attributed solely or even primarily to discrimination. Instead, the cultural conditioning from early childhood through post-Ph.D. seems to be the chief factor operating against the potential woman mathematician. How to succeed in spite of this and how to change the conditioning to ease the path for our successors is the task to which at least some are now addressing themselves.

While the issues I have mentioned are the shared concerns of many mathematicians, men and women, the opinions and impressions are my own; they do not represent the views of the panel, the MAA or any other organization.

HISTORY IN THE MATHEMATICS CURRICULUM: ITS STATUS, QUALITY, AND FUNCTION

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1. Status and quality of history. In words of that great American patriot, who contributed so magnificently to the pollution of our highways and the air we breath, viz., the late Henry Ford the First, "History is Bunk!" (Actually he said "History is more or less bunk," but like many quotations the abbreviated form is considered an improvement.) Similar sentiments were expressed by Napoleon, who characterized history as "a fraud," and Matthew Arnold, who termed it "That huge Mississippi of falsehood called history."

To judge from the present status of history in the mathematics curriculum, one might conclude that mathematicians feel much the same about history. As a rapid check, I selected 7 state institutions ranging from one of the largest universities to a small college, and 4 eminent private universities. From a search of their catalogs

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I determined that among the 7 state institutions, 3 have history of science departments, but no course in the history of mathematics was listed in any of these or in any of the 7 mathematics departments. Of the other 4, one listed a quarter course on the junior level ("up to the advent of calculus"), another listed a 1-semester course for teachers, another a 1-semester course in the history of elementary mathematics, and the fourth a quarter course covering material up to the 17th century and "selected topics from more recent mathematical history." Not a single one mentioned any history of modern mathematics, other than the "selected topics" cited.

After the program for this meeting was mailed out, I was pleased to receive a letter from Professor Arthur Hallerberg of Valparaiso University containing the results of a questionnaire which he had mailed out last year to 143 institutions having well-known mathematics departments. Of the 83 who replied, 41 offer no course in history of mathematics; of the rest, none requires it of mathematics majors, although 8 do require it of teaching majors. I wish I had time to include more of his results.

Further evidence may be found in the history content of the MONTHLY. By 1957, the "History" classification in the annual index had disappeared (actually it had not appeared in 1955, but was used for exactly one paper in 1956). With volume 76, 1969, under the new editorship of Harley Flanders, the index was expanded to 16 classifications. None of these is "History", which, presumably because of its rarity, appears in the classification called "General."

Now I doubt that the reason for this situation is to be found in the nature of history itself. Several reasons may be offered which seem to be more credible. In the first place, during the decline of history, mathematics itself has been undergoing rapid acceleration — some have termed it a "Golden Age." And it seems plausible that during a period of expansion in mathematical theory, interest in historical research should wane. Why bother with the past, when the future beckons so enticingly?

But I am sure this is not the whole story. To speak frankly, I have detected a current of disparagement, bordering on scorn, among research mathematicians, indicating that historical research has somewhere along the way fallen into disrepute. During the first third of the present century, it was not unheard of for a mathematics department to award a Ph.D. in history. Today the candidate for a degree in history is likely to be shunted into either the school of education, or into the department of the history of science. And department chairmen are notoriously unwilling to hire doctorates from other departments, thus adding to the unwillingness of the research-capable young man to follow up any possible interest in history.

Yet how is the student to develop an interest in history if no substantial courses are offered in it? And here is another possible clue to the decline, namely, that the courses originally taught were mostly in the history of elementary mathematics, with only brief, if any, incursions into modern history. I can't refrain from quoting from an article by the late E. T. Bell entitled "Possible projects in the history of mathematics" published in *Scripta Mathematica* in 1945 — over a quarter century ago:

"Some of the dreariest ramblings ever endured in university lecture rooms by bored students earning a fairly easy credit, were those perpetrated in the name of scientific history a generation ago by professed historians of mathematics. These well-meaning and unimaginative men transferred to history the pseudoscientific fatuity of accuracy to the sixth decimal long after a rapid succession of basic new discoveries had outmoded profitless meticulosity in science. Interest they reprobated as a vice and pedantry they lauded as a virtue, all with the supposed sanction of the scientific method, of which they were congenitally incapable of understanding anything. Their drab lectures appear to have had an unintended but predictable effect.

"Inspection of recent and current catalogues shows that the fraction of colleges, universities and teacher-training schools offering a course in the history of mathematics is negligible." (Recall that this was in 1945.)

It is clear where Bell placed the blame. And whether he was right or not I feel that, after allowing for Bell's penchant for exaggeration, he hit close to the truth.

In view of the already crowded condition of the mathematics curriculum, I know that no amount of expostulation and entreaty on my part would overcome the present apathy concerning the history of mathematics. I am convinced that only two things can enable history of mathematics to compete for a place in the curriculum. These are, first, to devise courses which will not only attract the student but be of intrinsic value to his future; and, secondly, to find a way of rejuvenating the history of mathematics so that the excitement of doing research in it will be just as great and rewarding as in mathematical research proper.

2. Function of history in the curriculum. But let me pause a moment to consider a question which I am sure some of you may be asking at this point, viz., "Why should there be more attention paid to history of mathematics? Maybe the situation is as it should be, considering how crowded the curriculum is and how difficult it is to give our students what they need for either a baccalaureate major in mathematics, or even the Ph.D. More explicitly, what function can history serve under these circumstances?"

Before I attempt to answer this, let me interpolate that if anyone had told me 30 or 40 years ago that I would one day be making a plea for history before the MAA, I would have replied "Impossible!" And my doing so is absolutely not the result of my looking around for a worthwile cause to support, or a possible title for an MAA address, but rather the attempt to answer criticisms which I have been hearing students make for years. These indicated to me there was something wrong with the present system — something which is apparently being aggravated by the recent upsurge of mathematical research. I refer to complaints from students that the various courses we are offering them are too self-centered, and that their teachers were making no effort to interrelate these courses. And they were asking, how *do* all these specialities relate to one another, and what significance do they individually have for the bulk of mathematics? Where is it all going, anyway?

A couple of years ago I participated in a CUPM panel which conducted interviews with a fairly representative group of mathematics majors, some of whom had already graduated. I was impressed by the fact that these same evidences of frustration repeatedly occurred in the criticisms made by these students.

I have often observed, too, that among some of the most capable, research-wise, of new Ph.D.'s, can often be found the greatest lack of knowledge concerning the background and significance of their work, as well as abysmal ignorance of the reasons for doing it and of the general nature of mathematics. In short, they are uneducated specialists. If you ask them why they are specialists, the best reason they can give is that this is the way to get results which merit publication and hence a good job.

Now of course they are right, and I am not one to decry specialization. In this modern day and age, we are all specialists of one sort or another. But I don't believe that courses in English literature, philosophy, or other so-called "humanities" which are commonly advocated for "broadening out" the specialist, are the answer here; their effects are too often soon smothered by the rapidly increasing burden of facts and details demanded by one's specialty. What is needed is something really germane to one's interest, which he won't forget because it really complements his interest, and which will actually be capable of serving both a humanistic purpose and a mathematical one. It should not only broaden one's outlook, showing the place of mathematics in one's culture, but it should inform him where his specialty fits into the general scheme of mathematics, how it arose in the first place, and give him a means of judging where it is likely to go. What I have in mind is the kind of knowledge about mathematics that will enable one to detect gaps where new concepts are needed; spot broad areas where new structures would provide unification and consolidation of seemingly diverse concepts; and recognize when a field has borne nearly all the mathematical fruit of which it is capable, so that it needs either to be rejuvenated by fertilization with ideas from other branches of mathematics, or possibly abandoned if its benefits to other fields are nil. The student should understand how and why the introduction of new conceptual materials may lead to the solution of long outstanding problems, as well as that once these materials are available, several working independently of one another will probably get the solution, and that he shouldn't blame himself if he was one of these and was preceded in publication. No doubt much of this kind of knowledge and perspective is acquired by experience and increasing mathematical maturity, although even in such cases I suspect that much of it is only intuitive.

3. Teaching of history. I have been pondering this situation for many years, and it is my firm conviction that the history of mathematics, when suitably conceived and adapted to the needs of our students, is precisely what is needed by many of the mathematical illiterates who pass through our departments. Now please let me make clear that I am not setting myself up as an authority on history; I am not. But the teaching of history is something with which I think all mathema-

ticians have a right to be concerned. And there are signs that this is happening. I have found out, during the past few months, that several mathematicians of excellent reputation, none of whom is a professional historian, are experimenting with history courses. One of these expressed to me the opinion that history of mathematics is "an idea whose time has come."

If this is the case, then it can be expected that there will be new workers in the field contributing their ideas regarding how it should be modernized. And I hope that my own remarks will be received in this light, viz., as a desire to contribute to the development of a history course that will perform the functions I have just mentioned.

Perhaps others share with me the feeling that we mathematicians have failed to consider the possibility of applying, to history, methods which have been so successful in the body of mathematics, viz., adopting a structural point of view. This has made it possible to consolidate and cross-fertilize seemingly unrelated parts of mathematics, thereby bringing them into a more manageable focus. Historians who must be contemplating with dismay the problem of recording all the developments of the 19th and 20th centuries might take a leaf from their mathematical colleagues' notebooks and consider whether a similar remedy might work for history.

But, you may ask, history is made by human beings, and how are you going to treat human beings by introducing higher level abstractions as we have done in mathematics? If we restrict ourselves to biographical, chronological and anecdotal details, I agree that we cannot. But if instead we treat the history of mathematics as a flow of concepts and ideas in the large, then we already raise it to the level of higher abstraction. Moreover this might make feasible the coordination and patterning of historical events in a manner quite similar to that employed in mathematics proper—but adapted to the historical point of view.

4. Cultural history. Actually, the standpoint from which I believe we should present the history of mathematics is at an even higher level than mathematics. By this I mean, to take a broad view of mathematics as a living, growing organism which is continually undergoing evolution; in short, we should study it as a culture. Only two months ago I came across a little book [**1**] embodying 3 lectures given in 1956 by Harry Shapiro, an anthropologist of the American Museum of Natural History. One of these lectures (the second) was devoted to the contributions which he thought the modern discovery of culture could make to historical research and writing. Although admitting that historians "have become increasingly aware of culture content," he deplored the fact that few (if any) historians "exhibit any familiarity whatever in their writing with principles that anthropologists have been able to extract from cultural data." His remarks were accompanied by examples from both Irish and American history.

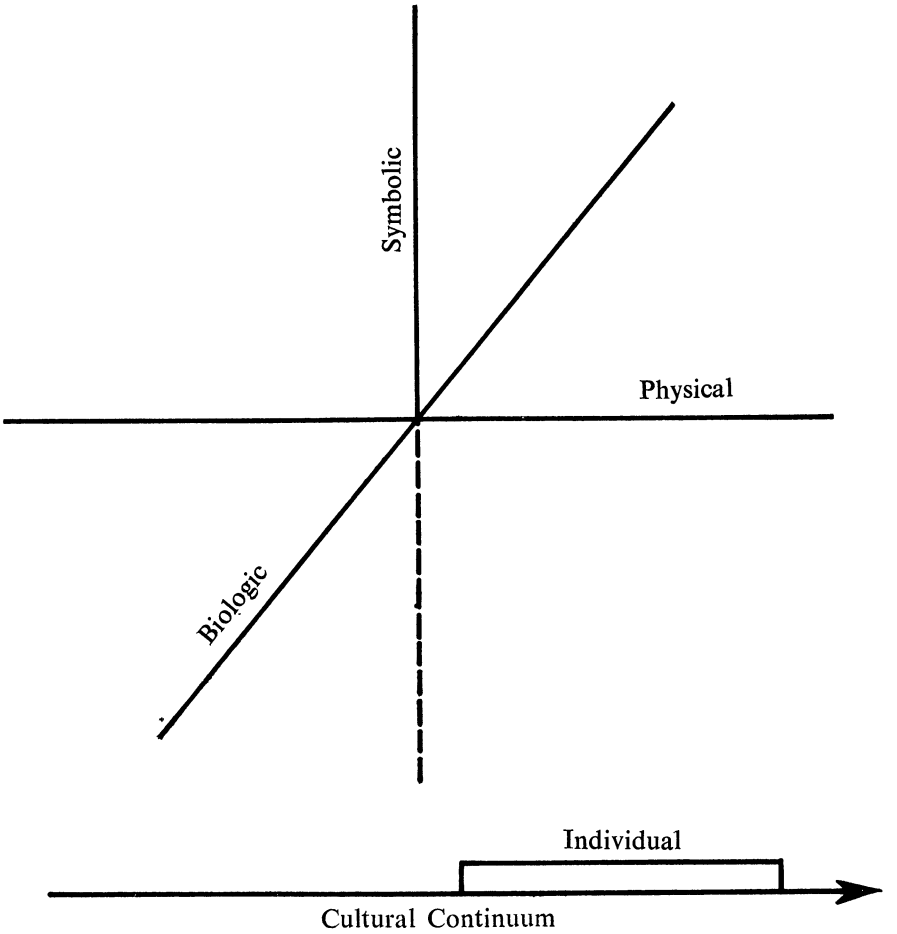
However, we cannot expect our students will have taken a course in cultural anthropology. In order to overcome this handicap, I have tried to devise a suitable substitute especially adapted to the point of view of the mathematician. This involves

making clear what is meant by a *symbol*. This is necessary since most mathematicians use the word "symbol" in a special sense, namely in the sense of so-called "mathematical symbol" or, in mathematical logic, "logical symbol." This, I have learned, has caused me to be gravely misunderstood heretofore, so I don't intend to make the same mistake now. For instance, I have been suspected of exaggerating the importance of "symboling" in view of the "glorious nonsymbolic achievements of Greek geometry" and "the Arabic development of a rhetorical algebra" [**2**]. But both Greek geometry and Arabic algebra were decidedly symbolic. My critic, a well-known historian but also a mathematician, was naturally taking it for granted that "symbol" meant "mathematical symbol" in the narrow sense.

The usual dictionary definition defines "symbol" as "something that stands for something else;" and this really sums up the matter in a nutshell. If I say the word "air" you probably think instantly of something you breathe, unless, of course, you think of one who is the beneficiary of an estate. At any rate, the word "air" stands for something else and hence is a symbol. Most words are symbols. But symbols don't have to be words; they can be traffic lights, geometric figures, finger and hand positions used by the deaf and dumb, or "peace" symbols for instance. Advertisers employ words, designs and pictures which they repeat over and over by radio, TV, print, and other forms of display, with the aim of creating symbols which will automatically pop into our minds whenever we want the sort of articles they offer for sale. "Snap, crackle, pop" is a symbol for a certain brand of cereal. It is no exaggeration to say that we are saturated by symbols. That we mathematicians customarily think of "symbols" in the narrow sense in which *we* use the term, is in itself an indication of how specialized we have become in our thinking.

Once we have learned what a symbol stands for, we usually develop a "habit" attitude toward it. An experienced driver habitually stops his car when he comes to a red light or a "STOP" sign; it isn't necessary for him to pause to inquire the meanings of these symbols. Indeed, for many symbols we get into the habit of treating them as though they were identical with their meanings — which leads to great efficiency but can be dangerous sometimes. In such a context they function only as *signs*. Animals other than man understand and react to signs. But they cannot, apparently, create symbols. To create a symbol, or as I shall say, *to symbol* (see [**3**]), one must be able to assign to some combination of sounds, events, structure, or other thing capable of being perceived, a *meaning*. We can teach a dog to follow closely at our heels on the command "heel!" But it is we, as humans, who invented this signal; the dog did not invent it and to him it is only a sign to be reacted to in a fashion to which he has been trained. Similar remarks can be made about chimps who professedly "count" up to 7. The experimenter assigns the meanings to the lights or colors which serve as the symbols, not the chimp. To use a biological term, the ability to symbol is *species-specific* (see [**4**], for instance), and can be used to distinguish humans from other animals; it is a necessary and sufficient condition for being a member of the species *homo sapiens*.

Diagram No 1



To teach the mathematics student the meaning of the word "culture," we can now proceed as follows: Consider diagram number 1. Aping Euclid, this is supposed to represent the world in which we live—only now it is the world of culture. Euclid's 3-dimensional world has been compressed into the one axis, labelled "Physical." Everything, living or not, has a physical form, but if a living thing, it has a place in the biological realm and is not confined to the 1-dimensional in this scheme, but has another degree of freedom in the plane of biological forms. But when we, as human beings, use our faculty of symboling in order to conceptualize, we then are enabled to enter a new dimension, not accessible to other life forms; this is the world of culture. Without symbols we could not enter it. The world in which we live is compounded of tools and technology, rituals and beliefs, architecture and the arts, literature and the sciences — including, of course, mathematics. All of these are based on symbols,

without which we would have no words to communicate, or with which to hand on to our progeny the vast conceptual world that we have created — a world which molds our beliefs, customs, and language as we are reared in our particular niche of this world of culture (see Note 1).

This reminds us that the world of culture is not a static thing; it is continually undergoing expansion and change. So another dimension should be added to the diagram. For sake of simplicity, I have separated this — having now compressed the world of culture into a single dimension — and by a single line represented the flow of culture, or the "cultural continuum" as it is frequently called by the anthropologist. The individual is introduced into this flow when he is born, is culturally conditioned while young, and eventually contributes to the cultural environment through his own inventions and creations, and ultimately dies. His ability to make his contributions was conditioned by his physical, biologic (or genetic) and cultural heritage; if he is poorly endowed with any of these, his contribution may be little or nil. Genetically he may be a genius, but if he is born into a culturally poor area of the world of culture, that genius may never show. But whatever he accomplishes will be dependent upon the labors of those who preceded him, and which reach him via the written or spoken word, i.e., by symbols.

But I must skip details. We are familiar with the fact that various forms evolved in the physical world, and later, living forms evolved. But the evolutionary process did not stop there. Just as the evolution of the living cell made possible the complexity of life forms familiar to the biologist, so did evolution of the ability to symbol in the species *homo sapiens* make possible the complexity of cultures that we see today. And just as the history of living forms could be expanded and made more meaningful by the Darwinian and post-Darwinian theory of evolution, so can the cultural history of man — and this includes the history of science and its subdomain, the history of mathematics — be supplemented by a theory of evolution. As was made clearly evident at the centenary celebration, in 1959, of the publication of Darwin's *Origin of Species* (whose proceedings have been published in 3 volumes [**5**]), modern anthropology has come to recognize that the evolutionary process did not stop with the biological, but continued with the cultural. The evolution of culture has become quite as active a field of investigation as has been the evolution of biological forms. And, I might add parenthetically, cultural evolution has been accompanied by virtually the same sort of disagreement in various scholarly circles as was the theory of biological evolution. This is why we still use the term "theory" in connection with it, although since it explains so many things that otherwise appear to have only vague mythical or philosophical explanations, there seems little doubt of its scientific utility and respectability.

I would like to suggest that a semester course in what I call "Evolution of Mathematical Concepts and Theories" will provide the student with answers to such questions as "How did mathematics get this way?" and inform him of what he is

likely to see in the future. Such a course would be based on history, but history in the sense of a continuously evolving subculture. The history involved could be either ancient or modern, or both, depending on the mathematical maturity of the students. It need not replace the more orthodox type of history course for the history major, although even he should profit by taking it before his other courses in history.

5. History as evolution. Now the history would provide principally the *stages* of the evolutionary process. But there is more to evolution than these. If we take a look at what the biologist has done, we shall notice that some of the major problems of biological evolution have been concerned with the dynamics of the process; i.e., with those forces that were instrumental in producing the stages. Darwin himself proposed the theory of natural selection — a survival of the fittest. Later biologists discovered gene shuffling and mutational forces. But probably due to its late arrival on the scientific scene, the theory of cultural evolution seems not to have advanced so far (see Note 2). Anthropologists have been unable to agree on the stages of general cultural evolution — we must recall that they must rely heavily (in addition to data on existing primitive cultures) on archaeological rather than on recorded evidence, and culture is simply not found in diggings but must be inferred from pots, bones, weapons and other physical evidence. Certain forces have, to be sure, been discovered, such as *diffusion* — the passing of such cultural elements as customs, religions and tools from one culture to another. But even here, much time and energy has been consumed in arguments over whether diffusion or independent invention accounted for similarities between different cultures. It has come to be recognized, however, that these are not mutually exclusive. For instance, counting probably originated independently in many different cultures, but once a primitive tribe comes in contact with a more advanced stage of civilization, diffusion of the counting practices from the more advanced to the less advanced usually occurs.

In the "Points for Discussion" for a panel on *Social and Cultural Evolution* during the Chicago Darwinian Centenary which I mentioned a while ago, can be found the following [**5**; vol. 3, p. 233]: "As to the macrodynamics of cultural evolution, its causes and principles, ... there is as yet no general agreement. For the near future this subject needs careful research. This is necessary as a basis for any attempt to predict or control the direction of cultural evolution."

Fortunately in mathematical history we have a wealth of recorded information. I use the word "wealth" in spite of the fact that historians bemoan the loss of most Greek mathematical works, for example. In comparison with the scarcity of early remains which the anthropologist has to work from, we are indeed lucky. It would be nice to know more about how counting and the number concept evolved, and just what individuals were responsible for the geometric discoveries and inventions presented to us in finished form in Euclid's *Elements*. But we should be grateful that we can infer pretty well just what the general outline of early mathematical development was like and of course in the case of modern mathematics, we indeed

have a wealth of recorded material. Regarding the stages through which mathematics has passed there is still some conjecture, especially on the elementary level. As to the forces involved, there seems little reason to think that they were much different (except for being fewer in number) from forces operating today.

In diagram number 2, early *stages* in the evolution of number are listed (see [**6**], p. 180). I must omit details. The first two stages we get from the anthropologist. Comparison by (1-1)-correspondence can be inferred from anthropological evidence regarding early number words, and tallying is evidenced in many early numerical records — the earliest being the find in 1937 of the radius of a young wolf from paleolithic times which is covered with notches so grouped as to be indubitably a tally

Diagram No. 2

Stages in Evolution of Number

One-two differentiation	Numeral Systems
One-two-many	Mysticism
Comparison: (1-1)-Correspondence	Operations with numerals
Tallying	Fractions
Number words	Zero
Ideographs	Negative, complex numbers
	Etc.

(see Note 3). The only item about which there may be some question is "Mysticism" (Note 4). Certainly most of us are familiar with Pythagorean numerology, but it had its counterpart in early Babylonian mysticism and I believe there is good reason for assigning it a part in the evolution of the concept of number — in short, with numbers becoming nouns, or things. It survives today, of course, in the host of numerologists, astrologers, and number lore. The number 13 has such a bad reputation as to induce many modern hotels to omit the 13th floor, although I am sure that their managers could not tell what the number 13 is as a concept. In fact, all of these stages have their modern counterparts, just as many early biological forms still exist in modern form.

Diagram No. 3

Forces of Mathematical Evolution

1. Environmental Stress
   (a) Physical
   (b) Cultural
2. Hereditary Stress
3. Symbolization
4. Diffusion
5. Abstraction
6. Generalization
7. Consolidation
8. Diversification
9. Specialization
10. Cultural Lag
11. Cultural Resistance
12. Selection

6. Forces of evolution; how mathematics grows. By way of contrast, consider the list of *forces* of mathematical evolution given in diagram number 3 (Note 5). Again I shall omit details, but shall briefly illustrate their nature. (See, however, the discussion in my book referred to above.)

Environmental stress is listed first, since it was unquestionably the first and most elementary of the forces involved in the evolution of mathematics. Indeed, it was likely active even before man evolved, since capability of one-two differentiation can be exercised by most animals and is not necessarily cultural in nature. In order to adapt, the animal must be able to sense whether he is facing one or more enemies, for example. Thus much of the initial environmental stress was physical in nature. However, with the evolution of culture in man, environmental stress of a cultural nature began to play a part, as might be expected since man was entering a new world. Comparison by matching, tallying, and, eventually, the invention of number words took place. And, when urban life evolved, the stress exerted by building, architecture, imposition of taxes and recording thereof and the like forced the invention of elementary calculating. And of course cultural stress still plays an active part in mathematical evolution, as those who were affected by the demands of the second world war can testify. And don't think that present economic conditions resulting in a lack of jobs for new Ph.D.'s is not going to have its effect!

I shall comment only briefly on *how* these forces individually work – actually I have not had time in my own studies to complete such an analysis (no geneticist has solved all the problems concerned with mutation). But I am sure that even superficial consideration of them will be enough to indicate their general function and importance. Symboling was already active in the invention of number words; as the mathematician has often done, primitive man first utilized words of ordinary discourse, as in the use of "hand" for the number 5 for instance. L. L. Conant's classic work of 1896, "The Number Concept," is revealing here [7]. And of course symboling is one of our chief tools, as are also abstraction and generalization. Diffusion, cultural lag and cultural resistance I have borrowed from the anthropologist. Diffusion I have already defined earlier; we wouldn't be using the Babylonian sexagesimal system for fractional measurement of angles if it hadn't diffused from one ancient culture to another, and, eventually, into our own Western culture. Even our journals can be considered as a means of diffusion of mathematical ideas.

Cultural lag can be thought of as a sort of "laziness," or indisposition to make the effort to adopt a more efficient tool. I just mentioned our use of Babylonian numeration in angle measurement, and I imagine cultural lag also played some part in this, although I'll leave that to the professional historian. A current example may be found in the plans for converting to the metric system in this country; the big problem will be overcoming cultural lag. Cultural resistance is a more overt obstacle to diffusion. Most missionaries have encountered it, and for a whole century the English mathematical community resisted adopting the Leibnizian differential notation presumably out of loyalty to Newton. I am sure some of you can recall

instances of cultural resistance in mathematical circles, in cases where one group of mathematicians refuses to adopt more efficient methods and concepts which have evolved in other groups; of course cultural lag may be operative in such instances also.

Two of the most important and profound of the forces listed are hereditary stress and consolidation. Only as mathematics has become more mature and complex has their influence become so great as to render them obvious. Hereditary stress is a cultural stress created by the accumulation, usually over a period of extended duration, of concepts and their interactions *within* a system. I find that historians have sometimes detected it. For example, the late historian of science, George Sarton [**8**; p. 444], stated: "The whole fabric of science seems . . . to be growing like a tree; in both cases the dependence upon the environment is obvious enough, yet the main cause of growth — the growth pressure, the urge to grow — is *inside* the tree, not outside [italics ours]." I believe, too, that what Struik has suggested [**9**] as a cultural force and called "cultural impetus," is largely a part of hereditary stress (although sometimes cultural stress of environmental type). Hereditary stress was active in the ultimate admission of complex numbers to mathematical respectability, although for a long time they were what Cardan termed *numeri ficti*, or *numeri falsi*. A prime example in modern mathematics is set theory which was born from the demands of the theory of functions. As each of us is introduced by his mentors into the mathematical culture stream, we inevitably react to hereditary stresses by recognizing where improvements, new theorems, and new concepts will contribute to the growth of the branch of mathematics in which we have elected to work. The psychological aspects of our reactions have been described by both Poincaré and Hadamard.

Although it is one of the most active forces in mathematics today, consolidation has operated throughout mathematical history. As far back as old Babylon, when the Akkadians conquered Sumer, they consolidated the old Sumerian terms for "multiplies by," "find the reciprocal of," with their arithmetic in the form of ideograms, thus initiating an important advance in mathematical symbolism. Derek Price cites the consolidation in Ptolemy's *Almagest* of the Greek *geometric* astronomy with the Babylonian *numerical* astronomy as the probable reason why Western science has reached such heights while this did not occur in other civilizations, such as China, which had the ingredients for such an achievement. He makes out quite a convincing case for this thesis in the first chapter of his book "Science since Babylon" [**10**].

Coming nearer to the modern era, an outstanding example of consolidation was that of number with line, as a result of which the analysts preceding the so-called "Arithmetization of analysis" were able to create a large body of good mathematics with the help of geometric intuition. And during the modern era, one of the most interesting examples was that of the consolidation of algebra and topology. Such fields as algebraic geometry, differential geometry, differential topology were formed by consolidation. It can be inferred, that as the body of mathematics grows, opportu-

nity for consolidation increases, and the greater power that is thus achieved can be seen in the solution of problems which had defied solution in their own fields. The process effects a kind of cross-fertilization.

It should be noticed that generally these forces do not act independently. Much as in biology, where adaptation often joins with gene mutation to effect survival, so in mathematics consolidation is frequently forced by hereditary stress; and in the process, diffusion, generalization and abstraction may play a part. It was the consolidation of the group-theoretic features of various mathematical theories that led to abstract group theory, and category theory is a nice example where generalizing from the features of the plethora of homology theories of modern algebraic topology resulted in a consolidation of common elements which is proving one of the most important modern tools in modern mathematics. If this sort of thing did not happen mathematics would simply grow like a tree with innumerable branches having no contacts with one another, with eventual chaos as the probable outcome.

7. Example of a course. It is impossible for me to make, in 50 minutes, the complete case for what I firmly believe is an area that offers much promise for research. I shall conclude with some comments on what I think can be done for the student on the basis of these ideas. First let me briefly exhibit some outlines indicating the nature of a one-quarter course I gave at the University of California in Santa Barbara a year ago. Diagram number 4 gives a list of the general topics covered. The students were supposed to be juniors and seniors, but a number of graduates were allowed to attend, including one who was working on the Ph.D. in philosophy. Since some of these topics may seem strange, I will exhibit outlines for two of them.

Diagram No. 4

A Course Outline

1. Symbols and symboling
2. Culture
3. Counting
4. Evolution of counting
5. Evolution of geometry
6. Evolution of real number system
7. Aspects of reality
8. Evolution of function, set concepts
9. Evolution of real analysis
10. Emergence of contradictions
11. Identification, analysis of evolutionary forces
12. Role of individual in evolution
13. Philosophies of mathematics
14. Evolutionary "laws"

Diagram number 5, "Aspects of Reality," may be roughly explained by pointing out that throughout the course, I repeatedly emphasized, as opportunity offered, that as a part of the world of culture, mathematics is just as real as any part of the physical world. But since it has a tendency to deal in ever higher levels of abstraction, we continually need reassurance that our creations do really add to the existing body of

mathematical reality. This has led to the use of models — which will explain why several of the items relate to model theory.

Diagram No. 5

7. Aspects of Reality

(a) Physical; perception of
(b) Extension to cultural environment
(c) Inception of use of models
  (i) Function to maintain contact with reality
  (ii) Beltrami, Klein models
(d) Evolution of model theory
(e) Role of models in axiomatics
(f) Mathematical reality
  (i) Reality of concepts after adoption by mathematical community

Referring to diagram number 6: The evolution of function and set was chosen as one of the topics partly because I could count on everyone having some acquaintance with these notions, and partly because they also offered an excellent example to show the interplay of the evolutionary forces.

Diagram No. 6

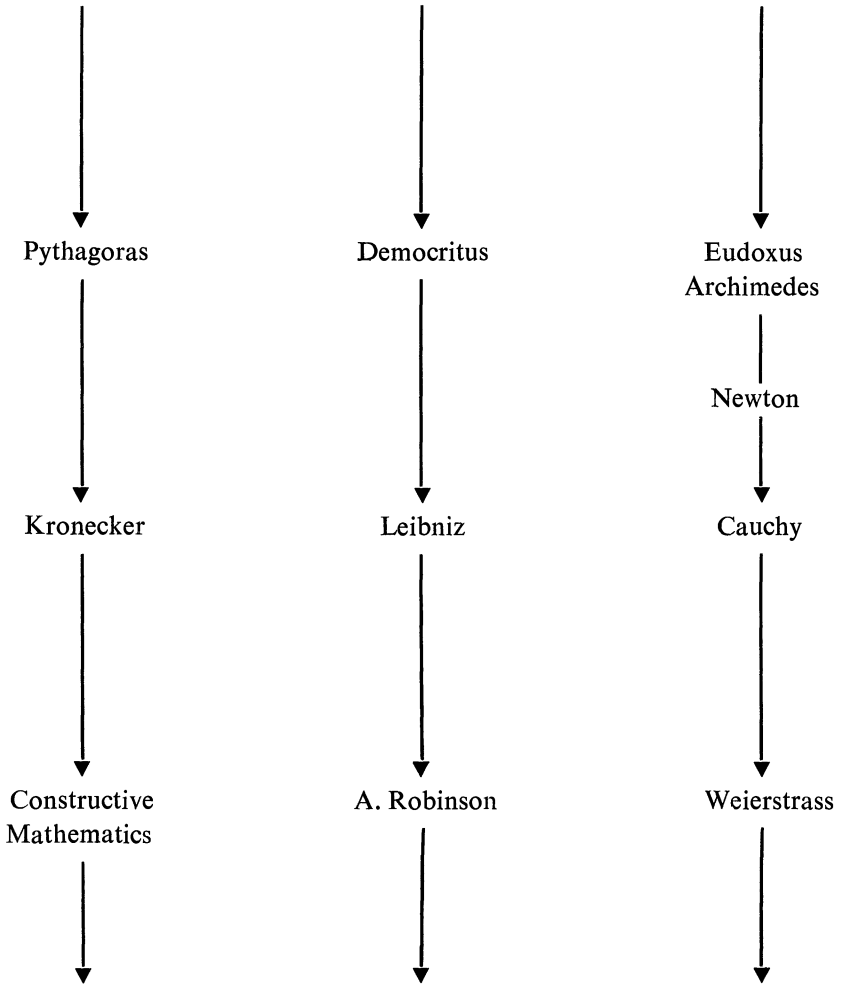
8. Evolution of function and set concepts

(a) Theory of sound; vibrating string
(b) D'Alembert: Euler, Bernoulli solutions
  (i) Disagreement over meaning of "function"
(c) Theory of heat; Fourier
  (i) "Uninhibited" notion of function
(d) Dirichlet's conditions
(e) Riemann's work on trigonometric series
  (i) Integrability conditions
  (ii) Influence on function concept
(f) Cantor's uniqueness theorem
  (i) Species of a point set
  (ii) Inception of set theory
(g) Emergence of new principles
  (i) Continuum hypothesis; axiom of choice

In showing how the processes of evolution work, I made extensive use of charts or diagrams, to show graphically the flow of influences of one part of mathematics upon another, as well as consolidations. Most of these are too complicated to squeeze into a compact diagram. Here is one (diagram number 7) containing some elements of conjecture — indulgence in reflecting on "What might have happened" was not frowned on, by the way. I chose to exhibit this one today because it is so simple (not historically complete, but purely indicative) and is somewhat topical in view of the subject of the lectures which Professor Robinson is giving at this meeting. Professor Robinson has discussed in his book [**11**] some of the reasons why the path depicted by the middle column was not pursued by analysis; the right-hand column represents the actual course of analysis, it will be observed.

I like to think that those who took the course acquired some understanding of how the various courses they were taking came into being, and how they were interre-

Some possible directions for the foundations of Analysis



lated — although I left much of this to the individual to reason out for himself using the ideas he had, hopefully, assimilated. Certainly each understood that mathematics is still undergoing evolution, and that if he was going to make it a career, his only chance for success was to enter the stream at some likely point of his own choice; but to expect that he would have to spend much of his future in keeping up with the changes that would inevitably occur.

Obviously this was not an orthodox history course. It was more in the nature of what the historian of science would call a science of the history of mathematics. If may be that a history course along more orthodox lines can be devised which will accomplish much the same ends in a more efficient manner. I have been pleased, during the course of preparing this material, to hear from several mathematical col-

leagues who are working on the problem of a suitable modern history course — so much so, that I earnestly look forward to the rejuvenation of history in a more up-to-date form in the classroom; and even that the subject will reach such a degree of acceptance as to be again considered worthy of the Ph.D. in mathematics.

8. Philosophical implications. One final word: When I was briefly discussing mathematical reality, perhaps some of you wondered where Platonism fits in? In particular, does a theory of mathematical evolution, based on the location of mathematical reality in the world of culture run counter to Platonism? The answer is emphatically "No"; no more than Darwinism destroyed existing religions, despite the fears of the clergy. The anthropologist studies religions as a part of culture; to him they form an adapting mechanism, and he takes no position, as a scientist, on whether they represent a reality outside the world of culture or not. Similarly, a theory of mathematical evolution can study, using the tools of science, the manner in which Intuitionism, Formalism, Constructivism, Platonism, or any other philosophy of mathematics evolved. But it takes no position on their so-called "Truth," or on what other possible types of reality they may represent. So if you are a Platonist, go ahead and enjoy it!

Except for minor changes and addition of literary references, this is a verbatim copy of the author's address before the summer meeting of the Association at Pennsylvania State University, 1971.

NOTES

1. See Ernst Cassirer, An Essay on Man, Yale Univ. Pr., New Haven, Conn., 1944. "As compared with other animals man . . . lives . . . in a new *dimension* of reality. . . . Physical reality seems to recede in proportion as man's symbolic activity advances. . . . He has so enveloped himself in linguistic forms, in artistic images, in mythical symbols or religious rites that he cannot see or know anything except by the interposition of this artificial medium" (ibid., p. 25).

2. See, however, L. A. White, Energy and the evolution of culture, Amer. Anthropologist, vol. 45 (1943), pp. 335–356, for a proposal regarding general cultural evolution and the forces governing it; also W. F. Ogburn, On Culture and Social Change, O. D. Duncan, ed., Univ. of Chicago Pr., 1964.

3. See the note in Isis, vol. 28 (1938), pp. 462–463, referring to a news item in the Illustrated London News of Oct. 2, 1937, concerning excavations made by Karl Absolon in Czechoslovakia.

4. Whether passage through a stage in which different numeral forms were used for various categories of objects and concepts, is conjectural, although there is much evidence for it. For instance, this phenomenon occurred among certain Plains Indian tribes, as well as among Northwest Indian tribes and other cultures; remains of such a classificatory numeral system are found in the Japanese language.

5. Except for the addition of "Specialization," this is the list of forces given on p. 169 of my book [6].

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MATHEMATICAL NOTES

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VARIATIONS ON THE BINOMIAL SERIES

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1. Introduction. This study began when one of us asked the other whether there exists a reasonable continuous analog of the equality

$$(1)\qquad (1+z)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k \quad (\alpha > -1, |z| = 1, z \neq -1),$$

where, for real u,

$$(2)\qquad \binom{\alpha}{u} = \frac{\Gamma(\alpha+1)}{\Gamma(u+1)\Gamma(\alpha-u+1)}.$$

It is natural to try to replace the right hand side of (1) by $\int_0^\infty \binom{\alpha}{u} z^u\, du$. This, however, led us up a blind alley.

We then observed that from a known formula (§3) one obtains

$$(3)\qquad \binom{\alpha}{u} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-iut}(1+e^{it})^\alpha dt, \ \alpha > -1, \ -\infty < u < \infty.$$

In particular, for k an integer, $\binom{\alpha}{k}$ is the kth Fourier coefficient of $(1+e^{it})^\alpha$. Since by (2), $\binom{\alpha}{k} = 0$ for $\alpha > -1$ and $k = -1, -2, \cdots$, we can interpret (1) as an equality (throughout $(-\pi, \pi)$) between $(1+e^{it})^\alpha$ and its Fourier series $\sum_{k=-\infty}^{\infty} \binom{\alpha}{k} e^{ikt}$. Therefore, a continuous analog of (1) appears to be obtained by inversion of the Fourier transform (3):