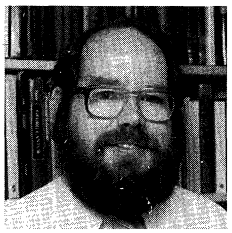
Isaac Newton: Man, Myth, and Mathematics

V. Frederick Rickey



V. Frederick Rickey received all of his degrees from the University of Notre Dame, earning his Ph.D. in 1968 with a specialization in logic. He then went to Bowling Green State University, where he has remained except for three sabbaticals. He has served as editor of the Notre Dame Journal of Formal Logic and on the editorial staff of The Philosopher's Index. Over the years his interests have changed to the history of mathematics, which is now his primary research interest, especially the history of the calculus. Currently, he is Chair of the Americas Section of the International Study Group for History and Pedagogy of Mathematics (HPM for short), a group which encourages teachers at all levels to use the history of mathematics in the classroom to motivate students.

Three hundred years ago, in 1687, the most famous scientific work of all time, the *Philosophiae Naturalis Principia Mathematica* of Isaac Newton, was published. Fifty years earlier, in 1637, a work which had considerable influence on Newton, the *Discours de la Méthode*, with its famous appendix, *La Géométrie*, was published by René Descartes. It is fitting that we celebrate these anniversaries by sketching the lives and outlining the works of Newton and Descartes.

In the past several decades, historians of science have arranged the chaotic bulk of Newton manuscripts into a coherent whole and presented it to us in numerous high quality books and papers. Foremost among these historians is Derek T. Whiteside, of Cambridge, whose eight magnificent volumes overflowing with erudite commentary have brought Newton to life again.

> By unanimous agreement, the *Mathematical Papers* [*of Isaac Newton*] is the premier edition of scientific papers. It establishes a new criterion of excellence. Every further edition of scientific papers must now measure itself by its standard. [26, p. 87]

Other purposes of this article are to dispel some myths about Newton—for much of what we previously "knew" about him is myth—and to encourage the reader to look inside these volumes and to read Newton's own words, for that is the only way to appreciate the majesty of his intellect.

Newton's Education and Public Life

Isaac Newton was born prematurely on Christmas Day 1642 (O.S.), the "same" year Galileo (1564–1642) died, in the family manor house at Woolsthorpe, some 90 km NNW of Cambridge. His illiterate father—a "wild, extravagant, and weak man"—had died the previous October. His barely literate mother, Hanna, married the Reverend Barnabas Smith three years later, leaving Newton to be raised by his aged grandmother Ayscough.

Newton attended local schools and then, at age 12, traveled 11 km north to the town of Grantham, where he lived with the local apothecary and his books while attending grammar school. The town library had two or three hundred books, some 85 of which are still chained to the walls. Of course he studied Latin, also some Greek and Hebrew. Four years later, in 1658, he returned home to help his now twice-widowed mother manage the farm. Recognizing that Newton was an absent-minded farmer, his uncle William Ayscough (M.A. Cambridge, 1637) and former Grantham schoolmaster, Henry Stokes, persuaded his mother to send him back to Grantham to prepare for Cambridge. Judging by a mathematical copybook in use at Grantham in the 1650s, Stokes was a most unusual schoolmaster. The copybook contained arithmetic through the extraction of cube roots, surveying, elementary mensuration, plane trigonometry, and elaborate geometric constructions, including the Archimedean bounds for π. This went far beyond anything taught in the universities of the period; consequently, contrary to tradition, Newton had a superior knowledge of mathematics before he went to Cambridge [33, pp. 110–111; 34, p. 101; updating 20, I, p. 3].

In 1661, eighteen-year-old Newton matriculated at Trinity College, the foremost college at Cambridge, as a subsizar (someone who earned his way by performing simple domestic services). This position reflected his wealthy mother's reluctance to send him to the university. At that time, Cambridge was little more than a degree mill. Lectures were seldom given. Fellows tutored primarily to augment their income. Although Newton did not finish any of the books from the established curriculum, which consisted mostly of Aristotelian philosophy, he did learn the patterns of rigorous thought from Aristotles' sophisticated philosophical system. A chance encounter with astrology in 1663 led him to the more enlightened "brisk part of the University" that was interested in the work of Descartes [28, p. 90]. The laxity of the university allowed him to spend the last year and a half of his undergraduate studies in the pursuit of mathematics. In 1665, Newton received his B.A. "largely because the university no longer believed in its own curriculum with enough conviction to enforce it." [28, p. 141].

In the summer of 1665, virtually everyone left the university because of the bubonic plague. The next March the university invited its students and Fellows to return for there had been no deaths in six weeks, but by June it was clear that the plague had not left, so the students who had returned left again. The university was able to resume again in the spring of 1667. Newton had left by August 1665 for Woolsthorpe. He returned on 20 March 1666, probably left again in June, but not until he had written his famous May 1666 tract on the calculus. He did not return to Cambridge until late April 1667, having revised the May tract into the October 1666 tract while back on the farm. "For whatever it is worth, the papers do not indicate that anything special happened at Woolsthorpe." [27, p. 116]. Much has been written about these plague years as the *anni mirabiles* of Newton, but the record clearly shows that he wrote the bulk of his mathematical manuscripts on the calculus while he was at Cambridge.

Myth: *At the Woolsthorpe farm, during the plague years, Newton invented the calculus so that he could apply it to celestial mechanics.*

The primary source for the myth [27, p. 110] of Newton's miracle years is this 1718 (unsent?) letter from Newton to Pierre DesMaizeaux:

> In the beginning of the year 1665 I found the Method of approximating series & the Rule for reducing any dignity [= power] of any Binomial into such a series. The same year in May I found the method of Tangents..., & in November had the direct method of fluxions & the next year in January had the Theory of Colours & in May following I had entrance into y^e inverse method of fluxions. And the same year I began to think of gravity extending to y^e orb of the Moon & (having found out how to estimate the force with w^{ch} globe revolving within a sphere presses the surface of the sphere) from Keplers rule... I deduced that the forces w^{ch} keep the Planets in their Orbs must [be] reciprocally as the squares of their distances from the centers about w^{ch} they revolve: & thereby compared the force requisite to keep the Moon in her Orb with the force of gravity at the surface of the earth, & found them answer pretty nearly. All this was in the two plague years of 1665 & 1666. For in those days I was in the prime of my age for invention & minded Mathematicks & Philosophy [= Science] more then at any time since. [27, p. 109]

Lucasian Professor. At Trinity College, Newton became a Minor Fellow in 1667 and a Major Fellow in 1668. On 29 October 1669, at the age of 26, Newton became the second Lucasian Professor of Mathematics at Cambridge, succeeding Isaac Barrow (1630–1677). This post gave him security, intellectual independence, and a good salary. According to the Lucasian statutes, Newton was to lecture once a week during each of the three terms and to deposit ten of the lectures in the library. Even though this position had been designed by its founder Henry Lucas as a teaching post, not a research position [20, V, xiv], Barrow had already turned the position into a sinecure and Newton did not work much harder at the teaching aspects of the post. He deposited 3–10 lectures per year for the first seventeen years as Lucasian Professor, and none thereafter.

As a teacher, Newton left no mark whatsoever. Years later, when he was duly famous, one would expect that many people would have claimed to have attended his lectures, yet we know of only three. Perhaps the situation is best summed up by Newton's amanuensis (a human wordprocessor), the unrelated Humphrey Newton:

> He seldom left his chamber except at term time, when he read in the schools as being Lucasianus Professor, where so few went to hear him, and fewer that understood him, that ofttimes he did in a manner, for want of hearers, read to the walls. [6, X, 44]

London and Beyond. In 1696, Newton accepted the post of Warden of the Mint (moving to London in March or April of 1696) and four years later became Master. In 1701, Newton resigned the Lucasian professorship. In 1703, he was elected President of the Royal Society, which he ruled with an iron hand until his death. In 1705, Newton was knighted by Queen Anne—not for his scientific advances, but for the service he had rendered the Crown by running (unsuccessfully) for Parliament in 1705 [28, p. 625]. For the rest of his life, Newton looked after the Mint and the Royal Society, twice revised his *Principia* (1713 and 1726), engaged in the infamous priority dispute with Leibniz, and toiled on secret research in religion and church history. His creative scientific life essentially ended when he left Cambridge.

Newton died 20 March 1727, at the age of 84, having been ill with gout and inflamed lungs for some time. He was buried in Westminister Abbey.

Newton's Nachlass. At the time of his death Newton was wealthy. Income from the Lucasian Chair and farm rents brought £250 per year, sufficient for a handsome living for a bachelor Don. When he became Master of the Mint, his salary jumped to £600 and he also received the perquisite of a commission on the amount of coinage. This amounted to some £1500 pounds per year, thus bringing his income to over £2000 pounds per year, a very substantial figure at that time. On his death his estate was valued at £30,000.

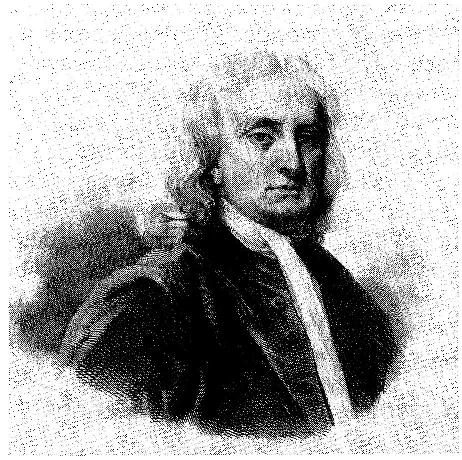


Figure 1. Newton at age 82.

Newton left his library of some two thousand volumes to his nieces and nephews. The books were quickly sold to the Warden of Fleet Prison for £300 for his son Charles Huggins who was a cleric near Oxford. On Huggins' death in 1750 they were sold to his successor, James Musgrave, for £400. They remained in the Musgrave family until 1920, when some of them were sold at auction as part of a "Library of miscellaneous literature," fetching only £170. Although the family didn't know what they had sold, the book dealers knew what they had bought. Newton's annotated copy of Barrow's *Euclid*, which sold for five shillings, was soon in a bookseller's catalogue for £500. In 1927, the remaining 858 volumes were offered for £30,000 but remained unsold until 1943 when they were purchased for £5,500 and donated to the Wren Library at Trinity College. Of the thousand or so that were dispersed in 1920, some still show up unrecognized in bookshops. As recently as 1975, one was purchased in a Cambridge bookshop for £4. The books are easily identified by Newton's peculiar method of dog-earing by folding a page down to point to the precise word that interested him.

From Newton's library, 1736 books have now been located. Since his was a working library, a subject classification of the nonduplicates provides some information about Newton's interests. (For additional details, see [10, p. 59], from which this table is condensed.)

Subject	Mathematics	Physics and Astronomy	Alchemy	Theology
Number of titles (%)	126 (7.2)	85 (4.9)	169 (9.6)	477 (27.2)

Subject	History	Other Science	Other
Number of titles (%)	143 (8.2)	158 (9.0)	594 (33.9)

Newton also had access to the library of Barrow until Barrow's death in 1677, and to the Cambridge libraries until he moved to London in 1696.

Whiteside has tracked down every available scrap of material on Newton's mathematics and published it in *The Mathematical Papers of Isaac Newton* [20]. To really appreciate Newton's mathematical genius, one must grapple with his mathematics as he wrote it. The best place to gain an overview for this project is in Whiteside's wonderful introductions to these volumes and to the various papers in them. They have been used extensively in preparing this paper.

This biographical sketch has been intentionally kept short. For further details about Newton and his work, see the article by I. B. Cohen in the *Dictionary of Scientific Biography* (DSB) [6, X, pp. 42–103]. This is the single most authoritative reference work about the lives and contributions of deceased scientists. To avoid frequent references to it, we give dates after the first occurrence of an individual's name if the DSB contains an article about him. Two excellent biographies of Newton are Westfall's full scientific biography, *Never at Rest* [28], and Manuel's psychobiography *A Portrait of Isaac Newton* [15], some conclusions of which must be taken with care. For mathematical details, consult the many papers of Whiteside, only a few of which are cited here.

Newton's Mathematical Readings

The year 1664 was a crucial period in Newton's development as a mathematician and scientist, for it was then that he began to extend his readings beyond the traditional Aristotelian texts of the moribund curriculum to the new Cartesian ideas. (For details of Newton's nonmathematical readings, see McGuire [16].) According to Abraham DeMoivre (1667–1754), the expatriate French intimate of Newton during Newton's last years, the immediate impulse for Newton taking up mathematics was:

> In 63 [Newton] being at Sturbridge [international trade] fair bought a book of Astrology, out of a curiosity to see what there was in it. Read in it till he came to a figure of the heavens which he could not understand for want of being acquainted with Trigonometry.
>
> Bought a book of Trigonometry, but was not able to understand the Demonstrations. Got Euclid to fit himself for understanding the ground of Trigonometry.
>
> Read only the titles of the propositions, which he found so easy to understand that he wondered how any body would amuse themselves to write any demonstrations of them. Began to change his mind when he read that Parallelograms upon the same base & between the same Parallels are equal, & that other proposition that in a right angled Triangle the square of the Hypothenuse is equal to the squares of the two other sides.
>
> Began again to read Euclid with more attention than he had done before & went through it.
>
> Read Oughtreds [Clavis] which he understood tho not entirely, he having some difficulties about what the Author called Scala secundi & tertii gradus, relating to the solution of quadratick [&] Cubick Equations. Took Descartes's Geometry in hand, tho he had been told it would be very difficult, read some ten pages in it, then stopt, began again, went a little farther than the first time, stopt again, went back again to the beginning, read on till by degrees he made himself master of the whole, to that degree that he understood Descartes's Geometry better than he had done Euclid.
>
> Read Euclid again & then Descartes's Geometry for a second time. Read next D^{r} Wallis's Arithmetica Infinitorum, & on the occasion of a certain interpolation for the quadrature of the circle, found that admirable Theorem for raising a Binomial to a power given. But before that time, a little after reading Descartes Geometry, wrote many things concerning the vertices Axes [&] diameters of curves, which afterwards gave rise to that excellent tract de Curvis secundi generis.
>
> In 65 & 66 began to find the method of Fluxions, and writt several curious problems relating to that method bearing that date which were seen by me above 25 years ago. [20, I, pp. 5–6]

These words of DeMoivre, which agree with the report of Conduitt [20, I, pp. 15–19], certainly have an air of authenticity to them, and we know, based on extant manuscripts, that they are substantially correct (modulo Stokes's copybook). In the years 1664–1665, Newton made detailed notes on the following contemporary high level books, which influenced him at the very beginning of his mathematical studies.

- Barrow's *Euclidis Elementorum* (1655)
- Oughtred's *Clavis Mathematicae* (1631) in the 1652 edition
- *Geometria, à Renato des Cartes*, 1659–1661 edition of Schooten
- Schooten's *Exercitationum Mathematicarum Libri Quinque* (1657)
- Viète's *Opera Mathematica*, 1646 edition of Schooten
- Wallis's *Arithmetica Infinitorum* (1655)
- Wallis's *Tractatus Duo* (1659).

Let us look carefully at each of them to see what Newton learned.

Euclid (fl. ca. 295 BC). As DeMoivre indicated, Newton read Euclid as a student, although he did not develop any deep knowledge of the work then. Recall the story [28, p. 102] that Barrow examined Newton on Euclid and found him wanting. Newton was mainly influenced by books II (geometrical algebra), V (proportion), VII (number theory), and X (irrationals). The primary thing that he learned from Euclid was the traditional forms of mathematical proof [20, I, p. 12].

William Oughtred (1575–1660). At age fifteen, Oughtred went to Cambridge where he studied mathematics diligently on his own, for there was then hardly anyone there to teach him. He graduated B.A. in 1596 and M.A. in 1600. In 1603, he became a (pitiful) preacher and soon settled in as rector at Albury where he remained until his death.

It was as a teacher that he was renowned. He taught privately and for free. People came from the continent to talk to him, so wide was his reputation in mathematics. To instruct a young Earl, Oughtred wrote a little book of 88 pages that contained the essentials of arithmetic and algebra. *Clavis Mathematicae* (*Key to Mathematics*) published in 1631, was "a guide for mountain-climbers, and woe unto him who lacked nerve." [2, p. 29]. The style was obscure, the rules so involved they were difficult to comprehend. Oughtred carried symbolism to excess, a habit acquired by his most famous pupil, John Wallis. Nonetheless, *Clavis* established Oughtred as a capable mathematician and exerted a considerable effect in England, for it was a widely studied book in higher mathematics [32, p. 73].

Oughtred's *Clavis*, in the 1652 edition, was one of the first mathematical books that Newton read. From it he learned a very important lesson: Oughtred taught that *algebra was a tool for discovery* that did not need to be backed up by geometry [13, p. 408]. Newton held Oughtred in high regard, describing him as "a Man whose judgment (if any man's) may be safely relyed upon." [19, III, p. 364]

René Descartes (1596–1650). René du Perron Descartes was born 31 March 1596 in La Haye (now La Haye-Descartes), France, a small town 250 km SSW of Paris. At the age of eight, he enrolled in the new Jesuit *collège* at La Flèche. There Descartes received a modern education in mathematics and physics—including the recent telescopic discoveries of Galileo—as well as more traditional schooling in the humanities, philosophy, and the classics. It was there, because of his then delicate health, that he developed the habit of lying abed in the morning in contemplation. Descartes retained an admiration for his teachers at La Flèche but later claimed that he found little of substance in the course of instruction and that only mathematics had given him any certain knowledge.

Descartes graduated in law from the University of Poitier in 1616, at age 20, but never practiced law as his father wished. By this time, his health improved and he enjoyed moderately good health for the rest of his life. Because he decided that he could not believe in what he had learned at school, he began a ten year period of wandering about Europe, spending part of the time as a gentleman soldier. It was during this period that Descartes had his first ideas about the "marvelous science" that was to become analytic geometry.

Although we have little detail about this period of his life, we do know that he hoped to learn from "the book of the world." Descartes reached two conclusions. First, if he was to discover true knowledge he must carry out the whole program himself, just as a perfect work of art is the work of one master. Second, he must begin by methodically doubting everything taught in philosophy and looking for self-evident, certain principles from which to reconstruct all science.

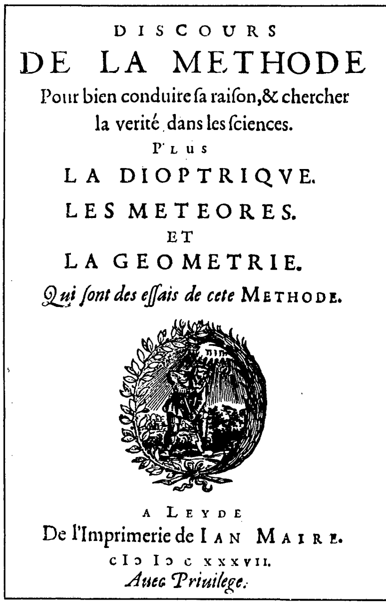
In November 1628, Descartes had a public encounter with Chandoux, who felt that science was founded only on probability. By using his method to distinguish between true scientific knowledge and mere probability, Descartes easily demolished Chandoux. Among those present was the influential Cardinal de Bérulle, who charged Descartes to devote his life to working out the application of "his manner of philosophizing ... to medicine and mechanics." To execute this design, Descartes moved to the Netherlands in 1628, where he lived for the next twenty years.

In Holland, Descartes worked at his system and, by 1634, had completed a scientific work entitled *Le Monde*. He immediately suppressed the book when he heard about the recent condemnation of Galileo by the inquisition. He learned this from Marin Mersenne (1588–1648), a fellow student at LaFlèche and later the hub of the scientific correspondence network in Europe. This reveals Descartes' spirit of caution and conciliation toward authority (he was a lifelong devout Catholic). Later he took care to present his less orthodox views more obliquely.

Three hundred and fifty years ago, in 1637, the *Discours de la Méthode* [Figure 2], with appendices *La Dioptrique*, *Les Meteores*, and *La Géométrie*, appeared anonymously in Leyden, although it was soon widely known that Descartes was the author. The opening *Discours* is notable for its autobiographical tone, compressed presentation, and elegant French style. It was written in French since he intended—as did Galileo—to aim over the heads of the academic community to reach the educated people. Today, it is this opening *Discours*, with its problem-solving techniques [Figure 3], that is read.

In 1644, Descartes published *Principia Philosophiae*, a work in which he presented his views on cosmology. He expounded a mechanical philosophy in which a body could influence only those other bodies that it touched. Thus, for example, Descartes imagined space filled with "vortices" that moved the planets. This world view quickly became dominant in Europe. After the publication of Newton's *Philosophiae Naturalis Principia Mathematica*, the two scientific outlooks competed until well into the eighteenth century. Significantly—and this is reflected in the titles —Newton made mathematics indispensable for understanding the universe.

Queen Christiana of Sweden, ambitious patron of the arts and collector of learned men for her court, had seen the works of Descartes and pleaded with him to



DISCOURS
DE LA METHODE
Pour bien conduire ſa raiſon, & chercher la verité dans les ſciences.
PLUS
LA DIOPTRIQVE.
LES METEORES.
ET
LA GEOMETRIE.
Qui ſont des eſſais de cete METHODE.

A LEYDE
De l'Imprimerie de IAN MAIRE.
cIↃ IↃ c xxxvii.
Auec Priuilege.

Figure 2.

Descartes' Rules:

The first was never to accept anything as true that I did not know evidently to be such; that is to say, carefully to avoid haste and bias, and to include nothing more in my judgments than that which presented itself to my mind so clearly and so distinctly that I had no occasion to place it in doubt.

The second was to divide each of the difficulties that I examined into as many parts as possible, and according as such division would be required for the better solution of the problems.

The third was to direct my thinking in an orderly way, by beginning with the objects that were simplest and easiest to understand, in order to climb little by little, gradually, to the knowledge of the most complex; and even for this purpose assuming an order among those objects which do not naturally precede each other.

And the last was at all times to make enumerations so complete, and reviews so general, that I would be sure of omitting nothing. [4, p. 16]

Figure 3. Pólya was very much influenced by Descartes [22, I, p. 56].

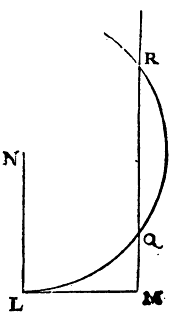
Figure 4. René Descartes

join her and teach her philosophy. She sent a man-of-war to fetch him but he was loath to go, in his words, to the "land of bears between rock and ice." But go he did. Being more of an athlete than a scholar, the 23-year-old Queen wanted her lessons at five in the morning in a cold library with windows thrown wide open. This harsh land, where "men's thoughts freeze during the winter months," was too much for Descartes. A few months later he caught pneumonia and died on 11 February 1650.

Contents of the *Geometry*. The *Geometry* of Descartes is available to us in two English editions, the well known Smith-Latham translation [3] and the only complete English translation of the whole *Discours de la Méthode* by Olscamp [4]. The latter should be consulted since the appendix on *Optics* contains much interesting material on the conics.

In the first book of the *Geometry*, Descartes gave new geometric solutions of quadratic equations. For example [Figure 5], to solve the equation $z^2 = az - b^2$ (where a and b are both positive), Descartes drew the base line LM of length b and a perpendicular line LN of length $a/2$. Then he drew the circle with center N and radius NL. This circle cut the line perpendicular to LM at M in two points. The line segments MR and MQ are the solutions of the equation, as the reader can easily check. Descartes was aware that if the circle misses (only touches) the perpendicular to LM at M, then there is no (only one) solution to the equation.

Observe [Figure 5] that we have adopted Descartes' notation. In fact, his *Geometry* is the oldest mathematics text that we can read without having great difficulties with the algebraic notation. Descartes introduced the use of x, y, z for variables and a, b, c for constants, and he also introduced the exponential notation (except that he sometimes writes "aa" for our "a^2"). The only significant difference is that Descartes uses the symbol $\propto$ for equality.

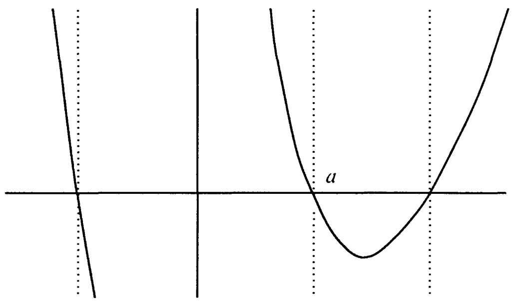


Enfin ſi i'ay
$z^{2} \propto az - bb$:
ie fais N L eſgale à $\frac{1}{2}$ a, & L M eſgale à b cõme deuãt, puis, au lieu de ioindre les poins M N, ie tire M Q R parallele a L N. & du centre N par L ayant deſcrit vn cercle qui la couppe aux poins Q & R, la ligne cherchée z eſt M Q, oubiẽ M R, car en ce cas elle s'exprime en deux façons, a ſçauoir $z \propto \frac{1}{2}a + \sqrt{\frac{1}{4}aa - bb}$, & $z \propto \frac{1}{2}a - \sqrt{\frac{1}{4}aa - bb}$.

Et ſi le cercle, qui ayant ſon centre au point N, paſſe par le point L, ne couppe ny ne touche la ligne droite M Q R, il n'y a aucune racine en l'Equation, de façon qu'on peut aſſurer que la conſtruction du probleſme propoſé eſt impoſſible.

Figure 5. From p. 303 of Descartes' *Géométrie*.

Another problem Descartes dealt with in the first book was the problem of Pappus (*fl.* A.D. 300–350), which he mistakenly believed was still open. The problem asks for the locus of points such that the product of the distances (measured at fixed angles) to half of a fixed set of lines is equal to the product of the distances to the other half (times a constant if the number of lines is odd). If there are three or four lines, Descartes showed that the locus is a conic. As an example with five lines, Descartes considered one horizontal line and four equally spaced vertical lines [Figure 6].



$$y = \frac{(x+a)(a-x)(2a-x)}{ax}.$$

Figure 6. Cartesian Parabola

He set the product of the distances to the first, third, and fourth vertical lines equal to the product of the constant distance a between the lines, the distances to the second vertical line and the horizontal line, and obtained the equation $axy = (x + a)(a - x)(2a - x)$. Newton later called this curve the Cartesian Parabola. Since there were very few curves in Descartes' day, each received its own fancy name. This curve was only the second cubic (that is, a polynomial in two variables of degree three) ever discussed. The first was the Cissoid of Diocles (*fl. ca.* 190 B.C.). Descartes used his new curve extensively in his third book to solve equations of the fifth and sixth degrees as intersections of it and a circle.

Geometrical vs. Mechanical Curves. The second book of Descartes' *Geometry* begins with a discussion of those curves which Descartes believed should be admitted into geometry. He does not consider the equation to be a sufficient representation of a curve, for equations are clearly algebraic objects. This forced him to always define curves by giving some geometric criterion. Later he derived the equation.

Descartes made a strict distinction between the curves that he called "geometrical" and those which he called "mechanical," but his explanation was none too clear. It has turned out that Descartes' geometrical (mechanical) curves are just the graphs of our algebraic (transcendental) functions. See Bos [1] for a full discussion. Descartes said that a curve is geometrical if it "can be conceived of as described by a continuous motion" [3, p. 43]. This excludes the spiral and the quadratrix because "they must be conceived of as described by two separate movements whose relation does not admit of exact determination" [3, p. 44]. Descartes allowed the use of a loop of thread to trace out a geometrical curve, as long as the shape of the string remained polygonal [3, p. 91]. Thus, the ellipse is a geometrical curve since it can be traced out using the familiar gardener's construction using string and pegs. In *La Dioptrique*, Descartes showed how to construct the hyperbola using straightedge and string [4, p. 135]. However, the curve generated by the moving end of a piece of thread as it unwinds from a spool is a mechanical curve, for the thread was curved while wound around the spool and straight after it unwinds.

> On the other hand, geometry should not include lines that are like strings, in that they are sometimes straight and sometimes curved, since the ratios between straight and curved lines are not known, and I believe cannot be discovered by human minds, and therefore no conclusion based upon such ratios can be accepted as rigorous and exact. [3, p. 91]

That straight and curved lines cannot be compared is an old dictum of Aristotle. Descartes' adoption of it was important for it set up the question of rectification of curves—that is, the problem of finding arc length of curves.

Let us now consider Descartes' argument for the Cartesian Parabola being a geometrical curve. He gave the following definition of a geometrical curve, then found its equation. Since its equation is the same as that of the Cartesian Parabola, the Cartesian Parabola is a geometric curve.

> I shall consider next the curve CEG [Figure 7], which I imagine to be described by the intersection of the parabola CKN (which is made to move so that its axis KL always lies along the straight line AB) with the ruler GL (which rotates about the point G in such a way that it constantly lies in the plane of the parabola and passes through the point L). [3, p. 84]

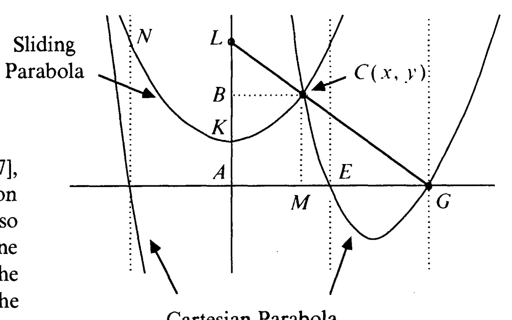


Figure 7.

If we let AB be the y-axis and AG be the x-axis (Descartes used the opposite convention), then the Cartesian Parabola is the locus of all points $C(x, y)$ of intersection of the parabola that slides up and down the y-axis and the ruler that pivots at the fixed point $G(2a, 0)$ and passes through the point L moving along the y-axis with the parabola. The parabola has equation $x^2 = az$, where $a = KL$ and $z = BK$ (the focus of the parabola is one-fourth of the way from K to L). Descartes found the equation of the curve using classical geometry: Since the triangles GMC and CBL are similar, $GM/MC = CB/BL$, that is, $(2a - x)/y = x/BL$. Thus, we have

$$BK = a - BL = a - \frac{xy}{2a - x}.$$

But the equation of the parabola CKN can be written $BK = x^2/a$. Equating these expressions for BK, and simplifying, we obtain,

$$x^3 - 2ax^2 - a^2x + 2a^3 = axy,$$

which is the equation of the Cartesian parabola. (Note that the name comes from the fact that a parabola is sliding up and down the line.)

Descartes' Subnormal Method. In our calculus classes, one important problem is to find an equation of the tangent line to a curve at a given point on the curve. Problems were not phrased this way in the seventeenth century, because equations of lines was not a well developed topic. They asked (equivalently) for the subnormal for a given point on the curve, that is, the length of the segment on the x-axis between the abscissa of a point on the curve and the x-intercept of the normal line at that point. The subtangent was defined analogously.

Descartes presented a method for finding the subnormal [Figure 8]. If we can find a circle, with center P on the x-axis, that cuts a curve in precisely one point C, then the radius at that point is normal to the curve. But if the center of the circle through the point C be moved "ever so little" along the x-axis, the circle will cut the curve at two points. This idea provided a means of finding the subnormal for any point (x_0, y_0) on the curve. Starting with the equation of the curve and the equation of a variable circle with center $P = (v, 0)$, find the equation giving their intersection. Then choose P so that the intersection equation has a double root.

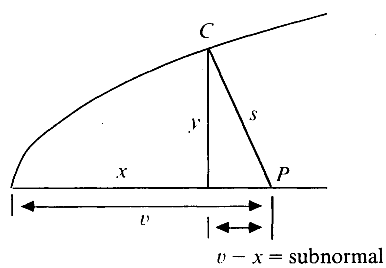


Figure 8

Let us consider the case of the parabola $y^2 = kx$. The circle having center $P = (v, 0)$ and radius s that passes through the point (x_0, y_0) has equation $(v - x_0)^2 + y_0^2 = s^2$. Since (x_0, y_0) is on the parabola, $y_0^2 = kx_0$, and we obtain

$$x_0^2 + (k - 2v)x_0 + (v^2 - s^2) = 0.$$

This equation will have a double root if and only if the discriminant is zero; in which case, $x_0 = -(k - 2v)/2$, or

$$v - x_0 = k/2.$$

This looks mysterious today, but any mathematically literate contemporary of Descartes would know that the parabola has constant subnormal. Perhaps we should check this result using the new calculus: If $y^2 = kx$, then $2yy' = k$. So $y' = k/(2y)$. Thus, the normal line at (x_0, y_0) has slope $-y_0/(k/2)$. To plot the normal line, we go down y_0 from the point (x_0, y_0) to land on the x-axis, and then go right the constant distance $k/2$. Thus, the subnormal for any point on this parabola does indeed have constant length, $k/2$.

Descartes was justly proud of this work, for he wrote:

> I have given a general method of drawing a straight line making right angles with a curve at an arbitrarily chosen point upon it. And I dare say that this is not only the most useful and most general problem in geometry that I know, but even that I have ever desired to know. [3, p. 95]

There is one final quotation from Descartes that is important here, for it deceived Newton—in a positive way:

> When the relation between all points of a curve and all points of a straight line is known [that is, when we have the equation of the curve]... it is easy to find... its diameters, axes, center and other lines [e.g., tangent and normal lines] or points which have especial significance for this curve... By this method alone it is then possible to find out all that can be determined about the magnitude of their areas, and there is no need for further explanation from me. [3, p. 92]

Newton believed Descartes' claim, that from the equation of a curve one can tell everything about it. This encouraged Newton to develop the variety of ad hoc techniques which he learned from the works of Descartes and Wallis into algorithms for solving problems about all curves. This was just one of the motivations that Newton had for inventing the calculus.

For further information about Descartes, see the DSB article by Crombie, Mahoney and Brown [6, IV, pp. 51–65]. The book by Scott [23] contains a detailed discussion of his mathematical work. Bos [1] gives an interesting study of Descartes' concept of curve. Of course, one should read the *Geometry* itself [3], [4].

Frans van Schooten (1615–1660). Schooten enrolled at the University of Leiden at age 16, where he was carefully trained by his father in the Dutch school of algebra. He met Descartes when the latter was in Leiden to supervise the printing of the *Discours de la Méthode* (1637). Schooten recognized the value of the work but had difficulty mastering its contents. So he went to Paris for further study, where he was cordially welcomed by Mersenne.

While in Paris, Schooten read the manuscripts of Pierre de Fermat (1601–1665) and Françoise Viète (1540–1603), and under commission of the famous Leiden publishing house of Elsevier, gathered all the printed works of Viète. This included Viète's most famous work, *In Artem Analyticam Isagoge* (*Introduction to the Analytic Art*) of 1591, which dealt mainly with the theory of equations. Because of this work, Viète is known as the father of algebra. Conscious of the great importance of the scattered works of Viète on algebra, geometry, and analysis, which had been published separately from 1579 to 1615, Schooten republished them with commentary as *Francisci Vietae Opera Mathematica* (1646). The work quickly became an indispensable collection of mathematical source materials, and Newton carefully studied a copy from the Cambridge libraries [20, I, p. 21].

Schooten returned to Leiden in 1643 and began working on a Latin translation of Descartes' *Géométrie*, which he published in 1649. Descartes had been dissatisfied with the form and argument of his *Géométrie* from the very day of its publication, and therefore encouraged the writing of commentaries clarifying its obscurities and developing its approach. Because of its valuable commentary and excellent figures, Schooten's edition was enthusiastically received. This success led him to prepare a much enlarged second edition that appeared in two volumes (1659–1661). It contained about 800 pages of commentary and new work, in addition to the 100 page translation of Descartes' *Géométrie*, and included [20, I, pp. 19–20]:

- Schooten's extremely valuable commentaries. Many of these details were derived directly from Descartes' own criticisms made in correspondence with Schooten.
- Florian Debeaune's (1601–1652) *Notae Breves*, a work which Descartes welcomed as a perceptive exposition of the more elementary aspects of his work. Debeaune posed the first inverse tangent problem.
- Jan Hudde's (1628–1704) studies on equations and extreme values. His rule for locating double roots of equations was useful in applying Descartes' tangent method. It was an important precursor of the derivative.
- Jan de Witt's (1629–1695) excellent tract on conic sections.
- An example of Fermat's extreme value and tangent method.
- Christiaan Huygens' (1629–1695) first publication, an improved method for finding the tangent to the conchoid.
- Hendrik van Heuraet's (1633–*ca.* 1660) rectification method, of which we shall say more below.

All of this shows the great effort that Schooten devoted to the training of his students and to the dissemination of their findings. Much of their work is available only in correspondence, careful studies of which are currently being made. It was from these editions of Schooten that mathematicians learned of the work of Descartes. It was the second Latin edition that Newton borrowed and annotated in the summer and autumn of 1664 (the copy he bought the following winter may have been the 1649 edition). It had an immense impact on his mathematical development; for after mastering it, he was current with research in the new analysis.

John Wallis (1616–1703). Before attending Emanuel College, Cambridge, the only mathematics Wallis knew was what he learned from his brother who was preparing for a trade. At Cambridge, mathematics "were scarce looked upon." He took his M.A. in 1640 and was ordained. In 1649, he was appointed Savilian professor of Geometry at Oxford, an appointment that must have surprised those who thought the only mathematics he had done was to decode a few messages for the Parliamentarians.

This is not quite true, but, wrote Wallis "I had not then [in 1648] seen Descartes' Geometry." [20, III, p. xv]. In 1647 or 1648, he chanced upon Oughtred's *Clavis*, mastering it in a few weeks, and then rediscovered Cardano's formula for the cubic. In 1648, at the request of Cambridge professor of mathematics John Smith, he reworked Descartes' treatment of the fourth degree equation by factoring it into two quadratics. As soon as he was appointed Savilian professor at Oxford, he took up the study of mathematics, with rare energy and perseverance, and soon became one of the best mathematicians in Europe. He held the post for 50 years.

Wallis's *Operum Mathematicorum Pars Altera* (Oxford, 1656) was a fat and rather motley two-part collection of his early mathematical lectures, commentaries, and researches [20, I, p. 23]. It contained his *De Sectionibus Conicis* (dated 1655), a treatise of 110 pages that was the first elementary text on the conics treated from the Cartesian viewpoint. In an appendix, Wallis tried to extend the approach to higher plane curves, especially the cubical parabola $a^2y = x^3$, where the constant a^2 was used to preserve dimensionality. He successfully found the subtangent, but had trouble with the graph because he did not feel comfortable with negative numbers. He also introduced the semi-cubical parabola $ay^2 = x^3$, a curve that played a very important role in the development of the calculus [30, pp. 295–298]. Quite suddenly the mathematical world had been presented with a powerful analytic geometry, only to find that there were few curves on which to practice it. The new perspective of Wallis—which took some time to be adopted by the mathematical community—was that any algebraic equation in two variables defines a curve [13, p. 238].

Together with his conic sections, Wallis published the work on which his fame rests, *Arithmetica Infinitorum* (dated 1656; printed 1655). This volume developed from his study of the *Opera Geometrica* (1644) of Torricelli (1608–1647). Wallis tried to apply these methods to the quadrature of the circle, but not even the study of the voluminous *Opus Geometricum* (1647) of Gregorius Saint Vincent (1584–1667), helped. Out of the project of squaring the circle, he did get his famous infinite product for $\pi/4$.

The *Arithmetica Infinitorum* exerted a singularly important influence on Newton when he studied it in the winter of 1664–1665. From it, Newton learned of the problem of quadratures, or, as we now say, finding areas under curves. Newton probably also read Wallis's *Tractatus Duo* (1659) that presented his research on the cycloid, cissoid, and other geometrical figures.

Rectification of Curves. By 1638, Descartes suspected that the logarithmic spiral might be rectifiable; that is, the length of an arc of the curve could be computed. Even if correct, this would not cause him any difficulties because the spiral is a mechanical curve, and Descartes only accepted Aristotle's dictum that straight lines and curved lines could not be compared for geometrical curves. In 1657, Huygens found the length of an arc of a parabola; but he used a mechanical curve in his solution, and thus Descartes' version of Aristotle's dictum was still intact. Also Huygen's method did not generalize.

The first geometrical curve to be rectified in a geometric way was Wallis's semi-cubical parabola $ay^2 = x^3$. As often happens, several people solved the problem simultaneously: William Neil in 1657, Hendrick van Heureat in 1659 [14], and Pierre de Fermat in 1660. Of course, a priority dispute erupted. Heureat's solution was the most influential because it was published in Schooten's second Latin edition of Descartes' *Geometry*. The proof used the new classical geometry of the seventeenth century and was fairly intricate (for details, see [8] or [13]). The method of proof was to replace the problem of rectification of the semi-cubical parabola by a simpler problem, the quadrature of an ordinary parabola.

This transformation of the problem to a simpler one shows up even when we do the problem today with the calculus, but it is so slick that it is easy to miss what happens. Starting with $y^2 = x^3$ (it is no accident that we still do this first today), we obtain $(y')^2 = 9x/4$. Thus, the arc length from, say, $(0,0)$ to $(4,8)$, is

$$L = \int_0^4 \sqrt{1 + (9x/4)}\, dx.$$

The substitution $u = 1 + (9x/4)$ transforms this into

$$L = (4/9)\int_1^{10} \sqrt{u}\, du.$$

The first of these integrals represents an arc length, whereas the second stands for the area under a parabola. Today, we just look at these as two simple integration problems, but in the old days B(efore) C(alculus), these were viewed as two separate kinds of problems.

Heuraet's method was entirely general. When Newton saw the proof, he realized the value of transforming one type of problem into another. This is one of the roots of the Fundamental Theorem of Calculus. It is the biggest swap of all—we trade integration for anti-differentiation. This is precisely what Newton did soon after he read Heureat's proof. (For a full history of the rectification problem, see Hofmann [11, Ch. 8].)

Concluding Remarks about Newton's Readings. In order to do creative work, a mathematician "needs an adequate notation, a competent knowledge of mathematical structure and the nature of axiomatic proof, an excellent grasp of the hard core of existing mathematics and some sense of promising line for future advance." [20, I, p. 11]. The works that Newton chose to read in 1664 and 1665 magnificently met these needs. He took his arithmetic symbolism from Oughtred, his geometrical form from Descartes. Of course, he grafted on new modifications of his own while creating the calculus. He learned elementary scholastic logic in grammar school and traditional forms of mathematical proof from Euclid. He learned the new analytic geometry of the seventeenth century from Schooten and de Witt, topics in algebra and the theory of equations from Viète, Oughtred, Schooten and Wallis. Most importantly, he learned of the twin problems of infinitesimal analysis: From Descartes, the method of tangents; from Wallis, quadrature. There were plenty of open problems for Newton to attack. Without doubt, the two strongest influences on Newton were Descartes and Wallis. [20, I, pp. 11–13]

It is of as much interest to note *what Newton did not read*. We miss the names of Napier, Briggs, Harriot, Desargues, Pascal, Fermat, Stevin, Kepler, Cavalieri, and Torricelli. Among the Greeks there is only Euclid, not Apollonius nor Archimedes. In fact, Newton seemed to dislike the method of exhaustion. There is great significance in this lack of knowledge of ancient mathematics and of the new classical (as opposed to analytic) geometry of the seventeenth century. He was not hampered by its knowledge. Had Newton gained a deep knowledge of classical geometry and the new classical geometry of his century, I conjecture it would have hindered his invention of the calculus (and similarily for Leibniz who was also ignorant of classical geometry).

As Westfall points out [28, p. 100] about Newton's readings:

> In roughly a year, without benefit of instruction, he mastered the entire achievement of seventeenth-century analysis and began to break new ground.

In fact, by mid-1665, Newton's urge to learn from others seems to have abated [20, I, p. 15].

Newton's Works

Newton was an extraordinary scientist because he made so many fundamental contributions to different fields:

- Mathematics, both pure and applied
- Optics and the theory of light and color
- Design of scientific instruments

• Synthesis and codification of dynamics
• Invention of the concept and law of universal gravity.

In addition, we now know, and are willing to admit, that he spent immense amounts of time working on:

• Alchemy
• Chronology, church history, and interpretation of the Scriptures.

The range and depth of Newton's intellectual pursuits never ceases to amaze us.

As a first step in understanding Newton's contributions, consider the chart below that indicates when Newton was involved in various research areas. One might think that Newton thought about everything all of the time, but the manuscript record shows that he worked on only a few areas at any one time, and these were not necessarily—in his mind at least—disjoint.

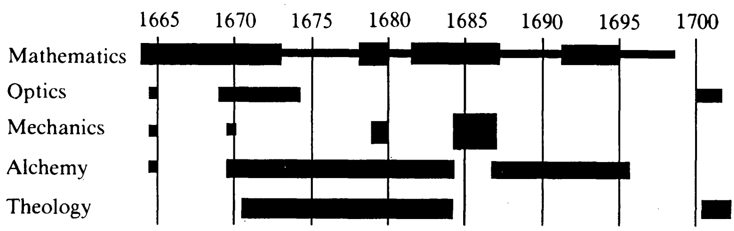


Figure 9. Newton's areas of activity.

We begin with a synopsis of Newton's mathematics as presented in Whiteside's edition of Newton's *Papers* [20]. This will be followed by a thumbnail sketch of each of these areas of Newton's intellectual efforts. Since it is impossible to discuss all of his contributions here, only a few examples of Newton's mathematical work will be discussed in detail. These were chosen with the teacher in mind, to provide examples that can be used in the classroom.

Volume I. (1664–1666). The volume begins with Newton's annotations on the works of Oughtred, Descartes, Schooten, Viète, and Wallis. The bulk consists of research on analytic geometry and the calculus. Newton turns Descartes' subnormal technique into the notion of curvature, and Hudde's rule for double roots into fluxions (differentiation). We see the calculus become an algorithm in mid-1665. This early work on the calculus was summarized in the October 1666 tract on fluxions. In a schematic diagram, Whiteside [20, I, p. 154] shows how all of these ideas came together to give birth to the calculus. The volume ends with miscellaneous work on trigonometry, the theory of equations, and geometrical optics.

Volume II. (1667–1670). Work on classification of cubics begins here and was published as an appendix to his *Optics* (1704). In this volume, we see Newton struggling with the graphs. The most important work on the calculus is the hastily composed 1669 tract *De Analysi* that summarizes all of his work thus far. He gave a copy of this to Barrow in 1669 to assert his priority over Nicolaus Mercator (1619–1687) whose *Logarithmotechnia* (1668), with its infinite series for the logarithm, had just appeared. Half the volume consists of his annotations on the *Algebra* of Kinckhuysen. One piece of Newton's advice here is too good not to pass on to our students:

> After the novice has exercised himself some little while in algebraic computation... I judge it not unfitting that he test his intellectual powers in reducing easier problems to an equation, even though perhaps he may not yet have attained their resolution. Indeed, when he is moderately well versed in this subject... then will he with greater profit and enjoyment contemplate the nature and properties of equations and learn their algebraic, geometrical and arithmetical resolutions. [pp. 423–425]

Volume III. (1670–1673). Although Barrow encouraged Newton to revise *De Analysi* for publication, the booksellers were uninterested. But he did combine the two earlier works on the calculus and many new results in a 1671 tract, with an important foundational change: he postulated a fluent variable of time for his fluxions; that is, all his derivatives are time derivatives. Also, here is an investigation of Huygen's pendulum clock and more research on geometric optics.

Volume IV. (1674–1684). Research in theology and alchemy kept him busy (Figure 9), though his work on mathematics never entirely stopped. This volume contains some of Newton's research on algebra, number theory, trigonometry, and analytical geometry. In the middle of this period, he became fascinated with the classical geometry of the Greeks. Only at the end of this period did Newton show great interest in fluxions and infinite series.

Volume V. (1683–1684). The bulk of this volume consists of Newton's ninety-seven self-styled "lectures," deposited as his Lucasian lectures on algebra for the period 1673–1683. The *Arithmetica Universalis* given here is an incomplete revision of the algebra lectures. Its published version was his most read work, not the papers on calculus.

Volume VI. (1684–1691). Halley's visit in August 1684 turned Newton's interest to the geometry and dynamics of motion, the subject of this entire volume. The work dates from the period 1684–1686, and is arguably as creative as the miracle years of 1664–1666.

Volume VII. (1691–1695). In the early winter of 1691–1962, Newton wrote *De Quadratura Curvarum*, on the quadrature of curves. He also dealt with classical geometry (1693), higher plane curves, and finite-difference approximations (1695). As always, Whiteside has "taken care to preserve all the significant idiosyncrasies, contractions, superscripts and archaic spellings" of the "ink-blobbed, much-cancelled and often rudely scrawled manuscripts." [p. ix]

Volume VIII. (1697–1722). Most mathematicians will find this the most interesting volume after the first, for it contains Newton's solution (simply stated without proof) of the brachistochrone problem as well as documents related to the priority dispute. (To see that this dispute involved much more than mathematics, read Hall's *Philosophers at War* [9].)

We calculus teachers should refrain from telling our students that Newton invented the calculus because he was motivated by physical considerations. Although applications are an excellent reason for studying the calculus, in Newton's case the record is clear: first mathematics, then applications.

The Binomial Theorem. On the frontispiece of the first volume of Newton's *Papers* we see the manuscript where he took up the age old problem of squaring the circle, or (to make the activity sound more respectable) the quadrature of the circle. He became interested in this problem after reading Wallis's *Arithmetica Infinitorum*. Newton learned there how to evaluate the integrals (here expressed in Leibniz's notation) $\int_0^x (1-x^2)^{n/2}\,dx$, where n is an even integer. Newton tabulated the values of these integrals in his attempt to find the area of a circle ($n=1$). To see how he did this consider the case when $n=6$:

$$\int_0^x (1-x^2)^{6/2}\,dx = 1(x) + 3(-x^3/3) + 3(x^5/5) + 1(-x^7/7).$$

The factors in parentheses are recorded in the rightmost column of the table below. The coefficients, $1, 3, 3, 1$, are recorded in the column labeled $n=6$. In general, to evaluate $\int_0^x (1-x^2)^{n/2}\,dx$, sum the products of the values in the nth-column by the corresponding terms in the rightmost column.

$n=0$	$n=2$	$n=4$	$n=6$	$n=8$	$\cdots$	times
1	1	1	1	1	$\cdots$	x
	1	2	3	4	$\cdots$	$-x^3/3$
		1	3	6	$\cdots$	$x^5/5$
			1	4	$\cdots$	$-x^7/7$
				1	$\cdots$	$x^9/9$
					$\vdots$	$\vdots$

Wallis had also tabulated these integrals, but since he used 1 rather than x as an upper limit, he did not see the pattern. But Newton recognized it as "Oughtreds Analyticall table," from his readings of Oughtred's *Clavis* [20, I, p. 452]. We, of course, now call this Pascal's triangle. Newton knew that each number in the table is the sum of the number to its left and the one above that, so he decided to extend the pattern backwards for all even values of n. Thus he obtained:

$\cdots$ $n=-2$	$n=0$	$n=2$	$n=4$	$n=6$	$n=8$	$\cdots$	times
1	1	1	1	1	1	$\cdots$	x
-1	0	1	2	3	4	$\cdots$	$-x^3/3$
1	0	0	1	3	6	$\cdots$	$x^5/5$
-1	0	0	0	1	4	$\cdots$	$-x^7/7$
1	0	0	0	0	1	$\cdots$	$x^9/9$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

To extend this table to odd values of n, Newton used a complicated proportionality argument (see [31] for details). Later, in a letter to Leibniz [19, I, pp. 130–131], Newton provided an easier explanation for the extension. When n is even, say, $n=2m$, the kth entry in the nth column is given by the binomial coefficient $\binom{m}{k-1} = m!/k!(m-k)!$. Newton ignored the restriction that n must be even and used the formula for binomial coefficients when n was odd. For example, the fourth entry in the $n=1$ column is given by

$$\binom{\frac{1}{2}}{4-1} = \frac{(\frac{1}{2})(\frac{1}{2}-1)(\frac{1}{2}-2)}{(1)(2)(3)}.$$

Thus, he obtained:

$\cdots n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$\cdots$	times
1	1	1	1	1	1	1	1	1	1	1	1	$\cdots$	x
$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\cdots$	$-\frac{x^3}{3}$
$\frac{15}{8}$	1	$\frac{3}{8}$	0	$-\frac{1}{8}$	0	$\frac{3}{8}$	1	$\frac{15}{8}$	3	$\frac{35}{8}$	6	$\cdots$	$\frac{x^5}{5}$
$-\frac{35}{16}$	-1	$\frac{5}{16}$	0	$\frac{1}{16}$	0	$-\frac{1}{16}$	0	$\frac{5}{16}$	1	$\frac{35}{16}$	4	$\cdots$	$-\frac{x^7}{7}$
$\frac{315}{128}$	1	$\frac{35}{128}$	0	$\frac{-5}{128}$	0	$\frac{3}{128}$	0	$\frac{-5}{128}$	0	$\frac{35}{128}$	1	$\cdots$	$\frac{x^9}{9}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Now, from the $n=1$ column, Newton was able to draw the conclusion that he sought:

$$\int_0^x (1-x^2)^{1/2}\,dx = x + (1/2)(-x^3/3) + (-1/8)(x^5/5) + (1/16)(-x^7/7) + \cdots.$$

For $x=1$, this gives an infinite series for the area of (a quadrant of) a circle. From this, Newton jumped to the conclusion that a similar "interpolation" could be done on curves (we would say, on functions) as well as on their quadratures (integrals), and then guessed the Binomial Theorem for fractional exponents. He checked this result several ways. First, he formally used the square root algorithm to obtain the series

$$(1-x^2)^{1/2} = 1 - (1/2)x^2 - (1/8)x^4 - (1/16)x^6 - \cdots.$$

Then he checked that it agreed with the Binomial Theorem. Next, he squared both sides of the above equation to see that an equality resulted. As a further check, he used formal long division to obtain an infinite series for $(1+x)^{-1}$. Note the wonderful research techniques he is using. Nonetheless,

> The paradox remains that such Wallisian interpolation procedures, however plausible, are in no way a proof, and that a central tenet of Newton's mathematical method lacked any sort of rigorous justification... Of course, the binomial theorem worked marvellously, and that was enough for the 17th century mathematician. [31, p. 180]

Newton became tremendously excited with his new tool, the Binomial Theorem, which became a mainstay of his newly developing calculus. He also did such bizarre computations as approximating log(1.2) to 57 decimal places.

$$\overline{P+PQ}\Big|^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + \&c.$$

Where $P+PQ$ is the Quantity, whoſe Root is to be extracted, or any Power formed from it, or the Root of any ſuch Power extracted. P is the firſt Term of ſuch Quantity; Q, the reſt (of ſuch propoſed Quantity) divided by that firſt Term, And $\frac{m}{n}$ the Exponent of ſuch Root or Dimenſion ſought. That is, in the preſent caſe, (for a Quadratick Root,) $\frac{1}{2}$.

Figure 10. First publication of the Binomial Theorem, 1685.

The Binomial Theorem was Newton's first mathematical publication. It appeared in Wallis's *Treatise of Algebra* (Figure 10) in a summary of Newton's two famous letters to Leibniz in 1676 [24, pp. 330–331]. These letters are readily available, with ample commentary, in Newton's *Correspondence* [19, II, pp. 20–47 and 110–161].

Optics. Newton's earliest work on optics was done at Cambridge and the experiments continued at Woolsthorpe during the plague, but was not put in near final form until he was preparing his Lucasian lectures for 1670–1672. It had long been known (see, for example, Descartes [4, p. 335]) that when light passed through a prism it was dispersed into a colorful spectrum. Newton was able to give a quantitative analysis of this behavior and to devise a new theory of light. In February 1671/72 (the slash date was used because England had not yet adopted the Gregorian calendar), this resulted in Newton's first publication in optics, the lengthy title of which also provides an abstract:

> A Letter of Mr. Isaac Newton, Mathematick Professor in the University of Cambridge; containing his New Theory about Light and Colors: Where Light is declared to be not Similar or Homogeneal, but consisting of difform rays, some of which are more refrangible than others: And Colors are affirm'd to be not Qualifications of Light, deriv'd from Refractions of natural Bodies, (as 'tis generally believed;) but Original and Connate properties, which in divers rays are divers: Where several Observations and Experiments are alleged to prove the said Theory. [18, p. 47]

This work engendered a controversy with Robert Hooke (1635–1703), who claimed to have published the ideas earlier. As a consequence, Newton became extremely reluctant to publish. In fact, the *Optics* was not published until 1704, the year after Hooke's death.

In developing his theory of light, Newton realized that lenses caused chromatic aberration. This set him thinking about telescope design, and he concluded that the problem could be avoided by using mirrors instead of lenses. Consequently he designed a reflecting telescope, built one himself, and then described it in the March 25, 1672 issue of the *Philosophical Transactions*. These first papers of Newton have been photoreproduced by I. B. Cohen [18], along with a valuable introduction by Thomas Kuhn. Rather than describe Newton's theory of light (which has been done by Alan Shapiro in the first volume of *The Optical Papers of Isaac Newton* [21]), we shall briefly discuss telescope design. This provides an interesting classroom example of the reflective properties of the conics.

The first reflective telescope was designed by James Gregory (1638–1675) and published in his *Optica Promota* of 1663, a work which Newton did not read until after he had invented his own telescope. Gregory's telescope consists of a concave primary mirror (on the right in Figure 11a) that is parabolic in shape, and a concave secondary mirror that is elliptical (strictly speaking, the surfaces generated by rotating these conics about the axis of the telescope). The incoming rays of starlight bounce off the parabolic mirror and are reflected through its focus. Beyond that focus is an elliptical mirror that shares a focus with the parabola and has its other focus behind a small hole in the primary mirror. Thus, after the reflected rays of starlight pass through the common focus of the parabola and ellipse, they are reflected off the elliptical secondary mirror and converge at the second focus of the ellipse. Gregory tried to have a telescope built to his design, but the opticians were unable to polish the mirrors properly.

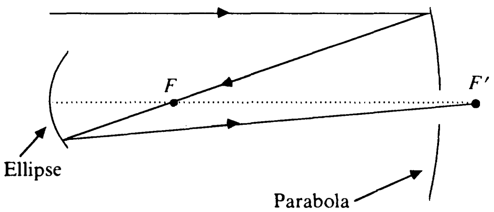


Figure 11a. Gregorian Telescope, 1663.

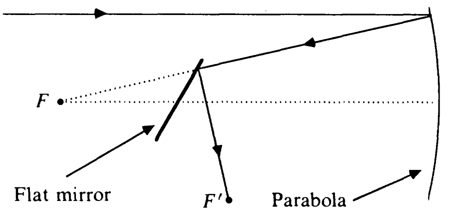


Figure 11b. Newtonian Telescope, 1668.

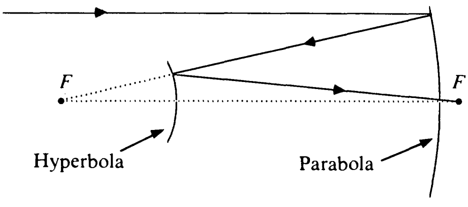


Figure 11c. Cassegrain Telescope, 1672.

In 1668, Newton placed a flat secondary mirror between the primary parabolic mirror and its focus [Figure 11b]. The eyepiece was located at the side of the telescope. Incoming rays of starlight reflect off the parabolic mirror and head for its focus F. Before they get there, they are reflected off the flat mirror. Then they converge toward F', the point symmetric to F with respect to the plane of the flat mirror. This invention remained unknown until Newton made another one (casting and polishing the mirrors himself) and presented it to the Royal Society of London on 11 January 1672. This so impressed the members that they elected him a Fellow of the Royal Society at that very same meeting.

Later in 1672, another telescope design [Figure 11c] was published by Guillaume Cassegrain (*fl. ca.* 1672) in France and abstracted in the *Transactions of the Royal Society*. The concave primary mirror is again a parabola with a hole in the center, and the secondary is a convex hyperbolic mirror which shares a focus with the parabola and has its other focus behind the hole in the parabola. Rays of starlight reflected from the primary parabolic mirror head toward the focus of the parabola. Before reaching that focus, they are reflected by the hyperbolic mirror toward the other focus of the hyperbola.

Cassegrain claimed that his design was superior to Newton's. In what was to become his typical style, Newton marshalled his evidence and attacked furiously. He claimed that Cassegrain's idea was not only a minor modification of Gregory's, but also optically inferior. Cassegrain retreated into anonymity. But Newton was wrong about the superiority of the Gregorian telescope. In 1779, Jesse Ramsden (1735–1800) showed that the combination of a concave and a convex mirror partially corrects the spherical aberrations, whereas in the Gregorian telescope, the aberrations of the two concave mirrors are additive. Today the Cassegrain model is used in most large reflectors.

A mathematical result of Newton's work on optics grew out of the problem of grinding a hyperbolic mirror (although he did not use one, the possibility of a hyperbolic lens was noted by Descartes [4, p. 139]). Newton, and independently

Christopher Wren (1632–1723), discovered that the hyperboloid of one sheet was a ruled surface. Newton used this result to show how to make a hyperboloid of one sheet on a lathe by holding the chisel obliquely to the axis of the lathe.

Religion. Despite inheriting his stepfather's theological library and buying several theological books when he came to Cambridge, Newton's serious study of theology began only in the early 1670's. No doubt this came about because the position of Fellow at Trinity required that one had to be ordained in the Anglican Church within seven years of receiving the M.A. In Newton's case, this was by 1675. Not being one to do anything halfway, Newton became engaged in an extensive reading program that took him through all the early Church Fathers. As a result, ordination became impossible for he had become a heretic.

Newton became an Arian or Unitarian—he denied the Trinity—of deep conviction and remained so for the rest of his life. His argument was: "Though Christ was the only begotten son of God, and hence never merely a man, he was not equal to God, not even after God exalted him to sit at his right side as a reward for his obedience unto death." [29, p. 130]. Newton arrived at this position through a careful analysis of Scripture. He believed that a deceitful Roman Church had manipulated the Emperor Theodosius to introduce the false doctrine of the Trinity into the Scriptures in the fourth century. The *Book of Revelation* was crucial to Newton's interpretation. He believed that the Roman Church was the "Great Apostasy" and never ceased to hate and fear it [28, p. 321].

By 1675, Newton was making plans to leave Cambridge for he knew that as a Fellow at the College of the Holy and Undivided Trinity he could not reveal his Arian views. To do so would be socially unacceptable, and he never in his life did so, except obliquely to a few people of similar persuasion. That he read the situation correctly is indicated by the dismissal of William Whiston (1667–1752), Newton's successor in the Lucasian Chair, for the uncompromising expression of Unitarian views. But just at this time, the Crown granted a special dispensation that the occupant of the Lucasian Professorship was not required to be ordained. Thus, Newton could stay at Cambridge. Newton's theological studies continued until work on the *Principia* interrupted [Figure 9]. In London, he was able to take up his theological studies again, and they continued for the rest of his life. (For further details, see [28, pp. 309–334] or [29].)

Alchemy. Newton's interest in alchemy has long been embarrassing to some scholars, while others delight in this trace of hermeticism and dub him a mystic. But there is now no doubt that he was a serious practitioner [Figure 9]. From 1669 (when he bought his first chemicals) until 1684 (when work on the *Principia* interrupted), Newton spent long hours in the "elaboratory." Newton again practiced alchemy from 1686 until 1696, but after he moved to London he never took it up again seriously [27, p. 121]. Newton did plan on adding alchemical references to the second edition of the *Principia* although he never did so. (For details on his alchemical work, see [5].) One benefit of this work was that he was able to cast the speculum for his first telescope.

In 1693, Newton suffered a nervous breakdown of uncertain duration and severity. There is no doubt that he frequently tasted his chemicals, but whether it was caused by mercury poisoning is debatable [7, pp. 88–90].

When Newton wrote to Oldenburg in 1673 that he intended "to be no further sollicitous about matters of Philosophy" [19, I, p. 294], and to Hooke in 1679 that "I had for some years past been endeavouring to bend my self from Philosophy to other studies in so much yt I have long grutched the time spent in yt study" [19, II, p. 300], we must take him at his word. During most of the decade of the 1670s, Newton preferred theology and alchemy to physics and mathematics.

The *Principia*. In his old age, Newton liked to reminisce and he himself started the story of the falling apple. We have four independent accounts of the tale [7, p. 29–31]. Here is Conduitt's [28, p. 154]:

> In the year 1666 he retired again from Cambridge... to his mother in Lincolnshire & whilst he was musing in a garden it came into his thought that the power of gravity (w^{ch} brought an apple from the tree to the ground) was not limited to a certain distance from the earth but that this power must extend much farther than was usually thought. Why not as high as the moon."

So let us grant that a falling apple started Newton thinking about gravity during the plague years; even if he made up the story it is harmless. But his retelling of this event, in his 1718 letter to DesMaizeaux (which we quoted earlier), is not harmless. Newton attempted to push back the date of his discovery of the law of universal gravitation to the plague years. His papers tell quite a different story.

In late 1664, Newton learned Kepler's third law: the square of the time that it takes a planet to make one elliptical revolution around the sun is proportional to the cube of the mean distance from the sun, that is, $T^2 \sim R^3$. The following January, Newton discovered the Central Force Law (see the letter to DesMaizeaux), which Huygens independently discovered and first published without proof in his *Horologium Oscillatorium* (1673) (see [12]). The Central Force Law states that the centrifugal (center fleeing) force acting on a body traveling about a central point is proportional to the square of the speed and inversely proportional to the radius of the orbit: $F = S^2/R$. Strictly speaking, this "force" is an acceleration, but we shall follow Newton's usage.

Newton was able to discover that the gravitational force between a planet and the sun must be inversely proportional to the square of the distance between them. If a planet travels with uniform speed around a circular (not elliptical) orbit of radius R in time T, then its speed S is $2\pi R/T$. Thus,

$$F = \frac{S^2}{R} = \frac{(2\pi R/T)^2}{R} = 4\pi^2\left(\frac{R^3}{T^2}\right)\left(\frac{1}{R^2}\right).$$

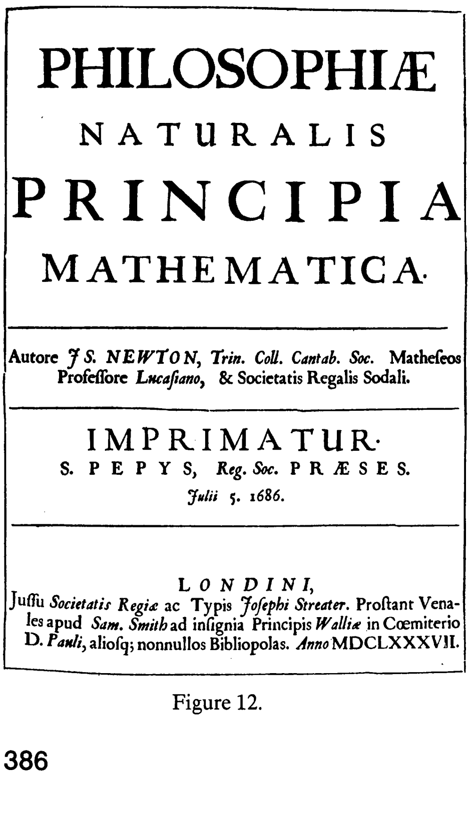
By Kepler's third law, R^3/T^2 is constant, and hence, $F \sim 1/R^2$. Newton left off at this point, devoting most of the next decade to alchemy and theology, though he was never completely divorced from mathematics (Figure 9).

Hooke, Halley, and Wren were able to make this same deduction by 1679, but the problem of explaining the elliptical orbits remained. On 24 November 1679, Hooke wrote to Newton suggesting a private "philosophical," that is, scientific, correspondence on topics of mutual concern. In this letter, Hooke mentioned *his* hypothesis of "compounding the celestiall motions of the planetts [out] of a direct motion by the tangent & an attractive motion towards the centrall body." [19, II, p. 297]. This does not seem to be much of a hint for proving that if the inverse square law holds, then the planets must move in elliptical orbits. But it started Newton thinking about the question again. Hooke gave further encouragement on January 17, when he wrote "I doubt not but that by your excellent method you will easily find out what that Curve must be." [19, II, p. 313]. Newton did succeed in finding the answer, but he kept it to himself.

It was also in 1679 that Newton learned of Kepler's second law: the radius from the sun to a planet sweeps out equal areas in equal times. It seems strange that Newton would have learned of the third law as a student in 1664, but not about the second until years later. The explanation is that the third law was generally accepted in the scientific community because it could be empirically verified, whereas the second was much more of a conjecture.

In August 1684, Edmond Halley (1656–1743) visited the 41-year-old Lucasian Professor at Cambridge. He asked the question that had been consuming him and his friends Hooke and Wren at the Royal Society in London: What path would the planets describe if they were attracted to the sun with a force varying inversely as the square of the distance between them? Newton replied at once that the orbits would be ellipses. Since this was the expected answer, Halley asked Newton how he knew. Newton astonished him by answering that he had calculated it. Halley asked to see Newton's computation, but as Newton seemingly saved every scrap of paper he ever wrote on, he (not surprisingly) could not find it. Perhaps he did not want to find it; his desire to be left alone to pursue his own interests, his fear of controversy, and his reluctance to publish would all make Newton want to carefully check his proof over again before he showed it to anyone. In November of 1684, Newton did send the computation to Halley in London, who was so excited that he prompted Newton to expand his work. (Weinstock [25] has challenged the common view that this proof actually appears in the *Principia*.)

Newton put aside his alchemical and theological studies to work on what was to become the most significant scientific treatise ever written: *Philosophiae Naturalis Principia Mathematica*. It took Newton eighteen months of intense intellectual effort



PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.

Autore *J S. NEWTON*, *Trin. Coll. Cantab. Soc.* Mathefeos Profeffore *Lucafiano*, & Societatis Regalis Sodali.

IMPRIMATUR.
S. PEPYS, *Reg. Soc.* PRÆSES.
Julii 5. 1686.

LONDINI,
Juffu *Societatis Regiæ* ac Typis *Jofephi Streater*. Proftant Venales apud *Sam. Smith* ad infignia Principis *Walliæ* in Cœmiterio D. *Pauli*, aliofq; nonnullos Bibliopolas. *Anno* MDCLXXXVII.

Figure 12.

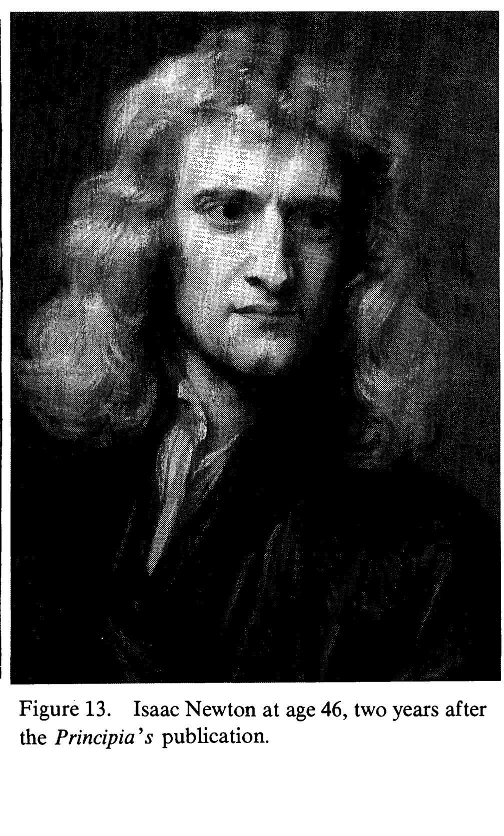


Figure 13. Isaac Newton at age 46, two years after the *Principia's* publication.

to compose his masterpiece, but the time was not just spent in writing up results that he had completed long ago. Many critical ideas in the *Principia* were not developed until the treatise was being written. In particular, Newton created the concept of universal gravity during this period.

> **Myth:** *Though Newton used the notation of the calculus in arriving at his results, he was careful in the Principia to recast all the work in the form of classical Greek geometry understandable by other mathematicians and astronomers.*

Newton started this myth himself in the midst of the priority dispute with Leibniz. If he could argue that he had used his calculus in composing the *Principia*, then he could claim that he did not steal the calculus from Leibniz who published his first paper on the (differential) calculus in 1684.

> The method of fluxions [Newton's calculus] is intrinsically algebraic rather than geometrical, and there is not the slightest reason—in the historical evidence or in logic—to suppose that the argument of the *Principia* was ever cast in an algebraic rather than the geometric mode in which it was published. [9, p. 28]

The geometrical format of the *Principia* is explained by the fact that around 1678, Newton became fascinated with classical geometry [Figure 9]. The *Principia* appears to be densely packed classical geometry, but that is only a façade. One need only read a bit to realize that it is packed with the informal geometrical *ideas* of the new analysis, the calculus. However, the formal machinery of the algebraic algorithms of the calculus is not to be found there. In order to make this point clear, it would help to look at the proof of a proposition from the *Principia*. Book I, Section II, Proposition I, Theorem I says:

> The areas which revolving bodies describe by radii drawn to an immovable centre of force do lie in the same immovable planes, and are proportional to the times in which they are described. [17, p. 40].

That is, if the gravitational force (whatever it might be) always acts toward a fixed point S, then Kepler's equal area law holds.

Newton's proof begins with classical geometry [Figure 14]. Suppose we consider equal time intervals, and that the body moves from A to B in one of those intervals. In the next interval it would move, on the same straight line, from B to c if no external force acts on it. The triangles SAB and SBc that are swept out in these equal time intervals have equal areas since the bases AB and Bc are equal, and the triangles have the same altitude. However, if at B "a centripetal [center seeking] force acts at once with a great impulse," then the body moves to some other point C (in the same plane) in the next time interval. The Parallelogram Law of Forces determines the location of the point C; the lines Cc and SB are parallel. Triangles SBc and SBC also have the same area, since they have a common base SB and their altitudes are equal, namely, the distance between the parallel lines SB and Cc. By transitivity, the triangles SAB and SBC have equal areas. Similarly for other triangles in the diagram. So far the proof was easy geometry. Next, Newton used an idea from his calculus: "Now let the number of those triangles be augmented, and their breadth diminished *in infinitum*... their ultimate perimeter ADF will be a curved line: and therefore the centripetal force, by which the body is continually drawn back from the tangent of this curve, will act continually" and the areas traced out in equal times will be equal [17, pp. 40–41].

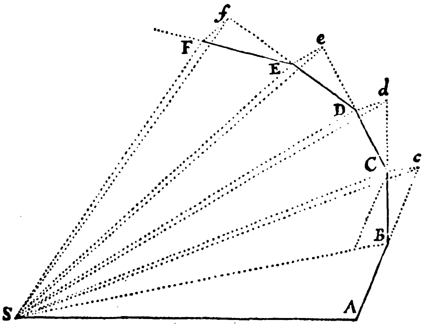


Figure 14. Newton's *Principia*, p. 37.

So that was really quite easy. The geometric ideas of the calculus are used constantly in the *Principia*, but the algebraic notations are not.

Conclusion. On 5 February 1675/76, Newton wrote to Hooke [19, I, p. 416]:

> What Des-Cartes did was a good step. You have added much... If I have seen further it is by standing on ye sholders of Giants.

While it is important to realize that Newton recognized the contributions of his predecessors, we must by now feel that Newton was the greatest giant of all. Just as Westfall, after twenty years of effort preparing *Never at Rest*, was more in awe of Newton when he finished than when he began, we too may realize that the closer we get to Newton, even when standing on the shoulders of Whiteside, the bigger the giant becomes.

Yes, Newton was a genius. That is undeniable. But he was not a Greek god. For all his faults, he displayed characteristics that we should tell our students about, for they are the keys to his, and their, success:

- He built on the best work of the past
- He had brilliant insights
- He worked "by thinking continually"
- He had stubborn perseverance
- He steadily expanded his inquiries
- He made mistakes—and learned from them.

Newton's success was a synergistic combination of innate genius and immense effort. This is the lesson of history.

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