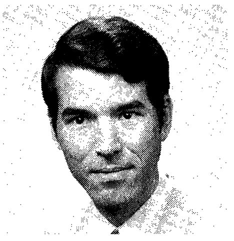
The Fractal Geometry of Mandelbrot

Anthony Barcellos



Anthony Barcellos is an analyst with the Commission on State Finance, California's principal economic forecasting agency. He previously served on the staff of the California Senate. Mr. Barcellos has a B.S. in mathematics from Caltech and an M.S. from the California State University, Fresno. While a graduate student and staff instructor in mathematics at the University of California at Davis, he was awarded both the departmental and campus awards for distinguished teaching. His interviews of Martin Gardner, Stanislaw Ulam, and Benoit Mandelbrot will appear in "Mathematical People" (Birkhäuser Boston), and he is presently completing a book on Conway's game of Life.

Although fractions have been numbers in good standing for thousands of years, there are still many places where they are not welcome. Occasionally they may seek to annex forbidden ground to their territory, but the results are so absurd that we take the ill-mannered intrusion in good humor and worry about it not at all. That is why we smile to hear that the average American family has 2.3 children, or that car ridership has increased to 1.7 passengers per vehicle on certain roads. (Not many of us would care to have the 7/10 of a person as part of our carpool.)

Of course, these fractions are merely statistical artifacts. We know what they measure and what they mean, so they don't bother us. It is easy, however, to think of examples where fractions (which we can construe generously to include irrationals) simply do not belong at all: What would you think of someone who said he was learning how to solve π equations in π unknowns? How about someone who claims to study objects of dimension $\log 4/\log 3$?

Unless you already knew, or guessed what was coming, you probably laughed equally at both absurd ideas. Nevertheless, we are about to take a serious look at the latter notion, for it is here that fractional numbers now appear to have made their greatest territorial gains since algebra accepted them as exponents. The one-dimensional line, two-dimensional surface, and three-dimensional solid have some strange new neighbors. Mathematics has been invaded by creatures from the fractional dimensions.

1. Snowflakes and Flowsnakes

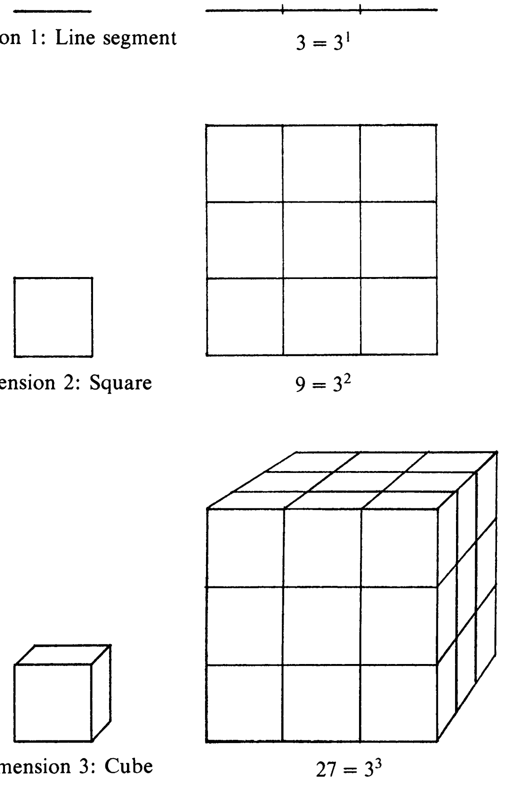
> But in the process of measurement it turns out, generally speaking, that the chosen unit is not contained in the measured magnitude an integral number of times, so that a simple calculation of the number of units is not sufficient. It becomes necessary to divide up the unit of measurement in order to express the magnitude more accurately by parts of the unit; that is, no longer by whole numbers but by fractions. It was in this way that fractions actually arose, as is shown by an analysis of historical and other data. [Aleksandrov, Vol. 1, p. 24]

Since most of us are accustomed to thinking of dimension as given by the number of independent directions or degrees of freedom, it is first necessary to find a characterization of dimension that will permit generalization. Fortunately, this turns out to be a simple consideration of the effects of dimension on the measure of similar geometric shapes.

Consider a line segment of unit length. If we triple its length—that is, expand it by a scaling factor of 3—we get a line segment of length three. This line segment, of course, contains three congruent components (that is, three copies of the original unit length). For reasons that will become clear very soon, we note that $3 = 3^1$.

Consider a unit square. If we expand the square by a scaling factor of 3—that is, we triple its sides—we get a square (Figure 1) whose area is 9 times as great. Equivalently, this means that the expanded square consists of nine congruent components (that is, nine copies of the original square). Observe that $9 = 3^2$.

Finally, consider a unit cube. If we expand it by a scaling factor of 3, we get a cube consisting of 27 congruent components. (Look at Figure 1 again or get out your Rubik's cube.) In this case, $27 = 3^3$.

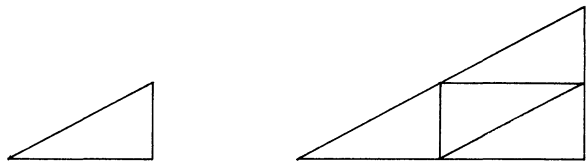


Dimension 1: Line segment $3 = 3^1$

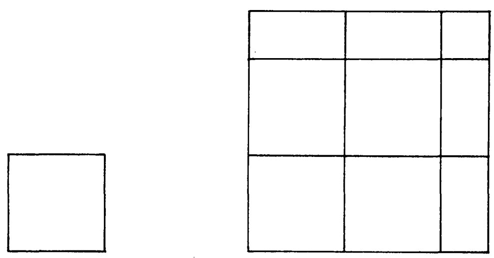
Dimension 2: Square $9 = 3^2$

Dimension 3: Cube $27 = 3^3$

Observe that in each of the three cases illustrated in Figure 1, we have a dimension d, a scaling factor s, and a number of components N, which satisfy the equation $N = s^d$. Other examples of this rule are shown in Figure 2, where a square expanded by the scaling factor 2.5 has been dissected into 6.25 components: $4(1) + 4(0.5) + 0.25 = 6.25$.



$4 = 2^2$



$6.25 = (2.5)^2$

$\sqrt{2} = (\sqrt{2})^1$

Figure 2.

For the present, we have been careful to confine our attentions to objects which are *self-similar*; that is, the expanded objects could be dissected into congruent components similar to the original object. (This is even true of the expanded square in Figure 2, whose larger components may be further broken up into copies of the small square in the upper right-hand corner.) Circles and cones are simple examples of objects which are not self-similar. Nevertheless, the relation which links measure (area, volume, etc.,) to dimension and scale continues to hold: tripling the radius of a circle increases its area by a factor of $9 = 3^2$, even though there is no way to cut the expanded circle into nine circles congruent to the original. Similarly, tripling the radius and height of a cone increases its volume by a factor of $27 = 3^3$. All the familiar geometric shapes seem to fall into this pattern.

As demonstrated in Figure 2, neither N nor s need necessarily be an integer for the equation $N = s^d$ to be valid. This is not particularly surprising because the idea of geometric similarity is easy to accept for nonintegral scaling factors. Dimension, however, is another matter; we expect d to be an integer. Let us see if it is possible to produce a geometrical object whose expansion by a factor s can be dissected into N components such that $d = \log N/\log s$ is not an integer. (It doesn't matter what base one chooses for the logarithms. A few examples worked on a calculator should be convincing of that, and the relation $\log_b x = \log_a x/\log_a b$ can be used to prove it.)

Perhaps unexpectedly, our first example of this new idea of fractional dimension turns out to be a senior citizen. In 1904, the Swedish mathematician Helge von Koch produced an interesting geometric construction which is called the "snowflake." The Koch snowflake (Figure 3) begins with an equilateral triangle of unit side; the first step in its construction replaces each side by a broken line of length 4/3. This step in the construction is equivalent to adjoining equilateral triangle of side 1/3 to each side of the original triangle. Each subsequent stage of the construction proceeds similarly, as illustrated, and the snowflake consists of the region obtained by taking the construction to the limit.

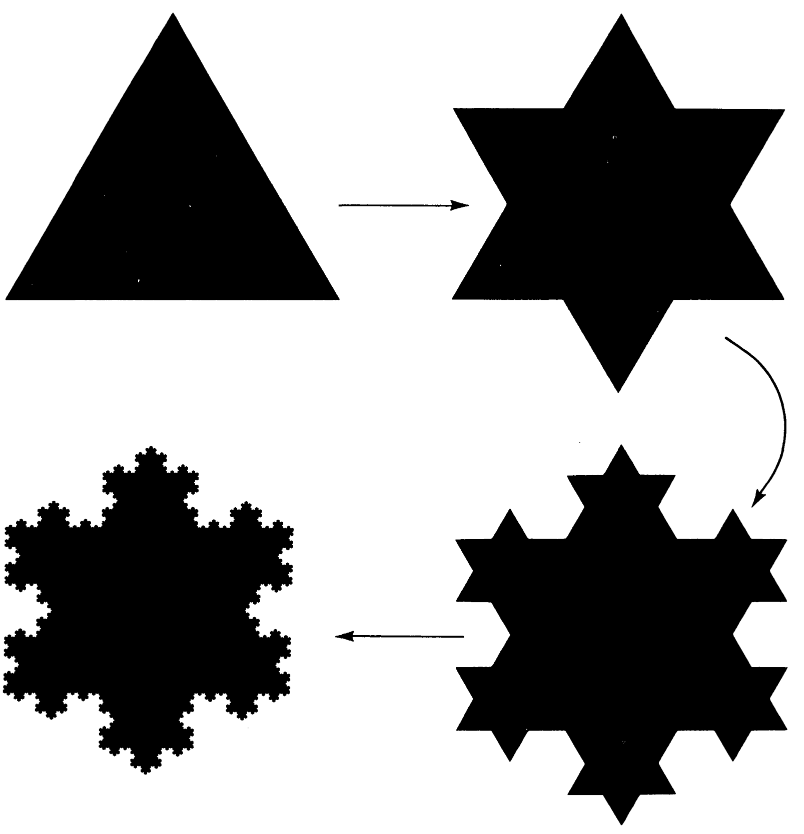


Figure 3.

The boundary of the snowflake is called the "snowflake curve," and its length is easily seen to be infinite: At each stage of the construction, the perimeter of the region is increased by a factor of 4/3. In the limit, the factor $(4/3)^n$ increases without bound. However, the area of the region enclosed by the curve is finite. In fact, some careful counting and the formula for the sum of a geometric series show that the area of the snowflake is

$$\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\left(\frac{1}{3} + \frac{4}{3^3} + \frac{4^2}{3^5} + \frac{4^3}{3^7} + \cdots\right) = \frac{2\sqrt{3}}{5},$$

which is $8/5$ the area of the original triangle.

Although this demonstrates that the snowflake is a peculiar object, it certainly doesn't compel us to believe that the boundary is anything other than what it appears to be: a surprisingly long (and rather crinkly) one-dimensional curve. Furthermore, while the snowflake curve is very pretty, it is not self-similar. To apply our formula, $d = \log N/\log s$, for the "self-similarity dimension," it is necessary to take a slightly closer look at a portion of the snowflake curve that *is* self-similar.

Figure 4 shows the construction for the snowflake curve confined to a line segment (which we could consider to be one side of the triangle in Figure 3). At each stage, we replace line segments with broken lines which are $4/3$ as long. We see that the limit curve consists of $N = 4$ components, each of which is a scaled-down version of this limit curve by the scaling factor $s = 3$. In other words, one-fourth of this limit curve can be scaled up by a factor of three to produce the entire curve again! Therefore, by our formula, this curve has self-similarity dimension

$$d = \frac{\log 4}{\log 3} \approx 1.2619.$$

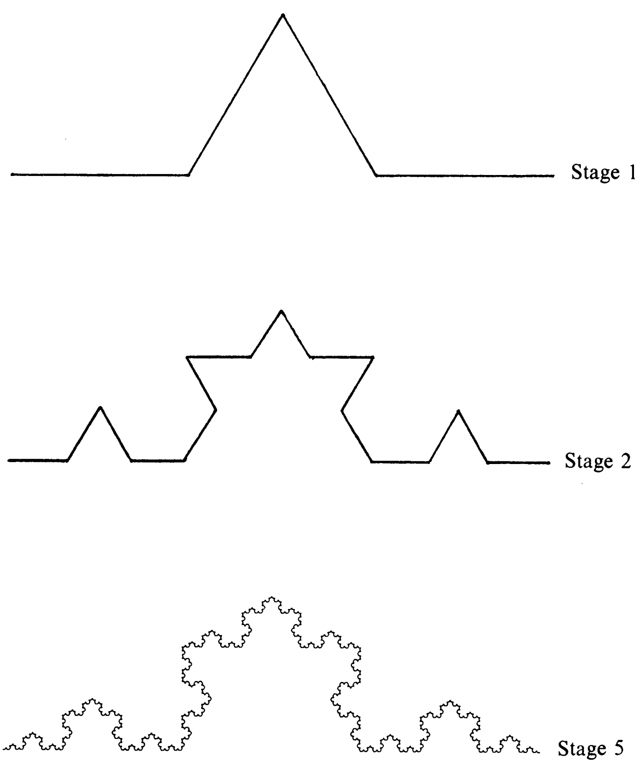


Figure 4.

The snowflake curve is thus our first example of a *fractal*, a term coined by Benoit B. Mandelbrot to describe objects of anomalous dimension. Mandelbrot may best be thought of as the true father of fractals, although he has gone to great lengths to draw together such early examples as the snowflake curve and to credit their discoverers as pioneers in the field. As a unified subject, however, fractal geometry simply would not exist had Mandelbrot not provided the framework, terminology, and techniques for its investigation. Later we shall see that Mandelbrot's definition of "fractal dimension" is not the same as the "self-similarity dimension" which we are using; they are usually equal, however, and the latter is easier to compute.

With but a single example in our fractal collection, it is too early to make claims concerning the use or nature of fractals—or to discuss Mandelbrot's view of the fractal nature of nature. Indeed, at this point it could just as easily be argued that the snowflake curve is a geometric curiosity and little more. It was so regarded until Mandelbrot advanced the unifying idea of "fractal" and many other examples began to appear. Let us look at an example whose construction is more complicated, but whose fractal dimension is easily computed: the flowsnake of R. William Gosper. [Gardner, 1976.]

In Figure 5(a), we have the initial step in the construction of the flowsnake curve: chord AC of the regular hexagon $ABCDEF$ is replaced by a broken line with seven segments, each of which is a chord of a similar hexagon. Inspection of the figure (note, for instance, that $\triangle AGM$ is congruent to $\triangle BHM$) shows that the seven small hexagons together have exactly the same area as $ABCDEF$; the scaling factor must therefore be $\sqrt{7}$. We can now repeat the process, replacing each of these seven chords by broken lines consisting of seven chords of smaller hexagons—as, for example, chord AH being broken into the seven chords shown in Figure 5(b).

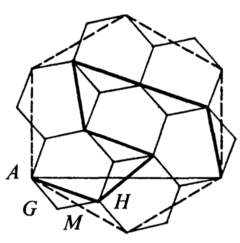


Figure 5(b).

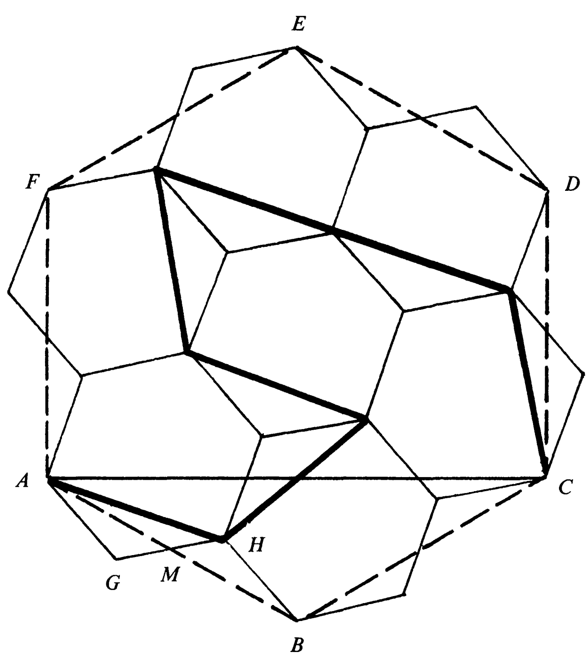


Figure 5(a).

The first four stages of the construction are displayed in Figure 6. At each stage, we replace each line segment from the previous stage by $N = 7$ segments scaled down by a factor $s = \sqrt{7}$. Therefore, the self-similarity dimension of the flowsnake is

$$d = \frac{\log 7}{\log\sqrt{7}} = 2.$$

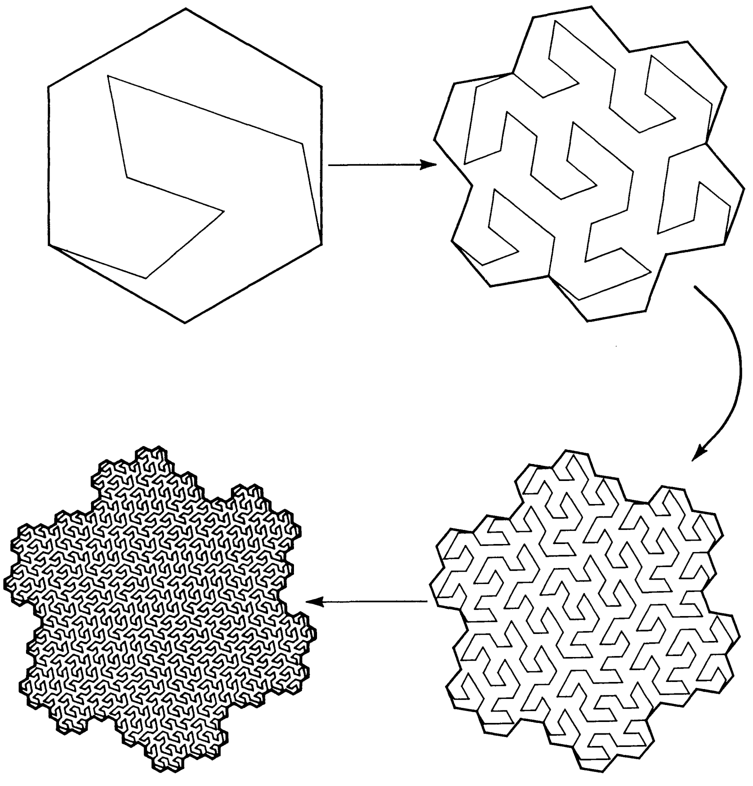


Figure 6.

It can be shown that the flowsnake is two-dimensional in another sense: It is an example of a Peano curve—a curve which passes through every point of a planar region.

Is the flowsnake a fractal? While it is a curious object (a limit curve which fills a planar region), its self-similarity dimension is a whole number. Therefore, by our use of the term, the flowsnake is not a fractal. We will later see, however, that Mandelbrot's formal definition of the term allows certain anomalous objects of integral dimension to be called "fractals," and we will use Gosper's flowsnake to construct an example of one. (This would seem to be a counterattack by the dispossessed whole numbers.)

We have yet to produce another object of nonintegral self-similarity dimension. As one such candidate, consider the *boundary* of the flowsnake—the curve which bounds the planar region filled by the flowsnake curve. This type of boundary, in the customary state of affairs, would be one-dimensional. Now look again at Figure 5. In constructing the flowsnake curve we were also constructing the flowsnake boundary: each side of the initial hexagon $ABCDEF$ is replaced by $N = 3$ segments scaled down by a factor of $s = \sqrt{7}$. (For example, AB is replaced by the path $AGHB$.) This is the basic iterative step; it is further illustrated in Figure 6, where the intermediate stages of the boundary are shown as being formed by the outer edges of the same hexagons being used to generate the intermediate stages of the flowsnake curve. According to our definition, the flowsnake boundary has a self-similarity dimension of

$$d = \frac{\log 3}{\log\sqrt{7}} \approx 1.1292.$$

It is possible to demonstrate—in a surprisingly straightforward way—that this fractional dimension really does make sense. Whereas the fractal dimension of the snowflake curve may have seemed to follow from a mechanical application of our formula for dimension, the fractal dimension of the flowsnake boundary has a compelling geometric demonstration.

As Figure 7 illustrates, the planar region filled by the flowsnake can be decomposed into $N = 7$ congruent subregions. Therefore, each of these subregions must be

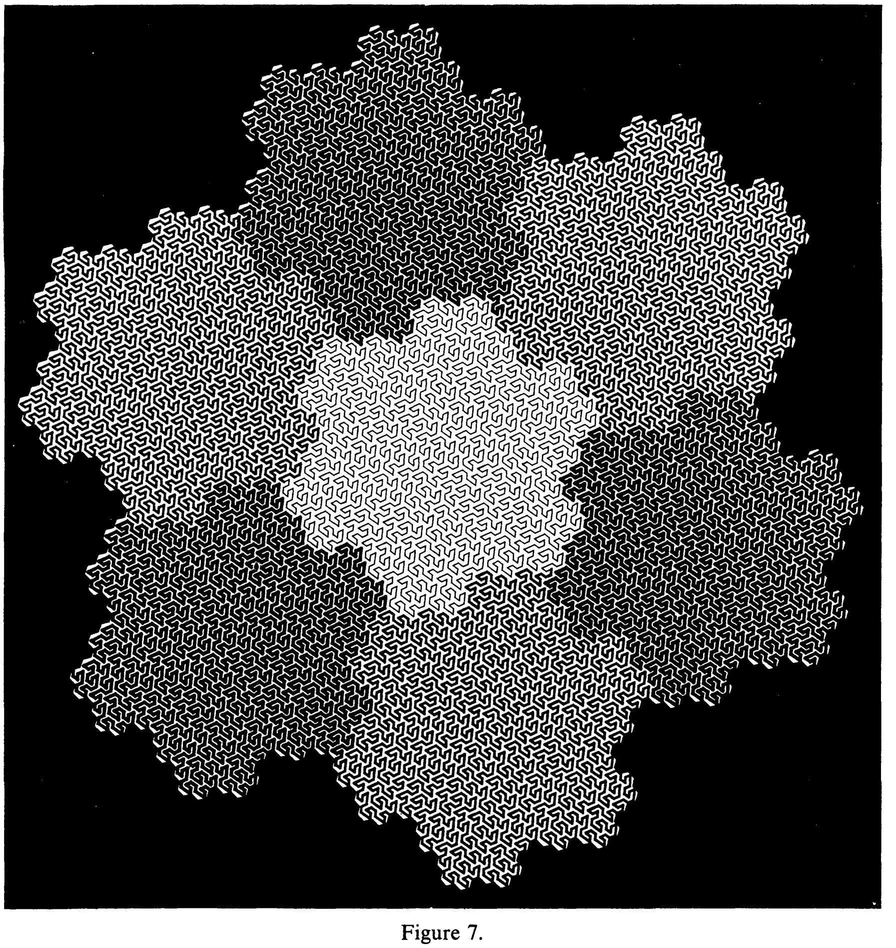


Figure 7.

scaled up by a factor of $\sqrt{7}$ in order to be as large as the whole region. Accordingly, we would further expect that the perimeter of the entire flowsnake region (that is, the length of the flowsnake boundary) to be $\sqrt{7}$ times as great as that of one of its subregions. (This would follow if the boundary were the usual one-dimensional curve.) Inspection of the figure, however, quickly establishes that the perimeter of the whole is actually 3 times as great! (Half the perimeter of each of the six outer subregions combines to form the whole of the flowsnake boundary.)

Our curious result can be paraphrased in this way: scaling up the linear dimensions by a factor of $\sqrt{7} \approx 2.646$ increases the boundary by a factor of 3. Therefore, the flowsnake boundary must have a self-similarity dimension d such that $3 = (\sqrt{7})^d$, and this yields our previous result $d \approx 1.1292$.

Alternatively, had we considered the perimeters first, the observation that the boundary of the whole flowsnake region was 3 times that of one of its subregions would have led us to believe that the area was greater by a factor of 9, as opposed to the true factor of 7. (Despite all the unlikely occurrences to this point, it is still true that 9 is not equal to 7.)

The flowsnake boundary is therefore a case where nothing but a fractional dimension can possibly make any sense. Now look back at the earlier example of the Koch snowflake and see whether the dimension of its boundary still seems as strange as it first did.

2. Mandelbrot's Definition of Fractal

Up to this point, we have been taking the naive definition of fractal as something with a fractional self-similarity dimension. Mandelbrot's formal definition is considerably more subtle, and actually permits some fractals to have integral dimension. We state the definition and then discuss the terms involved:

Definition [Mandelbrot 1982, p. 15]: A *fractal* is a set whose Hausdorff–Besicovitch dimension strictly exceeds its topological dimension.

Topological dimension is what most of us regard as the "usual" dimension. For our purposes, it suffices to consider topological dimension in terms of the usual examples: zero-dimensional points, one-dimensional lines or curves, two-dimensional surfaces, and three-dimensional solids. *Topological dimension is always expressed as a whole number.*

The Hausdorff–Besicovitch dimension is substantially more complicated; it is based on the idea of "measure" or "extent" of a set of points. (These are blanket terms for the concepts of "length," "area," and "volume," each of which is a measure associated with objects of a particular dimension.) The set whose dimension is to be calculated is covered in various ways by sets of known measure, and limit processes are used to compute the set's measure. This measure depends in a critical way on the dimension d assumed in the computation, there being at most one choice of d for which the measure will be neither 0 nor infinity. (See the accompanying sidebar, "A Closer Look at Hausdorff–Besicovitch Dimension.") The complexity of this process makes it unsuitable for our purpose and impractical as a computational tool.

Although the Hausdorff–Besicovitch dimension is not a simple matter to compute, the self-similarity dimension is much easier to calculate and fortunately turn out, for our purposes, to be equal to the Hausdorff–Besicovitch dimension. (This is not necessarily the case for all sets, but we will not encounter any of the counter-examples in our discussion of fractals.) Formally, a fractal is an object whose topological dimension is less than its Hausdorff–Besicovitch dimension. Less formally, we have been saying that a fractal is an object with nonintegral self-similarity dimension. However, Edward Szpilrajn showed that topological dimension cannot be greater than the Hausdorff–Besicovitch dimension [Hurwicz and Wallman]; thus (because topological dimension is always integral), if the Hausdorff–Besicovitch dimension is nonintegral, then it must exceed the topological dimension. Hence a fractal in our informal sense is also one by Mandelbrot's formal definition. The reverse is not true, because in the next section we will see a fractal whose self-similarity dimension is integral, but which satisfies the requirements of Mandelbrot's definition.

Having gone to the trouble of presenting the formal definition of fractals, we should also point out that Mandelbrot himself considers the idea of "fractal" to be broader than that which can be contained in any formal definition. In his own words [Mandelbrot 1982, p. 36]: "I continue to believe that one would do better without a definition."

3. You, Me, and Other Fractals

> "You know we built planets, do you?" he asked solemnly.
> "Well, yes," said Arthur, "I'd sort of gathered . . . "
> "Fascinating trade," said the old man, and a wistful look came into his eyes, "doing the coastlines was always my favorite. Used to have endless fun doing the little bits in fjords . . . "
> [Adams, p. 152]

The attentive reader will have already noticed that the snowflake and Gosper flowsnake are based on essentially the same type of iterative construction. This "Koch construction" serves to generate several new fractal curves whose self-similarity dimensions lie between 1 and 2. As before, we begin with the unit interval and replace it with a broken line whose segments are of equal length. Suppose that each segment is of length $1/4$. Then the scaling factor is always $s = 4$, while the number N varies with the construction. Four feasible choices of N are illustrated in Figure 8. (Compare Figure 4.) The effect of applying the construction for $d = 1.5$ to

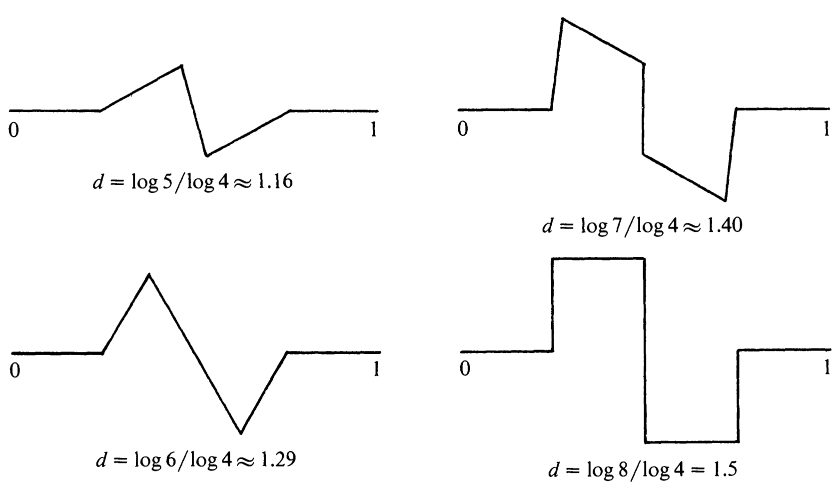


Figure 8.

the sides of a unit square is shown in Figure 9; Mandelbrot calls this a "quadric Koch island." [Mandelbrot 1982, p. 50.]

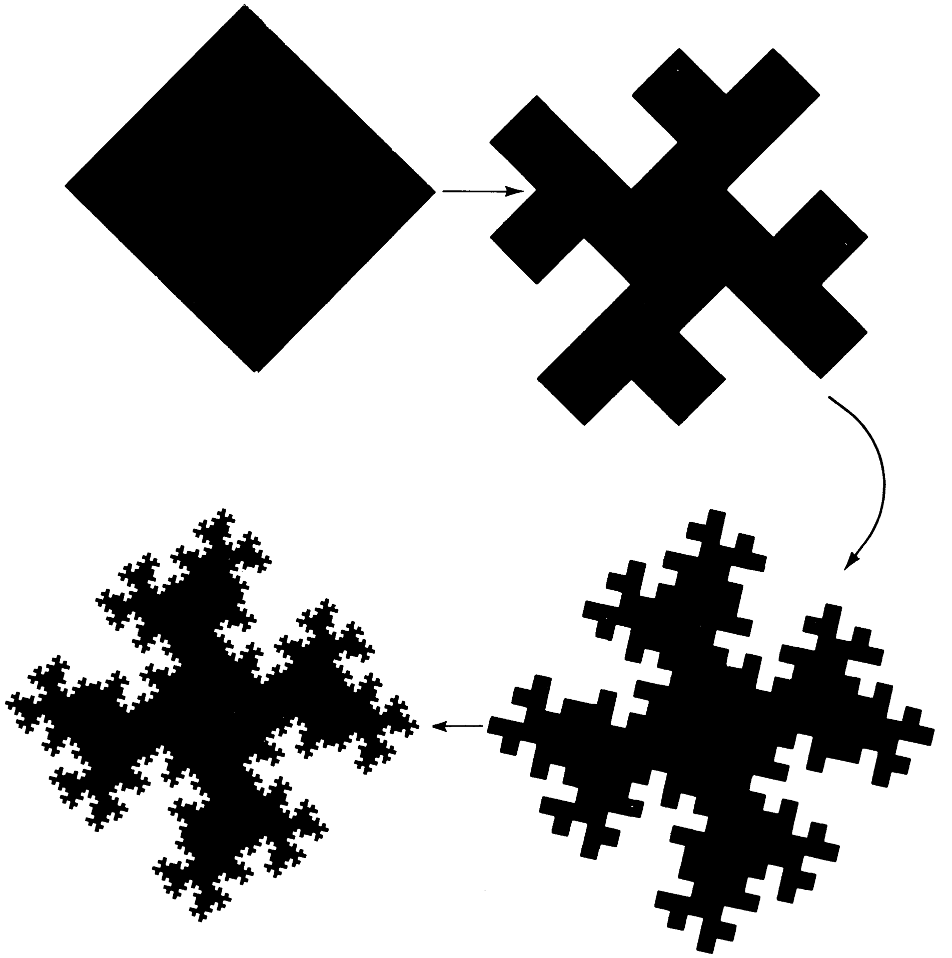


Figure 9.

For an example with $2 < d < 3$, consider a unit cube and the 27 subcubes you get by scaling down by a factor of 3. Remove 7 cubes: the central subcube of each of the six faces and the subcube in the very center. (See Figure 10 or get your Rubik's cube out again . . . . Yes, I know it doesn't really have a center cube.) We are left with $N = 20$ cubes and a scaling factor of $s = 3$. At each stage, we similarly dissect the scaled-down cubes so that, in the limit, we obtain what is called the Menger sponge. Figure 11 shows the fourth stage of its construction. The Menger sponge has self-similarity dimension

$$d = \frac{\log 20}{\log 3} \approx 2.7268.$$

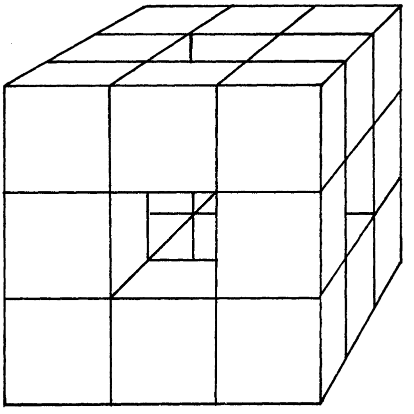


Figure 10.

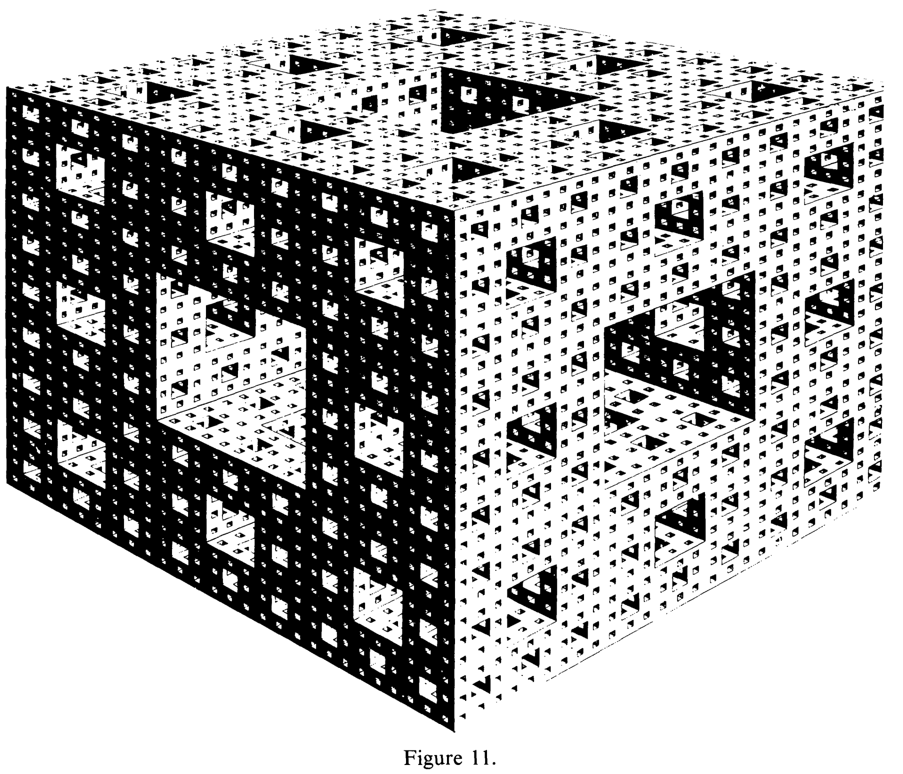


Figure 11.

The volume of the sponge is 0 (as can be shown by summing the volume of the material removed at each stage), but its surface area is infinite. The faces of the sponge are called Sierpiński carpets; since we have $N = 8$ and $s = 3$, the dimension of a carpet is $\log 8/\log 3 \approx 1.8928$.

While the Menger sponge and the other examples of fractals are intriguing, they appear to have little to do with what we call the "real world." In a sense, we have been misled by the artificial regularity of the Koch construction and over-reliance on self-similarity.

Mandelbrot's classic example of the fractal nature of coastlines ($d \approx 1.2$) is based on a less rigid notion of "statistical" self-similarity. This is akin to the observation that, given a suitable shift of scale, the coastline of Rhode Island "looks like" the coastline of California.

Mandelbrot has set himself the task of demonstrating that fractal geometry is an essential part of the structure of nature. Fractal geometry now provides a unified approach to problems involving systematic irregularity and similarity under changes of scale. In physics, fractals have been used to model Brownian motion and points of turbulence in fluid flow. In meteorology, fractals can be used to model clouds. Fractals have been applied to the distribution of matter in the universe; it seems that the universe itself may be a fractal!

In physiology, there is one example that hits particularly close to home: tissue cells and blood vessels. The latter provide the former with needed fuel (oxygen), so it is necessary that our system of blood vessels approaches closely to each tissue cell. Moreover, the vascular system carries nutrients to the cells and carries off waste products. Thus, each cell must be near an artery to receive sustenance and near a vein to discharge waste. Gosper's flowsnake can be used to construct a model of a planar analog of the body's vasculature.

Figure 12 shows the fourth stage of the construction of the flowsnake as a boundary between a white region and a black one. A "drainage system" or "watershed" is then modeled by drawing "rivers" along the midlines of the black and white "fingers" of the figure. The "rivers" are then differentiated into "arteries" (black) and "veins" (gray). Figure 13 illustrates a model in which the width of the vessels diminishes at a rate proportional to their length. The spaces between the vessels represent tissue.

A satisfactory model actually requires that the vessels narrow at a more rapid rate. It is possible to choose a rate such that, in the limit, the blood vessels will occupy only a small fraction of the planar region. The rest is then tissue. As Mandelbrot points out [Mandelbrot 1977, p. 78], it is a fractal:

> In the present planar reduction veins and arteries both have interior points, and small circles can be drawn entirely within them. On the other hand, the vessels occupy only a small percent of the overall area. The tissue is very different; it contains no piece, however small, that is not crisscrossed by both artery and vein. Such a tissue is a bona fide fractal curve: topological dimension of 1 and fractal dimension of 2.

Here, then, is the promised fractal of integral self-similarity dimension—a planar analog of our circulatory system. We should not be surprised now to learn that our body tissue can be regarded as a fractal surface. Body tissue is topologically two-dimensional because it is the interface between the three-dimensional arterial and venous systems. Its fractal dimension is three, however, because tissue actually occupies the bulk of the volume occupied by our bodies. In other words, you and I are also examples of fractals.

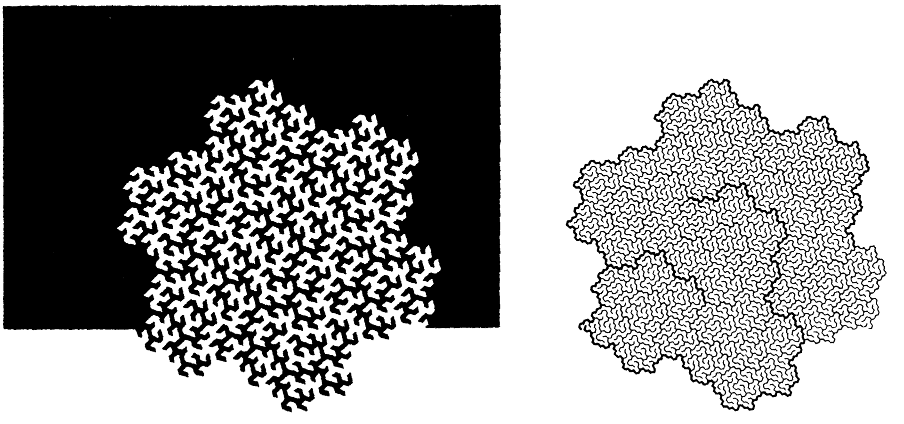


Figure 12.

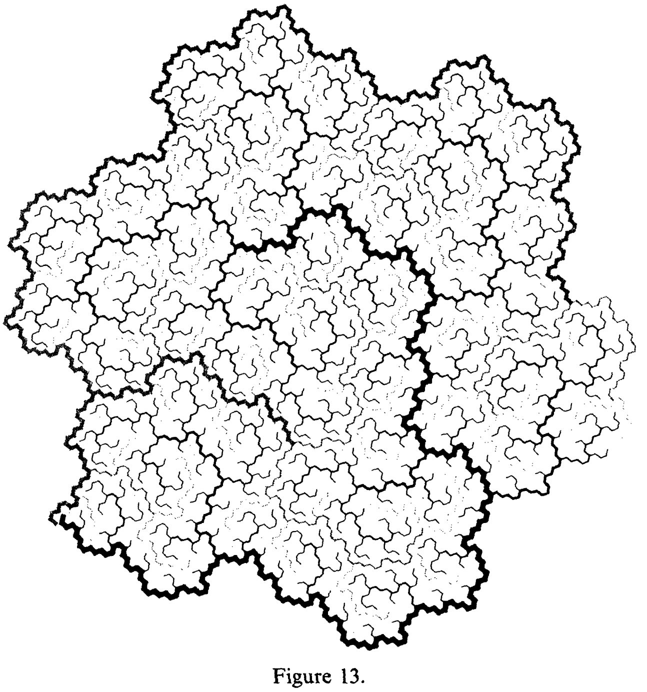
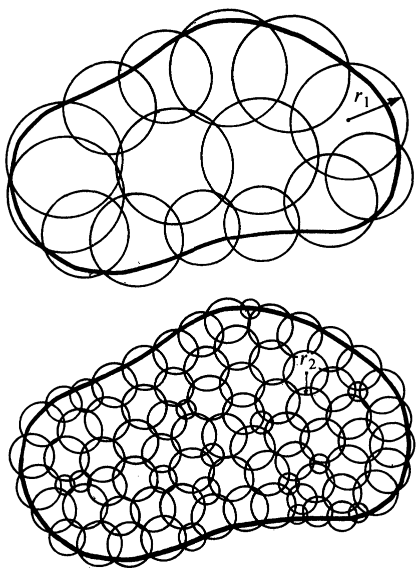


Figure 13.

A Closer Look at Hausdorff-Besicovitch Dimension

"By fluke, I had studied Hausdorff dimension—partly because a friend of mine, Henry McKean, Jr., had written his thesis on the Hausdorff dimension of certain random sets, when we both lived in Princeton. It was a very rarely used notion at that time, but I discovered that it had an application to reality." These remarks by Mandelbrot indicate how obscure the Hausdorff approach to dimension was before it became one of the foundation stones of fractal geometry. Part of the reason for this obscurity probably lies in the fact that the path to Hausdorff dimension is quite roundabout—one cannot go directly to it. Rather, it is first necessary to discuss the concept of "measure."



Coverings of the same region E for different choices of the maximum value of r ($r_2 < r_1$).

Figure 14.

To make the general approach clearer, consider the specific case of a two-dimensional region E (Figure 14). One possible way to measure this region's area is to cover E with disks whose radii do not exceed a given $r > 0$ and then sum up their areas. Since E can be covered in many ways, many different sums of these disk-areas may result. For our given r, the greatest lower bound of all such possible sums—denote this by glb(r)—gives an approximation of E's area. Note that when $r_0 < r$, every covering of E by disks of radii not exceeding r_0 is also a covering by disks of radii not exceeding r. Therefore, there are fewer r_0-determined coverings of E than there are r-determined coverings, and so glb(r_0) $\geqslant$ glb(r). This shows that glb(r) increases with decreasing values of r. The limit of glb(r) as r tends to zero is called the "Hausdorff measure" of E.

This Hausdorff measure is not precisely equal to the region's customary Euclidean area because every covering by disks, no matter how fine, always contains a certain amount of overlap. This deviation from our usual notion of area will turn out not to matter for our purposes.

In general, the Hausdorff measure of a region E in n-space could be calculated by covering it with n-balls of radii no greater than $r > 0$, computing glb(r), and taking the limit as r approaches zero. When $n = 1$, the balls are line segments; a line segment whose "radius" is less than r has Euclidean measure (length) less than $2r$. When $n = 2$, the balls are disks having Euclidean measure (area) bounded above by πr^2. For $n = 3$, we have the customary 3-dimensional balls with Euclidean measure (volume) less than $(4/3)\pi r^3$. The general expression for the ordinary n-dimensional Euclidean measure of an n-ball is $\gamma(n)r^n$, where $\gamma(n) = [\Gamma(1/2)]^2/\Gamma(1 + \frac{1}{2}n)$.† (Readers familiar with the gamma function, $\Gamma(x) = \int_0^\infty e^{-x}x^{n-1}\,dx$, can readily verify that $\gamma(1) = 2$, $\gamma(2) = \pi$, and $\gamma(3) = 4\pi/3$, as required.) Note that n does not have to be an integer for this expression for Euclidean measure to be meaningful.

This approach was extended by Hausdorff in 1919 to the calculation of *d-dimensional measures*, where $d \geqslant 0$ was not an integer. To compute the d-dimensional Hausdorff measure of a region E lying in n-space, we use the earlier approach of covering with n-balls of radius r, but now we sum the d-dimensional Euclidean measures, $\gamma(d)r^d$, of these balls.

Consider, for example, a line segment or curve of length L which lies in some n-dimensional space. For sufficiently small values of r, it will take approximately $L/2r$ n-balls of radius r to cover the curve. (Regardless of n, the diameters of the n-balls are $2r$.) Denote the sum of the d-dimensional measures of these balls by S_r. We have

$$S_r \approx (L/2r)\gamma(d)r^d = (L/2)\gamma(d)r^{d-1}.$$

Note that $0 \leqslant \text{glb}(r) \leqslant S_r$, since glb($r$) was defined as the greatest lower bound of all sums which estimate the measure of the curve, and S_r is only one such sum.

For $d > 1$, the approximating expression for S_r decreases to 0 as r tends to 0. Therefore, $\lim_{r\to 0}\text{glb}(r) = 0$. For $d = 1$, we have $S_r \approx L$. Although S_r must actually exceed L slightly (because of overlap and the fact that anything less than L would not be a covering), it can be made arbitrarily close by choosing a sufficiently small r and minimizing the overlap of the balls. Thus it follows that $\lim_{r\to 0}\text{glb}(r) = L$. (Note that in this simple case, the curve's Hausdorff measure agrees with its Euclidean measure.) For $d < 1$, the sum S_r tends to infinity as r decreases to zero. But for small r, we have $\text{glb}(r) \approx S_r$. Thus, $\lim_{r\to 0}\text{glb}(r) = \infty$ for $d < 1$.

As the preceding example shows, Hausdorff d-dimensional measure is zero or infinity when an unsuitable choice of d is made. Besicovitch pointed out that this could be used to define a notion of dimension for certain sets. By computing the

---

†*Editor's Note*: Readers may find useful here Marshall Fraser's formula for the volume of a n-ball of radius r:

$$V_n(r) = \begin{cases} \dfrac{\pi^{n/2}r^n}{(n/2)!}, & n \text{ even} \\[2ex] \dfrac{2^{(n+1)/2}\pi^{(n-1)/2}r^n}{n(n-2)\cdots 3}, & n \text{ odd.} \end{cases}$$

See "The Grazing Goat in n-Dimensions," this volume, pp. 126–134.

Hausdorff measure for different values of d, one might try to define as "dimension" the one value of d for which the Hausdorff measure is both positive and finite. If the measure is 0, then the value of d is too large; if infinite, d is too small. This definition is deficient in one significant respect, as can be shown by reconsidering our example of the curve of length L. Had L been infinite, the Hausdorff measure would also have been infinite—even for $d = 1$. "Hausdorff–Besicovitch dimension" is actually defined as the value of d which is the "cut point" separating infinite values of Hausdorff measure from zero values. The Hausdorff measure for the chosen value of d *may* be positive and finite, as we initially hoped, but the value at the cut point may also be zero or infinity.

Our use of Euclidean measure for the covering n-balls is more restrictive than Hausdorff's own approach. Hausdorff measure is a metric concept which uses balls of radius r and a "gauge" function $h(r)$. Our gauge function was $\gamma(n)r^n$. We could just as well have used r^n alone, or some other suitable function. A more detailed argument is provided on pp. 361–365 of Mandelbrot 1982.

BIBLIOGRAPHY

Douglas Adams, The Hitchhiker's Guide to the Galaxy, Pocket Books, New York, 1979.

A. D. Aleksandrov, et al, Mathematics: Its Content, Methods, and Meaning (3 volumes), The M.I.T. Press, Cambridge, 1963.

James D. Cannon, "Topological, combinatorial, and geometric fractals," The American Mathematical Monthly, Mathematical Association of America, Washington, D.C. (to appear).

Kenneth F. Falconer, The Geometry of Fractal Sets, Cambridge University Press (to appear).

Dave Fleischer, "Computer graphics: The art of hyper-reality," Science Digest, New York, February 1982.

A. Fournier, D. Fussell, and L. Carpenter, "Computer rendering of stochastic models," Communications of the Association for Computing Machinery, Volume 25, No. 6, June 1982. [See also Mandelbrot's reaction and the authors' response in the "Technical Correspondence" section of No. 8 (August 1982 issue) of the same journal.]

Martin Gardner, "In which 'monster' curves force redefinition of the word 'curve'," Mathematical Games, Scientific American, New York, December 1976.

Martin Gardner, "White and brown music, fractal curves and one-over-*f* fluctuation," Mathematical Games, Scientific American, New York, April 1978.

Douglas R. Hofstadter, "Strange attractors: mathematical patterns delicately poised between order and chaos," Metamagical Themas, Scientific American, New York, November 1982.

W. Hurwicz and H. Wallman, Dimension Theory, Princeton University Press, 1941.

Benoit B. Mandelbrot, Les objets fractals: forme, hasard et dimension, Flammarion, Paris, 1975.

Benoit B. Mandelbrot, Fractals: form, chance, and dimension, W. H. Freeman and Company, San Francisco, 1977.

Benoit B. Mandelbrot, "Fractals and the Geometry of Nature," 1981 Yearbook of Science and the Future, Encylcopaedia Britannica, Inc., Chicago, 1980.

Benoit B. Mandelbrot, The Fractal Geometry of Nature, W. H. Freeman and Company, San Francisco, 1982.

Bruce Schechter, "A new geometry of nature," Discover, Time-Life Magazines, New York, June 1982.

Ian Stewart, "The Chronicles of Primrose Polymath: Into the Two and a Halfth Dimension," Librairie Classique Eugene Belin, Paris.

Turner Whitted, "Some recent advances in computer graphics," Science, Washington, D.C., Volume 215, 12 February 1982.