Zaphod Beeblebrox's Brain and the Fifty-ninth Row of Pascal's Triangle

Andrew Granville

1. INTRODUCTION. A popular problem for an introductory combinatorics course is to prove that

The number of odd integers in any row of Pascal's triangle is always a power of 2. (1)

There seem to be two approaches to this question. The first uses the following remarkable observation of Kummer (which was made in 1855):

For any prime p and positive integers $n \geq m \geq 0$, the exact power of p that divides the binomial coefficient $\binom{n}{m}$ is given by the number of 'carries' when adding m and $n-m$ in base p. (*)

Thus the binomial coefficient $\binom{n}{m}$ is odd if and only if we have no carries when adding m and $n - m$ in base 2. A moment's thought and we see that this is equivalent to the statement that the set of 1's in the binary expansion of m is a subset of the set of 1's in the binary expansion of n. Therefore the number of odd binomial coefficients $\binom{n}{m}$ with $n \geq m \geq 0$ is given by the number of distinct subsets of the set of 1's in the binary expansion of n, which is precisely $2^{\#_2(n)}$, where $\#_2(n)$ is the number of 1's in the binary expansion of n. (This was first proved by Glaisher in 1899.)

The second, more elegant, approach is significant in the area of cellular automata (see [7]):

We start by replacing each entry of Pascal's triangle with an asterisk ("*") if it is odd, a blank (" ") if it is even. The problem above begins to count the number of asterisks in each row. Moreover, the normal rule of construction of Pascal's triangle (an entry equals the sum of the two immediately above) becomes a very simple binary rule:

An entry is an asterisk if and only if one of the entries immediately above is "*" and the other is blank. In FIGURE 1 we show this graphically:

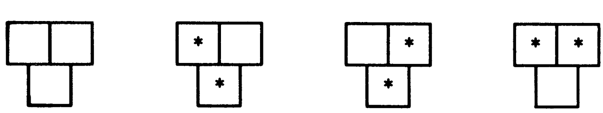

FIG. 1. The rules for addition (mod 2).

Thus Pascal's triangle itself looks like

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               *
              * *
             *   *
            * * * *
           *       *
          * *     * *
         *   *   *   *
        * * * * * * * *
       *               *
      * *             * *
     *   *           *   *
    * * * *         * * * *
   *       *       *       *
  * *     * *     * *     * *
 *   *   *   *   *   *   *   *
* * * * * * * * * * * * * * * *
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FIG. 2. The odd elements of Pascal's triangle (mod 2).

Continue FIGURE 2 for a few lines, stare at it, and a clear pattern begins to emerge: For every fixed $k \geq 0$, a triangle, T_k, is formed by the first 2^k rows (that is the coefficients $\binom{n}{m}$ (mod 2), for $0 \leq m \leq n \leq 2^k - 1$). T_{k+1} is then constructed by putting three copies of T_k in a triangle, with all blanks in the middle:

$T_{k+1} =$

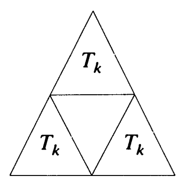


FIG. 3. The construction of the first 2^{k+1} rows of Pascal's triangle (mod 2) from the first 2^k.

The proof that row n has precisely $2^{\#_2(n)}$ odd entries, follows easily from induction on k: For given n, there exists a k such that row n belongs to T_{k+1} but not T_k. Now, as in FIGURE 3, we see that row n (modulo 2) is composed of two copies of row $m(:= n - 2^k)$ with some blanks in the middle. Therefore, row n contains twice the number of asterisks of row m, namely, $2.2^{\#_2(m)}$ (by induction) $= 2^{\#_2(n)}$.

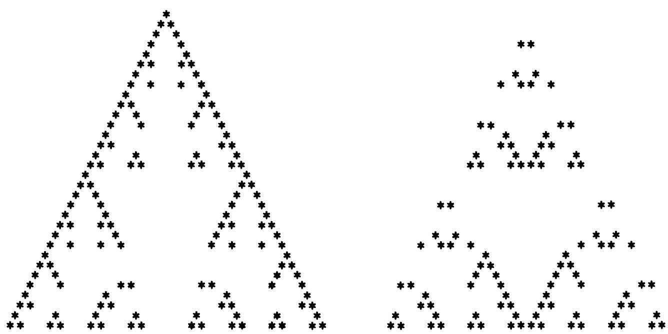
Many authors have worked on the corresponding problem of counting the entries of a given row of Pascal's triangle, that are not divisible by some fixed prime p. The first approach above works easily to give an exact count (see [**5**]); however the pictures generated by the second approach above are much more interesting, and are really rather pretty (see [**7**]).

In the autumn of 1988, I presented these ideas as part of an introductory combinatorics course at the University of Toronto. One student asked whether a similar result holds when one counts the number of entries that belong to the congruence class 1(mod 4), in a given row of Pascal's triangle. As I didn't know the answer, I suggested that the class compute the first few lines of Pascal's triangle (mod 4) to see if any pattern emerged. When they did so it transpired that the student had asked a very good question: We observed that the odd entries of row n of Pascal's triangle are either all $\equiv 1 \pmod 4$ or are split equally between the arithmetic progressions 1 (mod 4) and -1 (mod 4). Thus it seemed that the number of entries $\equiv 1 \pmod 4$ in row n is either $2^{\#_2(n)}$ or $2^{\#_2(n)-1}$, and the number $\equiv -1 \pmod 4$ is either 0 or $2^{\#_2(n)-1}$, respectively.

After class, I went to the library to find out whether this had previously been observed and how to prove it (it didn't seem to follow from any straightforward modification of either of the two methods above). Rather surprisingly this pattern had not been noticed, and as it seemed unlikely that such an attractive result would be unknown, I started to think that perhaps the pattern eventually ended. However, after computing the first 60 or 70 lines of Pascal's triangle (mod 4), I found that, not only did this pattern continue to emerge, but I could even guess how to distinguish between the two cases above:

The number of entries $\equiv 1 \pmod 4$ *equals the number of entries* $\equiv -1 \pmod 4$ *in row* n *if and only if there are two consecutive* 1*'s in the binary expansion of* n*; otherwise there are no entries* $\equiv -1 \pmod 4$ *in row* n. (2)

At the next class Rajesh Goyal, one of the computer science students attending, volunteered to draw two diagrams similar to FIGURE 2; the first with an asterisk only for entries $\equiv 1 \pmod 4$; the second with an asterisk only for entries $\equiv -1 \pmod 4$. We present these diagrams in FIGURE 4.



1 (mod 4) $\qquad$ −1 (mod 4)

FIG. 4. The odd elements of Pascal's triangle (mod 4).

As you can see, no predictable pattern leaps out, though certain members of the class were convinced that they could distinguish a maple leaf insignia! I suggested that Rajesh now draw Pascal's triangle again, this time placing all the odd entries in the same picture, but assigning different colours to the entries that were 1 (mod 4) and −1 (mod 4). Unfortunately no recognizable pattern evolved, and so the class returned to the course syllabus.

A few weeks later, still frustrated by this question, I came across a passage in Douglas Adams' science fiction/comedy novel *The Hitchhiker's Guide to the Galaxy*. There, Zaphod Beeblebrox, who has been acting unaccountably (even to himself), decides to run a series of tests on his two brains to see what is wrong. Having tried all the "standard" tests and having found nothing wrong, he proceeds to superimpose the X-rays of his two brains and look at the image through a green filter, which exposes, to his astonishment, the cauterized initials of the culprit who has been tampering with his heads!

It occurred to me to try a similar approach to our problem with Pascal's triangle. The idea was to colour those entries that are 1 (mod 4) *blue*, those that are −1 (mod 4) *yellow*, and leave the rest *blank*. Then, by superimposing different subtriangles of Pascal's triangle, to observe whether any pattern emerges (using the natural rules *blank* + *blank* = *blank*, any *colour* + *blank* = that *colour*, 2 times a particular *colour* = that *colour*, and *blue* + *yellow* = *green*). To my delight, this worked! To explain what happened, define U_k to be the triangle made up of the first 2^k rows of Pascal's triangle, coloured *blue*, *yellow* and *blank* as above (note that by altering the *blue* and *yellow* squares of U_k to asterisks, we get T_k). By FIGURE 3, and the fact that Pascal's triangle is symmetric about a vertical line drawn down its centre, we see that

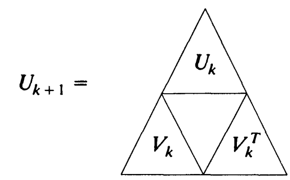


FIG. 5. The structure of Pascal's triangle (mod 4).

for some triangle V_k, where V_k^T is defined to be the reflection of V_k about a vertical line down its centre. So, if we wish to determine U_{k+1} then, by Figure 5, we must address the problem of determining V_k. By looking at a few such triangles V_k, it is easy to spot the pattern given in FIGURE 6.

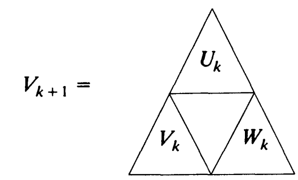


FIG. 6. The structure of V_{k+1} (mod 4).

Here W_k is some unknown pattern of *yellow* and *blue*. To try to find a simple way to derive W_k, I then used the *Beeblebrox* method to compare W_k with various other matrices and surprisingly found the important fact needed:

When we superimpose W_k onto V_k^T, every entry is either *green* or *blank*. In other words, the entry of W_k corresponding to a given entry e of V_k^T is *blank* if e is *blank*, *yellow* if e is *blue*, and *blue* if e is *yellow*. We represent this in FIGURE 7 for $k = 0, 1, 2, 3$, using ⊠ for *blue*, ⧇ for *yellow*, and ⧈ for *green* (since this journal is monochromatic!):

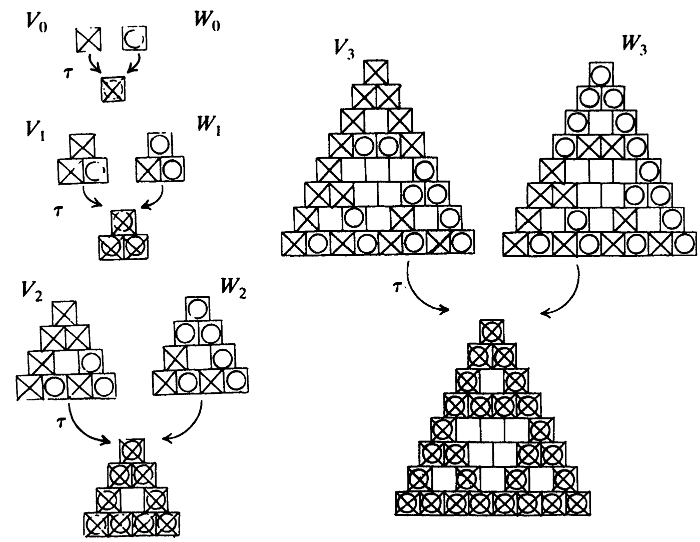


FIG. 7. *The Beeblebrox method*—Superimposing the transpose of V_k onto W_k.

Given the observation in FIGURE 7, which gives a complete description of the 'growth' of Pascal's triangle (mod 4), it is relatively simple to confirm (2). We again proceed by induction on n: Choose k so that $2^k \le n < 2^{k+1}$. Using FIGURE 5 we see that row n of Pascal's triangle is given (from left to right) by row $n_1 (:= n - 2^k)$ of V_k, some zeroes and then row n_1 of V_k^T. Thus the number of elements of row n, congruent to $j \pmod 4$ (for $j = 1$ or -1), is twice the number of such elements in row n_1 of V_k.

Now, if $n_1 < 2^{k-1}$ then using FIGURE 6 we see that row n_1 of V_k is precisely row n_1 of U_{k-1}, and so of Pascal's triangle itself. (2) then follows from the induction hypothesis, by noting that $(n)_2$ contains consecutive digits 11 if and only if $(n_1)_2$ does.

If $n_1 \ge 2^{k-1}$ then using FIGURE 6 we see that row n_1 of V_k is row $n_2 (:= n_1 - 2^{k-1})$ of V_{k-1}, then some zeroes, followed by row n_2 of W_{k-1}. Now by the observation in FIGURE 7, the number of elements of row n_2 of W_{k-1} that are congruent to $j \pmod 4$ (for $j = 1$ or -1) is precisely the number of elements of row n_2 of V_{k-1} that are congruent to $-j \pmod 4$, and so we see that row n_1 of V_k contains the same number of elements $\equiv 1 \pmod 4$ and $\equiv -1 \pmod 4$. Of course, as 11 are the left most digits of $(n)_2$ (as $n_1 \ge 2^{k-1}$), the equation (2) follows immediately.

It remains only to prove the truth of the observations explained by FIGURES 6 and 7, which we do in the next section, a fairly straightforward task.

Having established that the number of odd integers in any given row of Pascal's triangle is a power of 2, and that the number $\equiv 1 \pmod 4$ (or $\equiv -1 \pmod 4$)) is, likewise, either 0 or a power of 2, it now seems reasonable to investigate the numbers of integers in each row that are congruent to 1, 3, 5 or 7 (mod 8). Preliminary computations indicate what we might expect from the results that we have already obtained:

In each row of Pascal's triangle, the number of integers in each of the arithmetic progressions 1, 3, 5 *and* 7 (mod 8) *is either* 0 *or a power of* 2. (3)

Having computed that (3) holds true in the first 50 or so rows of Pascal's triangle, we will now try to prove (3) using the same sort of approach that we used to prove (2).

First, though, let's incorporate FIGURES 6 and 7 into one diagram, into the form we shall actually prove in Section 2: Define, for a subtriangle A of Pascal's triangle (mod 4), $-A$ to be the triangle A with *blanks* the same and the colours *blue* and *yellow* swapped around. Then we have

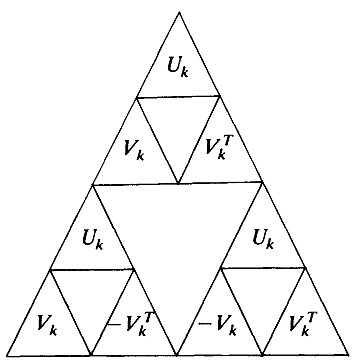


FIG. 8. The structure of U_{k+2} (mod 4).

Thus we may think of U_{k+2} being formed from U_{k+1} as follows: Cut U_{k+1} into the four triangles of FIGURE 5. Then multiply, element by element, the entries of these triangles by the triangle M (which is given in FIGURE 9 below) which gives V_{k+1}; similarly multiplying by M^T gives V_{k+1}^T. This is illustrated in FIGURE 9:

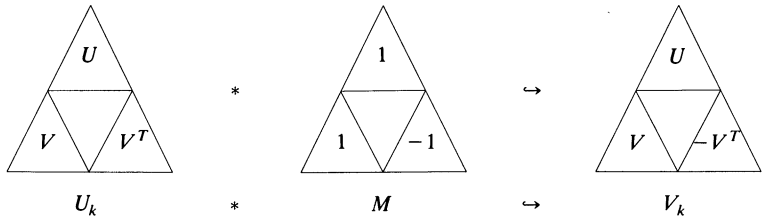


FIG. 9. The action of the growth triangle (mod 4).

This 'triangle', $M(= M_2)$, we will call the *growth triangle* (mod 4). So, to begin to prove (3), we attempt to find a growth triangle (mod 8); and later to start a proof of the corresponding statement (mod 16), we will find a growth triangle (mod 16).

For fixed positive integer b, define D_k to be the first $2^k - 1$ rows of Pascal's triangle (mod 2^b), with all even entries replaced by 0. Define E_k as in FIGURE 10.

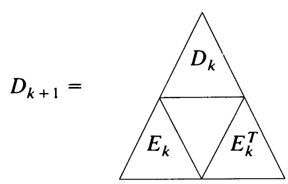
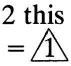


FIG. 10. Structure of D_{k+1} (mod 2^b).

In order to determine D_{k+1} from D_k we must be able to find E_k from D_k. For $b = 1$ and 2 this is done as above by noting that $E_k = D_k * M_b$ for each $k \geq b - 1$ where $M_1 =$  and M_2 is as in FIGURE 9. More generally, we shall prove, in section 2, that

$$E_k = D_k * M_b \quad \text{for each } k \geq b - 1, \tag{4}$$

where the triangle M_b (containing 3^{b-1} non-zero subtriangles) remains to be specified. We give the examples for $b = 3$ and 4 in FIGURE 11 (which you can test!).

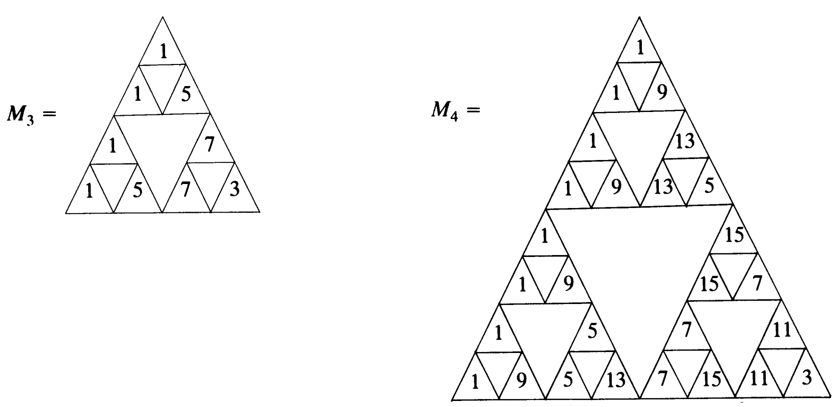


FIG. 11. The growth triangles (mod 8) and (mod 16).

Note that we could have formulated (2) in a similar way to (3):

> *In each row of Pascal's triangle the number of integers in each of the arithmetic progressions* 1 *and* $-1 \pmod 4$ *is either* 0 *or a power of* 2.

The reason that we gave the rather more precise statement (2) is that it fit easily into our induction hypothesis. Similarly, we will reformulate (3) so that the statement fits easily into an induction hypothesis. Note first, though, that in order to use the last row of M_3 it is necessary to 'cut up' the row of Pascal's triangle into four quadrants. Of course, it isn't really necessary for the second and third rows, as there are only two halves, and Pascal's triangle is symmetric. The last row of M_3 is only used when transforming row m into row $n = 2^{k+1} + m$ where m is of the form $m = 2^k + 2^{k-1} + r$ and $0 \le r \le 2^{k-1} - 1$. For such rows m we will describe only the first two quadrants as the other two may be deduced from symmetry.

If $(n)_2$ *contains no* 11 *and no* 101 *then all entries are* $\equiv 1 \pmod 8$.

If $(n)_2$ *contains no* 11 *but has a* 101 *then there are an equal number of entries* $\equiv 1$ *and* $5 \pmod 8$.

If $(n)_2$ *contains both a* 11 *and a* 101, *or it contains a* 1111, *then there are an equal number of entries in each of* 1, 3, 5 *and* $7 \pmod 8$, *and similarly in each quadrant (when relevant).* (3)

If n does not belong to any of the cases above then, in binary, it has the form given in FIGURE 12.

$$(n)_2 = \underbrace{1\ 1\ldots1}_{t_1\ 1\text{'s}}\ \underbrace{0\ 0\ldots0}_{u_1\ 0\text{'s}}\ \underbrace{1\ 1\ldots1}_{t_2\ 1\text{'s}}\ \cdots\ \underbrace{0\ 0\ldots0}_{u_m\ 0\text{'s}}\ \underbrace{1\ 1\ldots1}_{t_{m+1}\ 1\text{'s}}\ \cdots\quad\cdots$$

Fig. 12. The binary structure of n in the remaining cases. (Here each $u_j \ge 2$ and each $t_j = 1, 2$ or 3.)

If $t_1 = 2$ *and all other* $t_j = 1$, *then all entries of the first quadrant are* $\equiv 1 \pmod 8$ *and all entries of the second quadrant are* $\equiv 7 \pmod 8$.

If $t_1 = 2$ *and each other* $t_j = 1$ *or* 3 *(and at least one* $t_j = 3$*), then there are equal numbers* $\equiv 1$ *and* $3 \pmod 8$ *in the first quadrant and there are equal numbers* $\equiv 5$ *and* $7 \pmod 8$ *in the second quadrant.* (3)

If not as above and if each $t_j = 1$ *or* 2 *then there are equal numbers* $\equiv 1$ *and* $7 \pmod 8$.

If not as above and if each $t_j = 1$ *or* 3 *then there are equal numbers* $\equiv 1$ *and* $3 \pmod 8$.

Otherwise there are equal number of entries (in each quadrant, when relevant) $\equiv 1, 3, 5$ *and* $7 \pmod 8$.

The proof of this statement is straightforward, though lengthy. The advantage of (3)′ is that it is easy to prove and (3) can be deduced immediately; we leave checking the details to the reader!

After proving (2) and (3) one now wishes to generalize our result to the odd residue classes (mod 16), then (mod 32), etc. An obvious problem is that the statement corresponding to (2) and (3)′ for Pascal's triangle (mod 16) promises to be extraordinarily long. However, as such a statement might provide the clues necessary to guess at the correct statement in the general case—an odd arithmetic progression (mod 2 to an arbitrary power)—it seems worth finding. I worked on this problem for several days, but the statement just seemed to be getting ever longer!

Wishing to reduce the amount of work necessary, I asked my colleague, Yiliang Zhu, to run some programs on his computer to test a few ideas. The results that we got were unexpected—it seemed that most of our ideas failed. Nonetheless, certain that such a proof must exist, we did a number of other computations. Our efforts were not rewarded; nothing seemed to work. Finally, we simply printed out the first 128 rows of Pascal's triangle, (mod 16), and made a visual inspection to see if we could deduce any new patterns. And there it was, the reason that things didn't seem to work—Row 59. We give the first half in FIGURE 13:

1, 11, 15, 13, 0, 0, 0, 0, 7, 13, 1, 3, 0, 0, 0, 0, 1, 11, 15, 13, 0, 0, 0, 0, 15, 5, 9, 11, 0, 0

Fig. 13. Half of Row 59 (mod 16) (with 0 (mod 2) denoted by 0).

Unbelievably, there are exactly six entries of Row 59 in each of the congruence classes 1, 11, 13 and 15 (mod 16)! Our pattern has come to an end, but not before providing us with some interesting mathematics, as well as a couple of pleasant surprises.

2. THE GROWTH TRIANGLES. In the previous section all the assertions that we used in the proofs of (2) and (3)′ were justified there, except for the existence of the growth triangle, i.e. the formula (4). That is, we need to show that if

$$2^k \le n < 2^{k+1},\ j < n/2, \quad \text{and} \quad \binom{n}{j} \text{ is odd,} \tag{5}$$

then the ratio

$$\binom{n}{j} \Big/ \binom{m}{j} \pmod{2^b}, \quad \text{where } m = n - 2^k,$$

is fixed according to the position of $\binom{m}{j}$ in a similar triangle with 2^{b-1} rows. More precisely, we must prove

Proposition 1. *Let b be a positive integer and suppose j, k and n are integers satisfying (5), with $k \ge b - 1$. Define*

$$m' = [m/2^{k+1-b}],\ n' = [n/2^{k+1-b}]$$

and

$$j' = [j/2^{k+1-b}], \quad \text{where } m = n - 2^k. \tag{6}$$

Then

$$\binom{n}{j} \Big/ \binom{m}{j} \equiv \binom{n'}{j'} \Big/ \binom{m'}{j'} \pmod{2^b}. \tag{7}$$

M_b is therefore a triangle with 2^{b-1} rows. The nth row has $2n-1$ entries and the (n, k)th entry is given by

$$\begin{cases} \binom{n+2^{b-1}-1}{\frac{k-1}{2}} \Big/ \binom{n-1}{\frac{k-1}{2}} \pmod{2^b} & \text{if both } k \text{ and } \binom{n-1}{\frac{k-1}{2}} \text{ are odd;} \\ 0 & \text{otherwise.} \end{cases}$$

In order to prove Proposition 1 we shall prove a result that allows us to compute binomial coefficients in modular arithmetic: In 1878 Lucas gave a simple formula for any binomial coefficient $\binom{n}{m} \pmod p$, when p is prime, in terms of the digits of m and n when they are written in base p. When $\binom{n}{m}$ is not divisible by p, this formula can be rewritten as

$$\binom{n}{m} \equiv \binom{[n/p]}{[m/p]}\binom{n_1}{m_1} \pmod{p}, \tag{8}$$

where n_1 (and similarly, m_1) is defined as the least non-negative residue of $n \pmod p$ (note that (*) provides an easy way to determine whether p divides $\binom{n}{m}$). By iterating (8) it is very easy to compute $\binom{n}{m} \pmod p$. We will prove a generalization of this formula for binomial coefficients $(\text{mod } p^b)$ for arbitrary positive integers b.

Proposition 2. *Suppose that prime p is given. For each positive integer j, define n_j to be the least non-negative residue of $n \pmod{p^j}$. If p does not divide $\binom{n}{m}$ then*

$$\binom{n}{m} \equiv \binom{[n/p]}{[m/p]}\binom{n_b}{m_b} \Big/ \binom{[n_b/p]}{[m_b/p]} \pmod{p^b}, \tag{9}$$

for any positive integer b.

We notice two immediate consequences:

Corollary 1. *If p does not divide $\binom{n}{m}$ and $m \equiv n \pmod{p^b}$ then $\binom{n}{m} \equiv \binom{[n/p]}{[m/p]}$* $(\text{mod } p^b)$.

By iterating this we get

Corollary 2. *If p does not divide $\binom{n}{m}$ and $m \equiv n \pmod{p^k}$ where $k \geq b-1$ then* $\binom{n}{m} \equiv \binom{[n/p^{k+1-b}]}{[m/p^{k+1-b}]} \pmod{p^b}$.

From this we can easily give the

Proof of Proposition 1: By (*), 2 does not divide $\binom{n}{m}$ (as there are no carries when adding m and 2^k in base 2), and so $\binom{n}{m} \equiv \binom{n'}{m'} \pmod{2^b}$ by Corollary 2. By a similar argument $\binom{n-j}{m-j} \equiv \binom{n'-j'}{m'-j'} \pmod{2^b}$, and so the result follows as

$$\begin{aligned} \binom{n}{j}\Big/\binom{m}{j} &= \binom{n}{m}\Big/\binom{n-j}{m-j} \\ &\equiv \binom{n'}{m'}\Big/\binom{n'-j'}{m'-j'} = \binom{n'}{j'}\Big/\binom{m'}{j'} \pmod{2^b}. \end{aligned}$$

Finally we must prove Proposition 2:

Proof of Proposition 2: As p does not divide $\binom{n}{m}$ (by hypothesis), we know that each digit of n is at least as large as the corresponding digit of m when written in base p, by (*). Therefore each digit of $[n/p]$, $[n/p^b]$, n_b and $[n_b/p]$ is at least as large as the corresponding digit of $[m/p]$, $[m/p^b]$, m_b and $[m_b/p]$, respectively, (in base p), and so, by (*), none of the binomial coefficients in (9) are divisible by p. This also implies that

$$[n/p^b] - [m/p^b] - [(n-m)/p^b] = 0. \tag{10}$$

Now, for any positive integer n,

$$\begin{aligned} n! &= \prod_{j\geq 0} \prod_{\substack{r=1 \\ p^j \| r}}^{n} r \\ &= \left(\prod_{j \geq 0} \prod_{\substack{r \leq n/p^j \\ p \nmid r}} r \right) p^{\sum_{i\geq 1}[n/p^i]}, \end{aligned}$$

where $p^j \| n$ means that p^j is the highest power of p that divides n. Dividing this formula by the similar formula for $[n/p]!$ we get

$$n!/[n/p]! = p^{[n/p]} \prod_{r \leq n,\, p \nmid r} r. \tag{11}$$

Now, as $r \equiv r_b \pmod{p^b}$ for any integer r, we see that the product of those integers, coprime to p, between any two consecutive multiples of p^b, is congruent to $\prod_{r\leq p^b,\, p\nmid r} r \pmod{p^b}$. Similarly, the product of those integers, coprime to p, between cp^b and $cp^b + d$, (for any positive integers c and d), is congruent $(\text{mod } p^b)$ to the product of those integers, coprime to p, less than or equal to d. Therefore

$$\left(\prod_{r\leq n,\, p\nmid r} r \right) \equiv \left(\prod_{r \leq p^b,\, p \nmid r} r \right)^{[n/p^b]} \left(\prod_{r \leq n_b,\, p\nmid r} r \right) \pmod{p^b}.$$

The result then follows from combining this equation with (10) and (11) to evaluate

$$\frac{\binom{n}{m}}{\binom{[n/p]}{[m/p]}} \Bigg/ \frac{\binom{n_b}{m_b}}{\binom{[n_b/p]}{[m_b/p]}} \pmod{p^b},$$

and by using the fact (established above) that none of the binomial coefficients in (9) are divisible by p.

Actually Proposition 2 also provides another proof that if row n contains entries that are $-1 \pmod 4$ then n contains a '11' in its binary digit pattern: If row n contains an entry that is $-1 \pmod 4$, then choose k as large as possible so that row $q := [n/2^k]$ also contains an entry that is $-1 \pmod 4$. Suppose that the entry is $\binom{q}{r}$. By our choice of k, we see that

$$\binom{[q/2]}{[r/2]} \equiv 1 \pmod 4,$$

and so, by (9),

$$\binom{q_2}{r_2} \equiv -\binom{[q_2/2]}{[r_2/2]} \pmod 4.$$

By trying all possibilities for q_2 and r_2 (note that q_2 can only take the values 0, 1, 2 and 3), we see that this can only occur for $q_2 = 3$ and $r_2 = 1$ or 2. Therefore q_2 has a '11' in its binary digit pattern, and thus so does q and hence n.

3. GROWTH TRIANGLES FOR ARBITRARY PRIME POWERS. The idea of the growth triangles, used here for powers of 2, may be generalized to arbitrary prime power moduli. To prove this we need simply prove the following generalization of Proposition 1:

Proposition 3. *Let b be any positive integer, p be a given prime, and j, k and n be integers satisfying*

$$p^k \le n < p^{k+1}, \qquad k \ge b-1 \quad \text{and} \quad p \nmid \binom{n}{j}.$$

Then

$$\binom{n}{j}\bigg/\binom{n_k}{j_k} \equiv \binom{n'}{j'}\bigg/\binom{n'_k}{j'_k} \pmod{p^b}$$

where u_k is the least non-negative residue of $u \pmod{p^k}$, for $u = j$ and n, and $u' = [u/p^{k+1-b}]$ for $u = j$, j_k, n and n_k.

Proof: The proof is an almost immediate generalization of that of Proposition 1; the only real difference is that we re-express the binomial coefficients here in a slightly more complicated way:

$$\binom{n}{j}\bigg/\binom{n_k}{j_k} = \binom{n}{n_k}\binom{n-n_k}{j-j_k}\bigg/\binom{j}{j_k}\binom{n-j}{n_k-j_k}.$$

We leave it to the reader to complete the details of the proof.

The action of the growth triangle T_q, for $q = p^b$, is rather different than before, as we now create $(p^2+p)/2$ new non-zero triangles from each original one—see FIGURE 14.

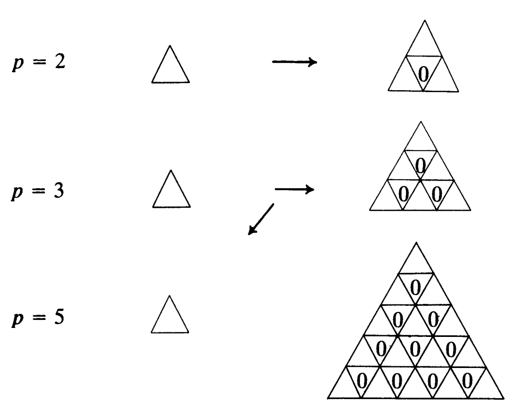


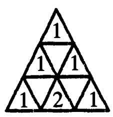
FIG. 14. The action of the growth triangle T_q, where q is a prime power of p.

T_q is composed of $(p^2+p)/2$ large subtriangles, arranged so that there is one such subtriangle on the top row, two on the second row,..., and p on the pth row, with zeroes in between (see FIGURE 14 for this structure). Each of these subtriangles has p^{b-1} rows, indexed by $0,1,\dots,p^{b-1}-1$, and the ith row contains $2i+1$ columns indexed by $j=0,1,\dots,2i$. The value of the (i,j)th entry in the $(m+1)$st subtriangle of the $(n+1)$st row (of subtriangles) $(0 \le m \le n \le p-1)$, reading left to right, is

$$\begin{cases} \binom{np^{b-1}+i}{mp^{b-1}+j/2} \Big/ \binom{i}{j/2} \pmod{p^b} & \text{if } j \text{ is even and } p \nmid \binom{i}{j/2} \\ 0 & \text{otherwise.} \end{cases}$$

Notice that if $q=p$ is prime (that is $b=1$) then i and j can only take the values $i=j=0$. Thus the $(m+1)$st subtriangle of the $(n+1)$st row of subtriangles has only one entry, $\binom{n}{m} \pmod{p^b}$. Therefore T_p is just the first p rows of Pascal's triangle (mod p), with a zero between each pair of consecutive entries on each row. (This result is, essentially, given in [5].)

We give some examples of T_q, where q is a power of 3, in FIGURE 15:

$T_3 =$ 

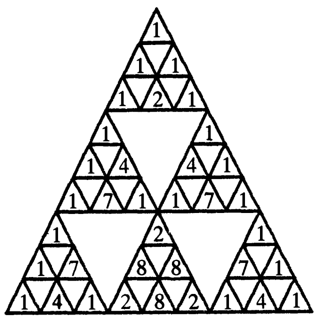
$T_9 =$ 

FIG. 15. Some examples of T_q when q is a power of 3.

The reader should note that if $q=2^b$ then T_q is formed from M_b as in FIGURE 16 below.

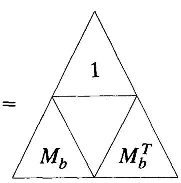
$T_{2^b} =$ 

FIG. 16. Constructing T_{2^b} from M_b.

4. SELF SIMILARITY MODULO *p*. A beautiful aspect of the picture of Pascal's triangle modulo 2 (FIGURE 2) is that the 'pattern' inside any triangle of black squares is similar in design to that of any subtriangle, though larger in size. If we extend Pascal's triangle to infinitely many rows, and reduce the scale of our picture in half each time that we double the number of rows, then the resulting design is called *self-similar*—that is, our picture can be reproduced by taking any subtriangle and magnifying it.

Many examples of self-similarity have been investigated by Mandelbrot [6]. Such pictures provide simple mathematical models for natural processes which are *self-organizing* (such as the growth of frost on a windowpane).

The process used to generate Pascal's triangle modulo 2 (FIGURE 1) may be modified to give further interesting, and sometimes self-similar, configurations: The patterns given by altering the 'rules' of FIGURE 1 and the number of dimensions in the picture, are known as *cellular automata*. Perhaps the most interesting example of these is Conway's *Game of Life* (see [3]).

Using the final remarks of the previous section we will obtain an interesting generalization, but in a different direction: In cellular automata, the 'cells' have two possible states—0 or 1 (*off* or *on*, *blank* or *asterisk*, *dead* or *alive*). However the entries of Pascal's triangle modulo k can be in any of k possible states—$0, 1, 2, \dots$ or $k-1$. As we shall see below, the idea of self-similarity has an interesting analogue when we allow many states. As a representation of natural processes, such cells may be thought of as containing more complex information than simply whether they are alive or dead; for instance colour, texture or even gender. Human cells are known to contain complex information, which is passed on (and sometimes modified) when they replicate: it may be that this process can be described by automata with a large number of possible states.

We start by reviewing the notion of self-similarity in terms of our growth triangles: The triangle formed by the first 2^{k+1} rows of Pascal's triangle (mod 2), is constructed from three copies of the first 2^k rows, positioned as in FIGURE 3. An easy proof of this may be given using induction: Include in the induction hypothesis the fact that the 2^kth row of Pascal's triangle (mod 2) is made up entirely of 1's. Then the $2^k + 1$th row has 1's on either end, with 0's all the way in between. Directly underneath each of these 1's we get a new triangle, which is formed in exactly the same way as the initial triangle of Pascal's triangle; these two new triangles are independent until they meet. Their meeting occurs at the 2^{k+1}th row, that is the 2^kth row of each of these new triangles and thus, by the induction hypothesis, this row is all 1's. This completes the induction hypothesis.

We now give a similar easy proof for the existence and structure of the growth triangles T_p, for each prime p: Start by noting that the $p + 1$th row of Pascal's triangle (mod p) has 1's on either end with 0's all the way in between (this is a consequence of the elementary fact that p divides $\binom{p}{i}$ for each i, $1 \le i \le p-1$, which may be deduced from (*)). Directly underneath each of these 1's we get a new triangle, which is formed in exactly the same way as the initial triangle of Pascal's triangle and these two new triangles are independent until they meet (which happens in the $2p$th row). Thus the $2p + 1$th row has 1 on either end, 2 in the middle, and 0's all the way in between. Again we find that directly underneath each of these 1's we get a new triangle, which is formed in exactly the same way as the initial triangle of Pascal's triangle, but the values underneath the 2 are twice the values in the initial triangle of Pascal's triangle. These three triangles meet in the $3p$th row, and thus the $3p + 1$th row has 1's on either end, 3's at one-third and two-thirds of the way across and 0's everywhere else. We get the same triangle forming underneath the 1's, but this time 3 times the initial triangle under the 3's. We continue this process, and we see that (by an easily constructed induction hypothesis) the $kp + 1$th row of Pascal's triangle (mod p) is a copy of the $k + 1$th row, with $p - 1$ 0's placed between consecutive entries. Finally, when we do this p times we will have constructed the first p^2 rows of Pascal's triangle (mod p) and we can start the whole process again, this time with the larger triangle formed by the first p^2 rows, as the $p^2 + 1$th row ($= kp + 1$th row with $k = p$) has 1's on either end with 0's all the way in between. We now see how FIGURE 14 explains the growth of Pascal's triangle (mod p).

The pattern just proved has a delightful consequence noted by Long [**5**]: Cut Pascal's triangle up into subtriangles of p^k rows, where these subtriangles have 1 entry in the top row, 2 entries in the second row,... and p^k entries in the p^kth row. The first p^k rows of Pascal's triangle give the only entry in the first row of a triangle of these subtriangles. Rows $p^k + 1$ to $2p^k$ of Pascal's triangle provide the two subtriangles of the second row of this new triangle, after missing out the large inverted triangle of 0's in between. Similarly, rows $(r - 1)p^k + 1$ to rp^k of Pascal's triangle provide the r subtriangles of the rth row of our triangle of subtriangles, after missing out the $r - 1$ large inverted triangles of 0's in between. This resulting triangle of subtriangles has the most extraordinary property—it still obeys the binary rule of FIGURE 1. That is that any two consecutive subtriangles on a row of this triangle add together, componentwise (mod p), to give the triangle immediately underneath.

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Department of Mathematics
University of Georgia
Athens, GA 30602