Circles and Spheres*

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1. On Drawing a Circle. As an aside in a course I took from him many years ago, Professor Murray Protter pointed out that if you want to draw a circle and its center freehand on a blackboard, you are better off first drawing the circle and *then* marking the center, rather than attempting the reverse process. Now, to an ancient Greek mind this piece of advice inevitably raises the question, *given a circle, how can one construct the center using compass and straightedge alone*?

This is a familiar geometric construction from our high school days. Since the perpendicular bisector of any chord passes through the center of the circle, all we have to do is draw two nonparallel chords and construct their perpendicular bisectors; the intersection of the perpendicular bisectors is the center of the circle.

One thing leading to another, as it often does in geometry, we might be led to inscribe a triangle in our circle. The sides of the triangle are chords of the circle, so the three perpendicular bisectors must intersect in one point, the center of the circle (Figure 1).

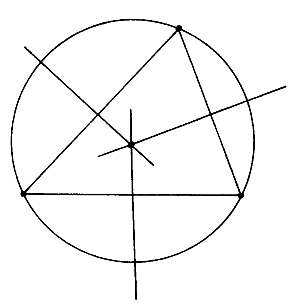

Figure 1.

* This is a revised version of a lecture given on April 28, 1979, at the third annual Saturday Morning Mathematics Conference sponsored by Dean L. H. Lange at San Jose State University.

Suddenly we find ourselves entering the seductive realm of remarkable points, lines, and circles associated with a triangle, where we see that the perpendicular bisectors of the sides of any triangle intersect in the center of the circumscribed circle, from which it readily follows that the altitudes intersect in one point, the orthocenter of the triangle. If we are unable to restrain ourselves, we are soon grappling with the famous nine-point circle, which passes through the feet of the altitudes, the midpoints of the sides, and the midpoints of the line segments from the orthocenter to the vertices.

The nine-point circle has innocently become the center of a minor pedagogical storm. Professor Jean Dieudonné, in lobbying for reform in the geometry curriculum in schools, does not look kindly upon the nine-point circle, and has held it up as an example of the sort of thing that often fills a well-needed gap in the curriculum. Professor Daniel Pedoe, on the other hand, has quite a different opinion about the worth of the nine-point circle, and refers to it as the first really exciting circle to appear in any elementary geometry course (see his book, "Circles" [15], which begins with three nice proofs that a single circle passes through the nine special points). In his monograph on the Schwarz function, Professor Philip Davis [6] uses the nine-point circle as the starting point for a rich and wide-ranging exploration of topics in complex function theory. The remarks on pages 1, 209–210, of his book are of interest with regard to the nine-point circle debate.

I shall avoid embroiling myself in controversy and omit further mention of the nine-point circle. My aim is to give a sampling of a few interesting results involving circles and spheres. Several themes will emerge along the way, and I hope the reader will find toward the end that they form a harmonious blend. A strong unifying thread will be the application of inversion and stereographic projection in analyzing certain problems.

In the course of the exposition occasional reference will be made to books and articles where most of what I say is given thorough treatment. Those wishing to pursue certain topics in depth will find in the final section a more detailed description of where to look.

2. Life Without a Straightedge. If, upon embarking on the task of constructing the center of a given circle, we find that our straightedge has been lost, leaving the compass as our only available tool, we need not despair. There is still hope, for it is known that *every construction possible with straightedge and compass can in fact be done with compass alone*. Naturally, we are no longer able to draw an entire straight line, but what we can do is obtain (as the intersection of circles) those points on a line, using only a compass, that we could have constructed using both straightedge and compass.

The fact that the straightedge can be dispensed with is often called Mascheroni's Theorem, although G. Mohr had published the discovery far earlier. This historical anomaly is just another instance of Boyer's Law: *Mathematical formulas and theorems are usually not named after their original discoverers.* One can find a discussion of Boyer's Law in an article by H. C. Kennedy [11], where it was first formulated.

In any case, we can be sure that *there is a way to find the center of a given circle using compass alone.* Showing this depends heavily on inversion, an important transformation treated in a later section. Although we will not discuss the construction here, the reader can do no better than read the lucid account in Courant and Robbins [3].

Our luck would not be so good if we mislaid the compass rather than the straightedge. It turns out that a straightedge by itself does not suffice to perform all constructions possible with both straightedge and compass. This unsurprising fact notwithstanding, we might still feel a faint glimmering of hope that the more modest task of constructing the center of a given circle is possible with straightedge alone. Alas, such is not the case. As we shall see later, *it is impossible to construct the center of a given circle using straightedge alone.* Stereographic projection will play a key role in demonstrating this demoralizing result.

Before proceeding to a discussion of inversion and stereographic projection, we gather together some useful properties of angles inscribed (and sometimes not inscribed) in circles.

3. Inscribed and Unscribed Angles. If one were to write a history of the circle, a prominent place would have to be given to the theorem of Thales. This is the statement that *an angle inscribed in a semicircle is a right angle*. Legend has it that Thales initiated our long and revered tradition of the logical development of mathematics with the revolutionary step of supplying a proof for his theorem. Since that time it has become customary to provide arguments in support of mathematical assertions, and in the course of history there have been numerous cases of success with the method.

Some economy will be gained in later statements by calling the size of the central angle subtended by an arc of a circle at the center the *bend* of the arc. If an arc has unit radius, then its bend is exactly its length. The inscribed angle theorem, which generalizes the theorem of Thales, says that *the size of an angle inscribed in a circle is half the bend of its intercepted arc*. Thus, in Figure 2 we must have $\sphericalangle APB$ equal to half the bend of the smaller arc from A to B. If the arc from A to B is a semicircle, then the bend is 180°, and so $\sphericalangle APB$ is 90°, giving Thales' theorem.

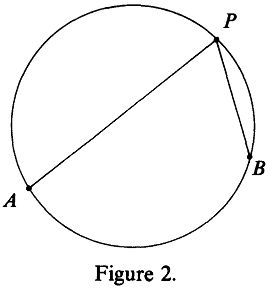

Figure 2.

As a consequence of the inscribed angle theorem, $\sphericalangle APB$ will maintain constant size as the point P moves along the arc of the circle between the fixed points A and B. The converse of this result provides a very useful characterization of circles. Namely, *if A and B are fixed, then the locus of points P such that $\sphericalangle APB$ is constant, with P remaining always on one side of the line through A and B, is an arc of a circle passing through A and B.*

But the inscribed angle theorem is itself a special case of an even more general result, which we shall refer to as the general theorem on intercepted arcs. This is illustrated in Figure 3 where α and β denote the bends of the respective intercepted arcs. In case P is inside the circle, the theorem says that $\theta = (\alpha + \beta)/2$, and in case P is outside we have $\theta = (\alpha - \beta)/2$.

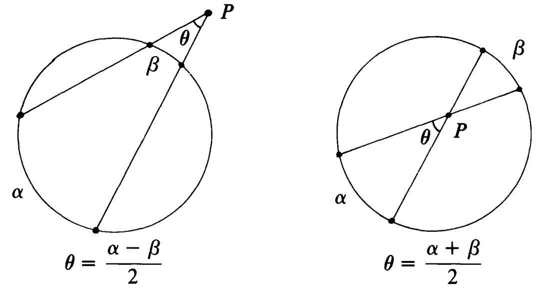

Figure 3.

In the special case where P is on the circle we see that $\beta = 0$, so $\theta = \alpha/2$ and we have the inscribed angle theorem.

A lovely result found in a paper of N. A. Court [4] and first drawn to my attention by Joseph Konhauser can be proved neatly using the general theorem on intercepted arcs. Consider a pair of circles intersecting in points A and B as in Figure 4. From a point P on the upper circle, rays are drawn through A and B respectively intersecting the lower circle in points X and Y.

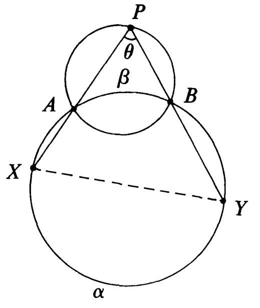

Figure 4.

Then we have that *the length XY remains constant as P is allowed to vary on the upper circle* (over those positions where the rays intersect the lower circle as in the picture).

The proof is immediate from Figure 4. As P moves on the upper circle, θ remains constant (by the inscribed angle theorem applied to the upper circle.). But $\theta = (\alpha - \beta)/2$ by the general theorem on intercepted arcs, and β is fixed; hence α must be constant as P varies. This implies that XY is constant.

Court's elegant proof uses the inscribed angle theorem directly. The preceding lightning stroke variant was shown to me by Steve Erfle.

A generalization of this to three dimensions is treated in Court's paper. Here one is confronted with a pair of spheres intersecting along a circle, forming a snowman as in Figure 5. Let P be a point on the upper sphere. Through each point on the circle of intersection extend a ray starting at P and intersecting the lower sphere in some point. The resulting collection of rays forms a cone, with vertex P, passing through the circle of intersection of the spheres and intersecting the lower sphere in another circle.

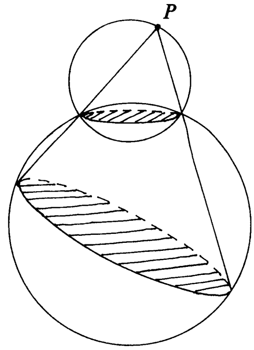

Figure 5.

As P varies on the upper sphere, our experience with the two-dimensional analogue leads us to suspect that the diameter of the lower circle remains constant. This is indeed the case, as is easily shown by intersecting the snowman with the plane determined by P and the centers of the spheres, thus reducing the problem to the situation depicted in Figure 4.

If you feel some uneasiness about the argument, it may be caused by our glib acceptance of the statement that the cone actually intersects the lower sphere in a circle. What is needed to complete the proof and put our minds at ease is a proof of the following intriguing fact: *if the generators of a cone enter a sphere along a circle, then they leave along a circle.*

How do we prove it? We finally turn to a discussion of that geometric transformation called inversion, which has been waiting impatiently in the wings, and whose properties will later enable us to establish the preceding result quickly and elegantly.

4. Inversion. Given a line λ in the plane, the operation of reflection across λ is a transformation of the plane onto itself leaving each point on λ fixed and sending every other point to its mirror image on the other side of λ. The transformation exchanges the positions of points that are mirror images of each other with respect to λ. We now describe an analogous transformation, with the line replaced by a circle.

Let γ be a circle of radius r centered at point O. For each point $P \neq 0$, let P' be the point on the ray $\overrightarrow{OP}$ such that

$$(OP)(OP') = r^2.$$

The transformation of the plane that sends each point $P \neq 0$ to its mate P' is called *inversion* with respect to γ. The center O of the circle is called the *center of inversion*.

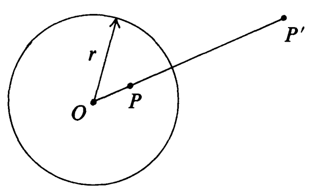

Figure 6.

To clarify the analogy with reflection across a line, first note that if P belongs to γ then $P' = P$, so inversion leaves fixed each point of γ. Note also that if P' is the image of P, then the definition implies that P is the image of P', and points inside γ exchange positions with their mates outside. The analogy is so compelling that inversion with respect to a circle γ is sometimes referred to as reflection across γ. A major difference however is that while reflection across a line preserves the Euclidean distances between points, inversion with respect to a circle does not. If P and Q are points near O, note that their images P' and Q' are very far apart.

An especially useful property for our purposes is that *the image of a circle or a straight line under inversion is again a circle or a straight line*. For example, it is obvious from the definition that a line through O is sent into itself. To see what happens in other cases, we make the following basic observation.

Let P' and A' be the images of P and A under inversion with respect to γ, as in Figure 7. Then

$$(OP)(OP') = r^2 = (OA)(OA'),$$

so

$$\frac{OP}{OA} = \frac{OA'}{OP'}.$$

Thus, by the side-angle-side similarity criterion for triangles, since $\triangle OPA$ and $\triangle OA'P'$ have $\measuredangle POA$ in common, we see that $\triangle POA$ is similar to $\triangle A'OP'$. In particular then

$$\measuredangle OPA = \measuredangle OA'P'.$$

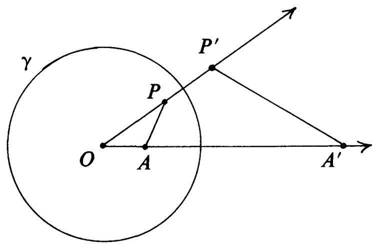

Figure 7.

As a quick application of this, note that if A is kept fixed and P allowed to vary over a circle passing through O and A, then $\measuredangle OPA$ will maintain constant size (because of the inscribed angle theorem). Hence P' will vary in such a way that the size of $\measuredangle OA'P'$ remains constant. But that means P' moves along a straight line through A'. Thus *the image under inversion of a circle passing through the center of inversion is a straight line.*

To see what inversion with respect to γ does to a circle not passing through O, suppose A' and B' are the respective images of A and B, where O, A, B are collinear, as depicted in Figure 8.

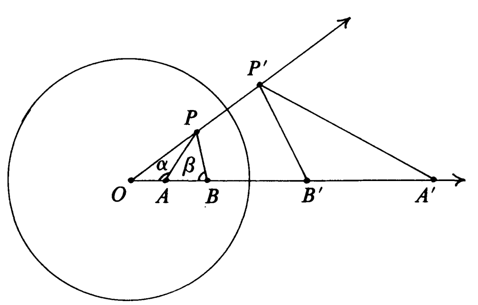

Figure 8.

By our preceding discussion we have $\measuredangle OP'A' = \measuredangle OAP = \alpha$, and $\measuredangle OP'B' = \measuredangle OBP = \beta$. Hence

$$\measuredangle A'P'B' = \measuredangle OP'A' - \measuredangle OP'B' = \alpha - \beta = \measuredangle APB,$$

where the last equality follows from the fact that the exterior angle α in $\triangle APB$ is the sum of the opposite interior angles. If now A and B are fixed and P allowed to vary over a circle passing through A and B, then $\sphericalangle APB$ will maintain constant size. Consequently $\sphericalangle A'P'B'$ will be constant, so P' will move along a circle passing through A' and B'. Thus *the image under inversion of a circle not passing through the center of inversion is a circle.*

Although inversion does not preserve Euclidean distances, it does have in common with ordinary reflection the property of *preserving angles* between curves. That is, if σ_1 and σ_2 are curves intersecting at P, and σ_1' and σ_2' their images under inversion, then the angle of intersection of σ_1' and σ_2' at P' is the same size as the angle of intersection of σ_1 and σ_2 at P (Figure 9. Note that the angle of intersection between two curves is by definition the angle between their tangent lines at the intersection point).

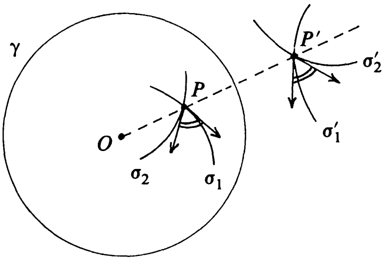

Figure 9.

We refer the reader to the listed references for a proof of this property.

The crucial role inversion plays in the Poincaré model for non-Euclidean geometry, discussed in the next section, depends on the following property. Suppose σ is a circle intersecting γ at points A and B, and σ is *orthogonal* to γ, that is, suppose σ intersects γ at right angles at A and B.

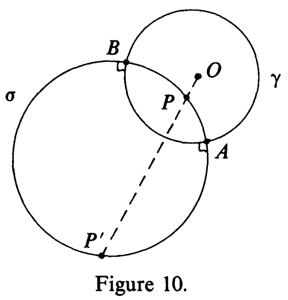

Figure 10.

Then *inversion with respect to γ sends σ onto itself*, with points on the outer arc of σ exchanging positions with points on the inner arc. The reader is invited to prove this as an exercise, using the fact that inversion preserves angles and sends circles to circles.

5. Reflections in Non-Euclidean Geometry. We digress for a minicourse in the rudiments of non-Euclidean geometry in the guise of the Poincaré model.

Fix a circle σ in the plane. The "points" of the Poincaré model, which will comprise our non-Euclidean "plane", are just the points interior to σ, The "lines" of the model are defined to be arcs of circles inside σ and *orthogonal to σ at their endpoints*. Diameters of σ are included among the "lines" of the model.

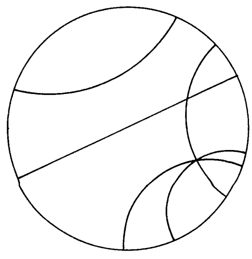

Figure 11.

The endpoints of these arcs and diameters are not considered part of a "line" since the "points" of the model are only those belonging to the *interior* of σ.

Given a "line" γ in the model, we define a non-Euclidean "reflection across γ" to be inversion with respect to the circle determined by γ. Since σ is orthogonal to γ, we see that "reflection" sends the interior of σ, the non-Euclidean "plane", onto itself, with points on γ remaining fixed and other points exchanging positions with their partners on the other side of γ. Thus, in a qualitative sense, non-Euclidean reflection behaves just like Euclidean reflection across a line.

We will not spend time discussing it here, but it is possible to define a non-Euclidean "distance" between points inside σ in such a way that non-Euclidean reflection is "distance" preserving. If we define the non-Euclidean "angle" between "lines" to be simply the ordinary Euclidean angle, then "reflections", being inversions, are in addition "angle" preserving, and we have a complete analogy with Euclidean reflections, both quantitatively and qualitatively.

With "point", "line", "angle", and "distance" so defined, it can be checked that this model satisfies all the usual axioms of Euclidean geometry save one, the Parallel Postulate. A glance at Figure 11 should convince you that given a "line" γ and a point P not of γ, there is not just one "line," but infinitely many "lines" through P not meeting γ, thus violating the Parallel Postulate.

A "circle" in non-Euclidean geometry is defined, as in Euclidean geometry, to be the locus of points "equidistant" (using the non-Euclidean distance) from a given point P, called the "center". In view of the fact that "distance" in the Poincaré model is not Euclidean, one might expect "circles" in the model to look somewhat peculiar. It is mildly surprising that this is not the case. In the Poincaré model, *"circles" are ordinary Euclidean circles*. To be sure, the non-Euclidean "center" of a "circle" is not the same as its Euclidean center *except* when it is a circle concentric with σ. Figure 12 depicts a family of "concentric" non-Euclidean "circles" with their non-Euclidean "radii" streaming out from their common "center".

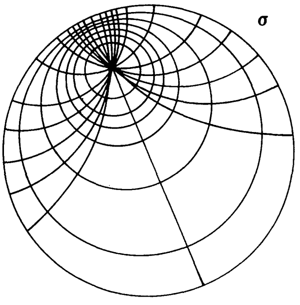

Figure 12.

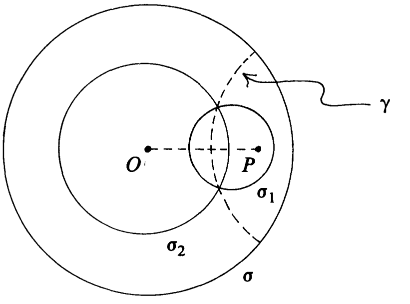

Figure 13.

Suppose σ_1 is a circle sitting inside σ. Let P be the non-Euclidean "center" of σ_1 and let O be the center of σ. Let γ be the non-Euclidean "perpendicular bisector" of the line segment OP (Figure 13). Then "reflection" across γ will send P to O and will send σ_1 to another circle σ_2 centered at O.

Interpreting what all this means in purely Euclidean terms, we see that inversion with respect to the circle determined by γ has transformed the pair of *nonconcentric* circles σ and σ_1 to a pair of *concentric* circles σ and σ_2. In other words, *given a pair of nonintersecting circles, one inside the other, there is an inversion that sends them to a pair of concentric circles.*

6. Steiner's Porism. Steiner's porism is not a dermatologist's nightmare, as you might imagine, but a beautiful theorem about the chain of circles one can draw between two given circles.

Consider two nonconcentric circles γ_1 and γ_2, with γ_2 interior to γ_1. Draw a circle between them, tangent to both. This is the beginning of a chain of circles formed by drawing successive circles between γ_1 and γ_2, each tangent to the preceding one and also to γ_1 and γ_2. The result is a *Steiner chain.*

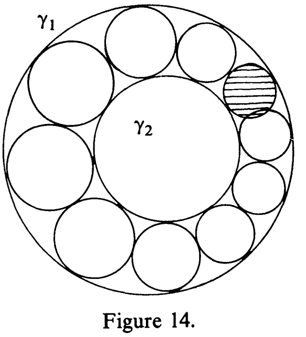

Figure 14.

In Figure 14 the initial circle of a chain is indicated by shading, and the chain is built moving around in the counterclockwise direction. Note that our chain has not closed up perfectly. The last circle we drew was not tangent to the initial circle. Bad luck! Perhaps if we move the initial circle a bit and start over, we will obtain a perfectly closed chain on the next go-around? Steiner's porism tells us not to waste our time. With γ_1 and γ_2 given, *if the chain does not close up perfectly starting with some initial circle, then it will not close up no matter where we start. Moreover, if the chain closes from one starting position, it will close up starting from any other.*

The following proof is a nice application of what we learned in the previous section. Inversion with respect to some circle sends the pair γ_1 and γ_2 to a pair of concentric circles. Since inversion sends circles to circles (none passing through the center of inversion), our Steiner chain is sent to a Steiner chain between the concentric circles. But it is obvious that Steiner chains between concentric circles either always close up perfectly or always do not, so the same is true for chains between γ_1 and γ_2.

Observe, by the way, that the Steiner chain might close up after winding around several times. The proof shows that the same would happen from any starting position.

The general technique used in the proof is worth noting. A complicated configuration can often be brought to a simpler form by an appropriate transformation. Analysis of the simpler situation then enables us to solve our original problem. This "transform-solve-invert" procedure is nicely formulated in an article of M. S. Klamkin and D. J. Newman [12], where numerous examples from analysis, algebra, and geometry, including Steiner's porism, are given.

7. Inversion in Space. This has to do with the 3-dimensional generalization of the geometric transformation we discussed in Section 4 and has nothing to do with smog control on Jupiter.

Let Γ be a sphere of radius r centered at a point O in space. We define *inversion* with respect to Γ exactly as we did in the plane. This transformation sends each point $P \neq O$ in space to a point P' on the same ray with endpoint at O such that $(OP)(OP') = r^2$. The definition immediately implies that each point on Γ is left fixed and points inside Γ are interchanged with points outside. There is a close analogy with the operation of reflection across a plane in space.

A plane passing through O is sent into itself under inversion with respect to Γ, and in fact its points undergo a transformation which is precisely inversion with respect to the circle in which it intersects Γ. More generally, it can be shown that *inversion with respect to* Γ *sends spheres and planes to spheres and planes.*

For example, if Σ is a sphere passing through the center of inversion O, then inversion with respect to Γ sends Σ to a plane π. This is illustrated in Figure 15, which we would obtain by intersecting the spheres by a plane containing their centers. What we actually see in the figure is a circle passing through the center of inversion being sent to a straight line under planar inversion. Rotating the figure about the vertical line through the centers shows that a sphere through the center of inversion is sent to a plane.

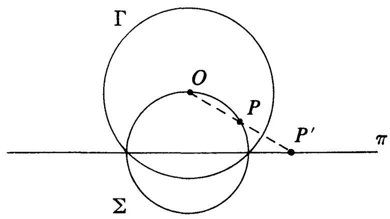

Figure 15.

Since circles and straight lines can be represented as intersections of spheres and planes, it follows that *the image of a circle or straight line under inversion with respect to* Γ *is again a circle or a straight line.*

Next we consider what happens to a sphere orthogonal to the inversion sphere (two spheres are orthogonal if their radii to each point on the circle of intersection are perpendicular). Begin with two intersecting circles σ and γ lying in the same plane, with σ orthogonal to γ, as in Figure 10. As we rotate the figure about the line joining the centers of the circles, σ sweeps out a sphere Σ and γ a sphere Γ, with Σ orthogonal to Γ. Since inversion with respect to γ sends σ onto itself (as indicated in Figure 10, any point P on σ is sent to a point P' again on σ), it is clear that inversion with respect to Γ sends Σ onto itself. To summarize, *if* Σ *is a sphere orthogonal to* Γ, *then inversion with respect to* Γ *sends* Σ *onto itself.*

Now consider a given sphere Σ and a given point O outside Σ. It will be easy to convince yourself that there is a sphere Γ centered at O which is orthogonal to Σ (hint: a tangent line segment drawn from O to Σ will be a radius of Γ). We have just seen that inversion with respect to Γ sends Σ onto itself. Thus if P is a point of Σ inside Γ, then its image under inversion with respect to Γ is a point P' on Σ outside Γ (Figure 16).

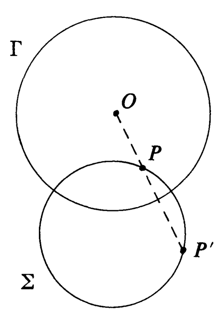

Figure 16.

Imagine now that P moves along a circle lying on Σ. Then the corresponding image points P' must also trace out a circle, since inversion sends circles to circles (none passing through the center of inversion). But the rays $\overrightarrow{OP}$ are the generators of a cone, with vertex O, entering Σ along a circle, and what we have just observed is that they must leave Σ along a circle. So we have proved the result we wanted at the end of Section 3, namely that *a cone entering a sphere along a circle must exit along a circle.*

In case you are interested in more substantial applications, consult the definitive work of H. Petard [16], where numerous results from higher mathematics are applied to the problems of big game hunting, and in particular to catching a lion in the desert. Inversion provides a powerful weapon here. Simply take a spherical cage to the desert and lock yourself inside. Now perform an inversion with respect to the cage. You are then outside and the lion is trapped inside. Caution: keep away from the center of the cage while performing the inversion.

8. Stereographic Projection. Imagine a sphere Σ with a horizontal plane π passing through its center. Viewing π as the equatorial plane of Σ, let O be the North Pole. *Stereographic projection* is the transformation from Σ into π that sends each point $P \neq O$ on Σ to the point P' where the ray $\overrightarrow{OP}$ intersects π. One of the most remarkable properties of this transformation is that *it sends any circle lying on* Σ *and not passing through* O, *to a circle in* π (Figure 17). A circle on Σ passing through O is sent to the straight line in which its plane intersects π.

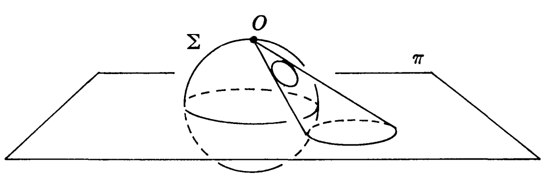

Figure 17.

To prove the circle preserving property, let Γ be a sphere centered at O and passing through the equator of Σ. Then we have a situation like that depicted in Figure 15. Since Σ is a sphere passing through O, inversion with respect to Γ sends Σ into π. Thus a point P on Σ is sent to a point P' in π by stereographic projection if and only if P is sent to P' under inversion with respect to Γ. Since inversion sends circles on Σ, not passing through O, to circles in π, the same is true for stereographic projection, as we wanted to prove.

As a rather peculiar corollary of this we see that *there exists a cone with two nonparallel cross-sections that are circles.* In Figure 17 the cone with vertex O has a circular cross-section with the base plane π and also with the plane of the circle on Σ. As a matter of fact we have already encountered this phenomenon in Figure 5, since there the planes of the two circles are not parallel unless P belongs to the line through the centers of the spheres. It can be shown that for such a cone the line joining the centers of the nonparallel circles does *not* pass through the vertex. We use this observation to prove that *it is not possible to construct the center of a given circle using straightedge alone.*

The proof is by contradiction. Start with a cone with vertex P having two nonparallel circular cross-sections by planes π and π' as in Figure 18.

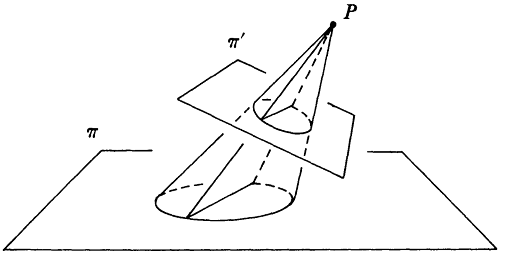

Figure 18.

Given any line in π, it is contained in a plane passing through P and intersecting π' in another line. This establishes a one-to-one correspondence between lines in π and lines in π'. Now assume that we have a method for constructing the center of a circle by drawing a sequence of lines with our straightedge. Application of our method to the base circle in π results in a sequence of lines, two of which, say t_1 and t_2, intersect in the center of this circle. But our one-to-one correspondence gives a corresponding sequence of lines *formed by the same construction method* in π'. Consequently the lines in π' corresponding to t_1 and t_2 should intersect in the center of the upper circle. But that could happen only if the line joining the centers of the two circles passed through P, which we know is not true. This contradiction shows that the construction is impossible.

9. Transform-Solve-Invert

Problem. Given points A, B, C, D in a plane, not all on the same circle or line, show that there exist two nonintersecting circles in the plane such that one passes through A and C and the other passes through B and D.

Solution. This allows a model application of the transform-solve-invert technique. We transform the problem to the case where the points are on a sphere, find circles solving the easier problem there, and transform back to get circles solving the original problem. To do all this, use Figure 17 and view A, B, C, D as points in the plane π. They are the images under stereographic projection of points A_0, B_0, C_0, D_0 on the sphere Σ. Now A_0, B_0, C_0, D_0 could not all lie on a plane, since otherwise they would lie on a circle (the intersection of that plane with Σ) and the image of this circle under stereographic projection would give a circle or line through A, B, C, D. Hence the line through A_0 and C_0 does not intersect the line through B_0 and D_0. Thus there exist *parallel* planes π_1 and π_2 with A_0C_0 in π_1 and B_0D_0 in π_2 (convince yourself that two skew lines in space are contained in some pair of parallel planes). These planes intersect Σ in a pair of nonintersecting circles, one through A_0 and C_0, the other through B_0 and D_0. Stereographic projection now sends these circles to a pair of circles in π that solve our problem.

The reader may have noted that the solution is marred by a minor technical difficulty, namely, one of the planes π_1 or π_2 might pass through O. However, a slight rotation of the plane eliminates this annoyance and still gives us circles in π satisfying our requirements.

The following example, from Ogilvy [14], gives a nice application of the transform-solve-invert technique using inversion.

Problem. Suppose γ_1, γ_2, γ_3 are circles in a plane, and suppose they intersect in a common point O. Suppose further that the common chord of γ_1 and γ_2 passes through the center of γ_3, and the common chord of γ_2 and γ_3 passes through the center of γ_1. Prove then that the common chord of γ_3 and γ_1 must pass through the center of γ_2.

Solution. Exercise for the reader (hint: perform an inversion with respect to a circle centered at O. The configuration in the problem is sent to a familiar configuration consisting entirely of straight lines).

Problem. Prove that the points of contact of adjacent circles in any Steiner chain all lie on a common circle.

Solution. This is too easy. We do not even give a hint for it.

10. The Reference Desk. The books of Coxeter and Greitzer [5], Eves [7], Ogilvy [14], and Pedoe [15] contain especially readable accounts of almost everything in this article. For further material on constructions, including an extensive treatment of constructing with compass alone, consult Courant and Robbins [3] and Eves [7]. These books also contain much material on inversions and non-Euclidean geometry. See Greenberg [8] for a further treatment of non-Euclidean geometry and the Poincaré model. The Steiner porism and some beautiful related material is developed in an essay of Honsberger [10] and, of course, in Coxeter and Greitzer, Eves, Ogilvy, and Pedoe. The booklet of Bakel'man [1] is devoted to inversions (obviously). In addition to the article of Klamkin and Newman, the reader should consult the book of Eves for numerous examples of the transform-solve-invert technique in geometry. Some examples also occur in an article by L. H. Lange and the author [2]. J. D. North [13] deals with stereographic projection in its application to the astrolabe. Hilbert and Cohn-Vossen [9] gives an account, unsurpassed in elegance and purity, of geometry in general and of inversions, non-Euclidean geometry, and stereographic projection in particular.

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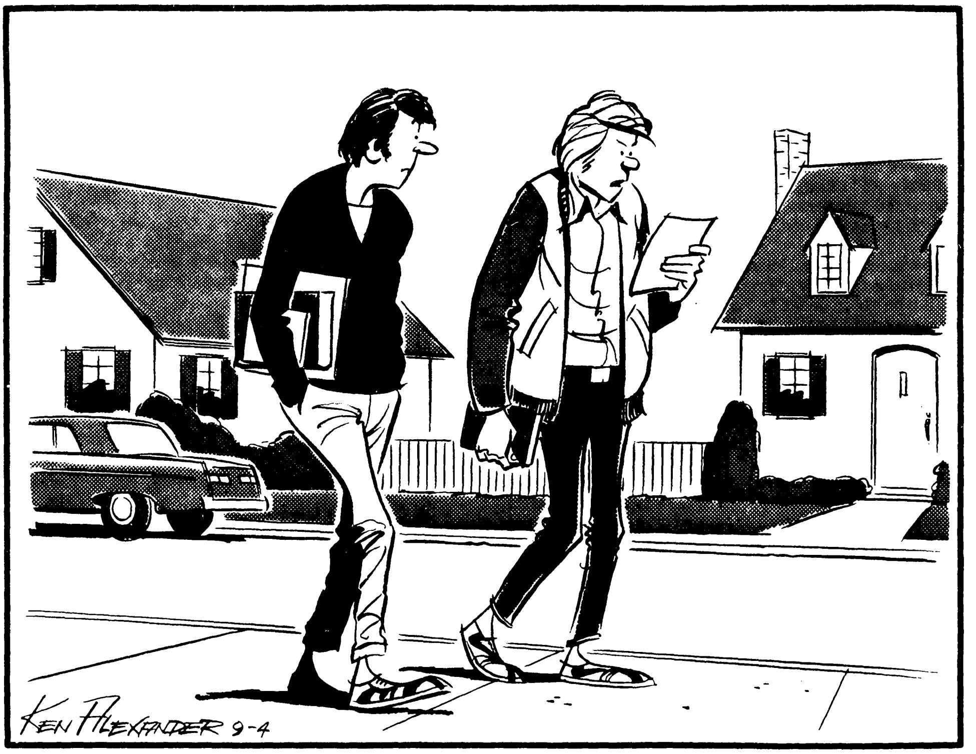

'I got pottery, art appreciation, choir, family life, PE and something called algebra'.