning the question of stability to which Plateau and others have devoted a considerable amount of experimental ingenuity and theoretical attention. In practical importance this question may well surpass that of the other problems mentioned. In his experiments with liquid cylinders of variable length, Plateau observed the phenomenon of instability. Instability occurs as soon as the length l of the cylinder exceeds its circumference $2\pi r$. Then even the smallest perturbations, caused by vibrations, convection currents, an electric field, or other unavoidable disturbances, lead to ever increasing bulgings and contractions. The cylinder changes into intermediate unduloidal shapes and finally disintegrates into a sequence of disconnected spheres appearing in a rather regular arrangement. The speed of this decomposition process depends on the viscosity of the liquid and may be very slow. The process itself can be observed in many disparate situations — in liquid jets, in spider threads, in the melting of an electrically overheated wire, etc. Plateau gave several derivations of the stability limit $l = 2\pi r$; none can be accepted as rigorous. A mathematical approach to the question of (mechanical) stability proceeds as follows. One subjects the cylindrical shape to small, but volume preserving, distortions and studies the change of the surface area under these distortions. If any one particular perturbation is capable of decreasing the surface area, then the cylinder cannot be in stable equilibrium. Strictly speaking, we have here a variational problem with subsidiary conditions. When we consider that the first variation of this problem always vanishes for a figure in equilibrium, we are led to a study of its "second variation." There are close ties between the theory of the second variation and the eigenvalue problem for a second order elliptic partial differential equation. This was first observed by H. A. Schwarz in 1885, who showed in his pioneering investigation of a concrete problem that the size of the smallest eigenvalue is crucial for the determination of stability limits. An extraordinary development in the calculus of variations, which cannot be discussed here, has taken place since.

While the question of stability for a liquid cylinder in equilibrium has in the meantime become an elementary exercise, it can still serve to illustrate one point, namely, the specification of physically realistic boundary conditions if the contact line is permitted to move. Particularly in view of problems of a more general nature, this specification deserves a careful examination. Consider a circular cylinder

$$\Sigma = \{x = u,\ y = \cos v,\ z = \sin v;\ 0 \leqq u \leqq l,\ 0 \leqq v \leqq 2\pi\}$$

of radius one and length l which is suspended between the vertical planes $x = 0$ and $x = l$. A small perturbation leads to the distortion of Σ into a surface

$$\Sigma' = \{x = u,\ y = [1 + \zeta(u,v)]\cos v,\ z = [1 + \zeta(u,v)]\sin v;\ 0 \leqq u \leqq l,\ 0 \leqq v \leqq 2\pi\}.$$

Here $\zeta(u,v)$ denotes a function which together with its first derivatives is assumed to be sufficiently small, so that $|\zeta| < \varepsilon$, $|\zeta_u| < \varepsilon$, $|\zeta_v| < \varepsilon$. The perturbation will be

volume preserving, if it satisfies the condition

$$2\int_0^l \int_0^{2\pi} \zeta(u,v)dudv + \int_0^l \int_0^{2\pi} \zeta^2(u,v)dudv = 0.$$

The area $A(\Sigma)$ of the lateral surface of Σ is equal to $2\pi l$, while the corresponding area of Σ' comes to

$$\begin{aligned} A(\Sigma') &= A(\Sigma) + \tfrac{1}{2}\int_0^l \int_0^{2\pi} [\zeta_u^2 + \zeta_v^2 + \zeta^2]dudv + O(\varepsilon^3) \\ &= A(\Sigma) + \tfrac{1}{2}\int_0^{2\pi} [\zeta(l,v)\zeta_u(l,v) - \zeta(0,v)\zeta_u(0,v)]dv \\ &\quad + \tfrac{1}{2}\int_0^l \int_0^{2\pi} [\Delta\zeta + \zeta]\zeta\, dudv + O(\varepsilon^3). \end{aligned}$$

In this formula the volume constraint is incorporated, and the expression $O(\varepsilon^3)$ embraces all terms whose order of smallness is ε^3 at least. The boundary conditions normally considered are of one of the forms

(i) $\zeta(0,v) = \zeta(l,v) = 0$,

(ii) $\zeta_u(0,v) = \zeta_u(l,v) = 0$,

(iii) $\zeta(0,v) = \zeta(l,v)$, $\zeta_u(0,v) = \zeta_u(l,v)$.

All three cause the disappearance of the single integral in the second expression for $A(\Sigma')$. The first set of boundary conditions corresponds to the case of a cylinder with fixed end circles, an experimentally realizable situation. The third set is appropriate for a configuration with periodicity features. It is the second choice of boundary conditions, expressing a transversality property, which appears to be in need of some interpretation. Visualize an experiment in which our cylinder Σ is freely suspended between vertical plates situated in the planes $x = 0$ and $x = l$. We assume, of course, that the interfacial tensions of the materials involved allow for a contact angle of 90° at the plates in the first place. A perturbation, described by a function $\zeta(u,v)$ as above, will distort the originally circular curves in which the lateral surface of Σ meets the planes $x = 0$ and $x = l$. Owing to the hysteresis which is experimentally observed in connection with the contact line between media, the boundary conditions $\zeta_u(0,v) = \zeta_u(l,v) = 0$, according to which the lateral surface of Σ' intersects the bounding plates at the angle of 90° also, and which thus are based on the assumption of an instantaneous reaction to disturbances of the contact line, merely represent a (more or less justified) idealization. It would be more realistic to take into account the course of the hysteresis as well as the time dependence of the disturbance. The corresponding physical law, if it would be established, might then impose a dependence between ζ and ζ_u, in the simplest case possibly a linear relationship of the form

(iv) $\zeta_u(0,v) = a\zeta(0,v)$, $\zeta_u(l,v) = -a\zeta(l,v)$.

The search for suitable boundary conditions and the determination of stability limits for equilibrium configurations subject to these conditions — in the case of a cylinder

as well as in more general cases (drops, menisci, etc.) — would seem to be of physical and mathematical interest.

For a number of equilibrium configurations, among them planes, spheres, cylinders as well as unduloids and nodoids, whose stability properties have also been extensively discussed by Plateau, the equations of the undisturbed surfaces are known — a fact which allows the explicit use of surface coordinates. Often, particularly in more complex arrangements in which the influence of gravity cannot be neglected (a special example is briefly mentioned by Plateau in §424 of his book), the investigator is less fortunate. Here the energy function to be minimized consists of two parts stemming from the surface tension and from the gravitational forces, and the shape of equilibrium figures cannot generally be expressed with the help of explicit equations. Scriven and his collaborators, Huh and Pujado, have employed the following approach. In conjunction with the stepwise numerical generation of the (axisymmetric) shape of a liquid drop or meniscus, they computationally test at every step the validity of the Jacobi condition which, in controlling the occurrence of a conjugate point, constitutes a well-known criterion for stability. For a one-parameter family of pendant drops of increasing volume they find that an axisymmetric drop with a fixed, or freely movable, contact line cannot be stable, i.e., cannot any more be suspended from a horizontal plane and rather disintegrates, if its meridional profile contains a point of inflexion. The crucial role of an inflexion point, which will always be present in a sufficiently voluminous drop, has been suspected for a long time. It would be worthwhile to develop a purely mathematical proof of this result which only relies on the fact that the meridional profile of the drop satisfies a certain non-linear differential equation as well as suitable boundary conditions, but which does not require the (unattainable) knowledge of this profile in an explicit form.

[**Added in proof, August 1, 1974:** Such an analysis has now been carried out by E. Pitts (J. Fluid Mech., 63 (1974) 487–508). The axisymmetric drop is stable (i.e., the second variation of the energy function remains positive) as long as its volume increases with its height. The occurrence of an inflexion point seems to have no bearing on the situation.]

What has been said here, of course, is not intended to provide a full picture of the stability problem. Commenting on the general investigations concerned with the determination of stability limits for liquid masses in equilibrium, Plateau states: "These investigations appear to me not devoid of interest, even from the purely mathematical point of view. They will probably present very great difficulties, and I shall leave the trouble of carrying them out to the geometers." Now, a hundred years later, many problems have been settled, while others still wait for their solution; but all are important.

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