Triquetras and Porisms

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In the May 1916 issue of *The American Mathematical Monthly*, Roger A. Johnson states and proves, without preamble, this appealing theorem [**7**]:

Theorem 1. *If three circles of equal radius intersect in a point O, their remaining intersection points lie on a circle of the same radius. Moreover, the orthocenter of these three points is O.*

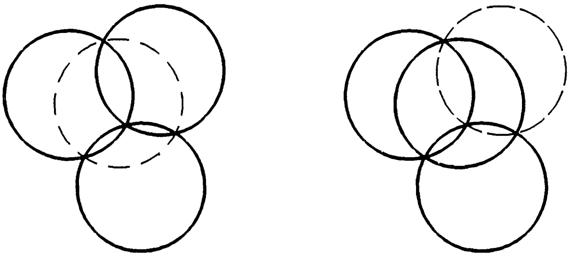


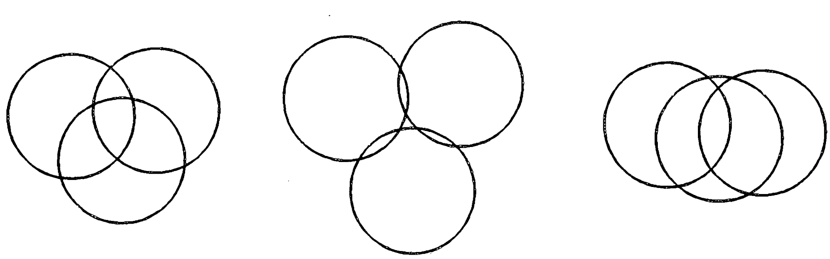
Figure 1
Two possible arrangements of the circles in Theorem 1

For the reader who does not have his or her May 1916 issue of the *Monthly* handy, here is an easy proof, which makes unabashed use of vectors.

Proof. Choose coordinates with the origin at O. Let w_1, w_2, w_3 be the coordinates of the centers of the three circles. Then $|w_i| = r$. Let A, B, C denote the three remaining points of intersection of the circles. It is easily seen that (if the centers are appropriately numbered) $w_1 + w_2 = C$, $w_1 + w_3 = B$, $w_2 + w_3 = A$. These points all lie on the circle of radius r centered at $w_1 + w_2 + w_3$. To verify that O is the orthocenter of $\triangle ABC$, observe that $(w_1 + w_2) \cdot (w_1 - w_2) = 0$, hence $\overrightarrow{OC} \perp \overrightarrow{AB}$. A similar argument obviously works for the remaining two sides. $\square$

"Singularly enough, this remarkable theorem appears to be new," commented Johnson 75 years ago. "A rather cursory search in several of the treatises on modern elementary geometry fails to disclose it, and the author has not found any person to whom it was known. On the other hand, the figure is so simple (especially as it can be drawn and the theorem verified with a coin or other circular object) that it seems almost out of the question that the fact can have escaped detection." Unfortunately, Johnson's theorem seems to be no better known now than it was then. I was unable to find it in any standard text on geometry (but see *Mathematical Reviews* 81c:51010, where the result is attributed to G. Titeica, a contemporary of Johnson), although the converse is readily found: if γ is the circumscribed circle about $\triangle ABC$, then the three circles obtained by reflecting γ through the sides of $\triangle ABC$ intersect at the orthocenter. (See [**9**, p. 51] or [**5**, p. 39].)

On discovering a shiny nugget like Theorem 1, one wonders if there is a gold mine nearby. In mathematical terms, one looks for generalizations. To this end, we will retain the hypothesis that the radii of our three circles, which we will call γ_1, γ_2, and γ_3, are equal (let us call this common radius r), and remove the hypothesis that γ_1, γ_2, and γ_3 are concurrent. In general the three circles will intersect in six points, as illustrated in Figure 2. It is natural to group the six points in two sets of three, and try to relate the circumradii a and b of these two triangles. For example, if the circumradius of the "inner" triangle in Figure 2(a) or (b) is known, can we compute the circumradius of the "outer" triangle? According to Theorem 1, if the "inner" circumradius is 0, then the "outer" circumradius must be r. In general, however, we need one additional datum: the radius of the circle on which the centers of γ_1, γ_2, and γ_3 lie. (This information is already known in Theorem 1 because if γ_1, γ_2, and γ_3 all intersect in a point, the radius of the circle on which their centers lie is also equal to r.) The following theorem, which is the main new result of this article, answers our question in the affirmative.



(a) (b) (c)

Figure 2
Three possible arrangements of the circles in Theorem 2

Theorem 2 (The Triquetra Theorem). *Given three intersecting circles γ_1, γ_2, and γ_3 of equal radius r, such that their centers lie on a circle of radius c. Choose any three intersection points, not all on one of the circles γ_i, and let their circumradius be a. Let the circumradius of the remaining three intersection points be b. Then one of the equations*

$$\pm ab \pm ac \pm bc = r^2 \tag{1}$$

must hold, where two of the $\pm$ signs are positive and one is negative.

Why do I call Theorem 2 "the Triquetra Theorem"? In heraldry (a subject with a fascinating language and set of rules, dating as far back as the Middle Ages),

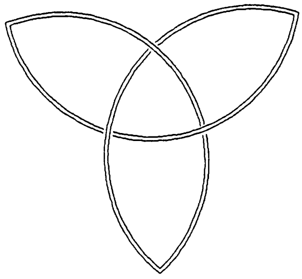


Figure 3
A heraldic triquetra

"triquetra" is a term that describes an emblem like the one illustrated in Figure 3. According to the reference book [**3**], a triquetra consists of "Three equal interlaced arcs. Normally used as a symbol of the Blessed Trinity." I propose *triquetra* as a mathematical term to describe a figure consisting of three mutually intersecting circles of equal radius; as such, it generalizes the notion of a *triangle*, which is a triquetra composed of circles of infinite radius.

The next theorem provides an explanation of the rule for determining the signs in equation (1); it uses the idea of a *signed circumradius* of three points. Intuitively, the circumradius of $\triangle P_1P_2P_3$ (in that order) is given a positive sign if P_1, P_2, and P_3 lie in counterclockwise order on the circle that contains them. If they lie in clockwise order, then the circumradius is given a negative sign.

We also refer below to the notion of *outer* and *inner* intersection points of a triquetra. In the case where the centers of the three circles of the triquetra form an acute triangle, as in Figure 2(a) and (b), it is intuitively obvious which are the outer and inner intersection points. Nevertheless, we need a formal definition. Let C_1, C_2, C_3 be the centers of γ_1, γ_2, γ_3, and O be the circumcenter of $\triangle C_1C_2C_3$. Let M_k be the midpoint of $\overline{C_iC_j}$. It is easy to see, since the radii of the circles γ_i are equal, that the intersection points of γ_i and γ_j lie on the line $\overleftrightarrow{OM_k}$. The intersection point that lies on the ray $\overrightarrow{M_kO}$ is called the *inner intersection point*, and the one that lies on the opposite ray is the *outer intersection point*.

The notation for the following theorem is illustrated in Figure 4.

Theorem 3 (The Triquetra Theorem, Version 2). *For $i = 1, 2, 3$, let the circles γ_i have center C_i and equal radius r. Assume that $\triangle C_1C_2C_3$ is acute, and has signed circumradius $c > 0$. Let $\gamma_1 \cap \gamma_2 = \{A_3, B_3\}$, $\gamma_1 \cap \gamma_3 = \{A_2, B_2\}$, and $\gamma_2 \cap \gamma_3 = \{A_1, B_1\}$. The labels "A" and "B" are chosen so that at least two of the points A_i are outer intersection points. Let a be the signed circumradius of $\triangle A_1A_2A_3$ and let b be the signed circumradius of $\triangle B_1B_2B_3$. Finally, let $\delta = \operatorname{sign}(r^2 - c^2)$. Then*

$$(a - c)(b + \delta c) = |r^2 - c^2|. \tag{2}$$

To see the connection between equation (2) and equation (1), it is only necessary to observe that the c^2 terms on both sides of equation (2) always cancel. It is also worth pointing out that in the case $c > r$ it is unnecessary to keep track of outer and inner intersection points, since equation (2) is then symmetric with respect to a and b.

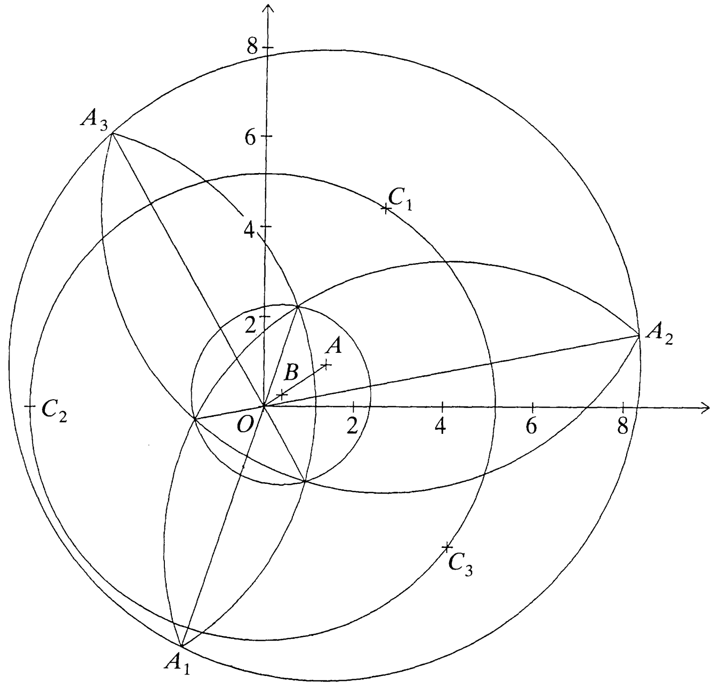


Figure 4

This figure illustrates the labeling of points for Theorems 3 and 5. For those who wish to verify Theorem 3, the coordinates of the points in the figure are as follows (all coordinates are plus or minus 0.001):

$C_1 = (2.688, 4.414)$	$A_1 = (-1.760, -5.330)$	$O = (0, 0)$
$C_2 = (-5.168, -0.013)$	$A_2 = (8.354, 1.622)$	$B = (0.392, 0.267)$
$C_3 = (4.144, -3.086)$	$A_3 = (-3.411, 6.055)$	$A = (1.378, 0.937)$

$a = 7.009 \qquad r = 6.316 \qquad c = 5.168 \qquad b = 1.994$

Theorem 1 can be deduced from the Triquetra Theorem by substituting $b = 0$ and $r = c$. In fact, one can even obtain Theorem 1 from the equation $b = 0$ alone, by applying the following inequalities.

Theorem 4 (The Triquetra Theorem, Version 2, Continued!). *Furthermore, if A_1, A_2, A_3 are all outer intersection points, then*

$$\begin{aligned} &0 \le a, b \le r \le c \le a + b \quad \text{if} \quad \delta < 0; \\ &0 \le b, c \le r \le a \le b + c \quad \text{if} \quad \delta > 0. \end{aligned} \tag{3}$$

No reader who examines Figure 4 carefully could fail to notice the first part of the next theorem. More will be said about the significance of the second part in the final section.

Theorem 5. *If C_i, A_i, and B_i are defined as in Theorem 3 ($\triangle\, C_1C_2C_3$ need not be acute), O is the circumcenter of $\triangle\, C_1C_2C_3$, B is the circumcenter of $\triangle\, B_1B_2B_3$, and A is the circumcenter of $\triangle\, A_1A_2A_3$, then O, A, and B are collinear.*

If $\triangle C_1C_2C_3$ *is acute and* A_i *are the outer intersection points, then*

$$\begin{aligned} &\text{if } c<r, \text{ then } OA^2=\frac{ac}{b}(b+c-a), \quad OB^2=\frac{bc}{a}(b+c-a);\\ &\text{if } c>r, \text{ then } OA^2=\frac{ac}{b}(a+b-c), \quad OB^2=\frac{bc}{a}(a+b-c). \end{aligned} \tag{4}$$

In the next section we will give the proofs of Theorems 2–5. The proof of Theorem 2 is rather long and computational; perhaps some enterprising reader will be able to find a more enlightening geometric proof. However, there are still some interesting and unexpected tricks in this proof. The identity (13) is noteworthy, as it gives a remarkably succinct necessary condition for three vectors to be vertices of a triquetra centered at the origin. (In fact, this condition is also sufficient.)

In the final section we will investigate some interpretations of the Triquetra Theorem, to satisfy those readers who (like the author) find more pleasure in a picture than in a formula. The notion of a porism will come into play there.

Proof of the Triquetra Theorem

Throughout this section, $\{i,j,k\}$ will denote any *cyclic* permutation of $\{1,2,3\}$.

Proof of Theorem 2. Define the points C_i, O, A_i, B_i, M_i, and the signed circumradii a, b, c as in the introduction. The assumption that two of the points A_i are outer intersection points is not important for Theorem 2, but it does no harm to assume this, since one of the two sets of three intersection points must contain two outer intersection points. We begin by choosing coordinates with origin at O. Let v_1, v_2, v_3 be the coordinates of the points A_1, A_2, A_3. In the triangle $\triangle C_1C_2C_3$, let $\theta_i=\angle C_i$. We will assume that the angles θ_i are all acute. At the end of the proof we will explain how to modify the argument in the case where $\triangle C_1C_2C_3$ is obtuse (the changes required are minor). We define "signed distances" $a_i=\pm OA_i$ by letting a_i be positive if A_i lies on the ray $\overrightarrow{OM_i}$, and negative if A_i lies on the opposite ray. We define signed distances $b_i=\pm OB_i$ in precisely the same way. Note that if $r<c$ then a_i and b_i are all positive, since O lies outside each circle γ_i, and therefore does not lie on the segment $\overline{A_iB_i}$. If $r>c$ then the outer intersection point has a positive distance from O, and the inner intersection point has a negative distance.

Now since the triangle $\triangle C_1C_2C_3$ is acute, O is in its interior. By examining the quadrilaterals $OM_iC_kM_j$, it is easy to show that $\angle M_iOM_j=\pi-\theta_k$. Moreover, since $\triangle C_iC_kC_j$ is inscribed in a circle centered at O,

$$\theta_k=\angle C_iC_kC_j=\tfrac{1}{2}\angle C_iOC_j=\angle M_kOC_j=\angle M_kOC_i.$$

Applying the law of cosines to $\triangle A_iOC_j$, we have

$$a_i^2+c^2-2ca_i\cos\theta_i=r^2. \tag{5}$$

(If a_i is negative this remains true.) By computing the power of the point O with respect to γ_1, γ_2, γ_3, we find that

$$a_ib_i=c^2-r^2, \tag{6}$$

hence

$$a_i + b_i = 2c\cos\theta_i. \tag{7}$$

From the formula for the circumradius of a triangle (see [**1**, Chap. 9]),

$$a = \frac{|v_1 - v_2|\,|v_2 - v_3|\,|v_3 - v_1|}{4\text{Area}\,(\triangle A_1A_2A_3)}. \tag{8}$$

(Here "Area" is interpreted as a signed area.) When $a_i > 0$, $\angle A_iOA_j = \theta_i + \theta_j = \pi - \theta_k$, while $a_i < 0$ implies, by supplementary angles, that $\angle A_iOA_j = \theta_k$. Thus in either case, $(A_iA_j)^2 = a_i^2 + a_j^2 + 2a_ia_j\cos\theta_k$, and equation (8) can be rewritten:

$$a = \frac{\left(a_1^2 + a_2^2 + 2a_1a_2\cos\theta_3\right)^{1/2}\left(a_1^2 + a_3^2 + 2a_1a_3\cos\theta_2\right)^{1/2}\left(a_2^2 + a_3^2 + 2a_2a_3\cos\theta_1\right)^{1/2}}{2(a_1a_2\sin\theta_3 + a_1a_3\sin\theta_2 + a_2a_3\sin\theta_1)}. \tag{9}$$

From the formula for the circumradius of $\triangle B_1B_2B_3$ which corresponds to (8) and (9), replacing b_i by $(c^2 - r^2)/a_i$, we find that

$$b = \frac{|v_1 - v_2|\,|v_2 - v_3|\,|v_3 - v_1|}{2a_1a_2a_3(a_1\sin\theta_1 + a_2\sin\theta_2 + a_3\sin\theta_3)}|r^2 - c^2|. \tag{10}$$

For later reference, note that

$$\theta_1 + \theta_2 + \theta_3 = \pi, \tag{11}$$

which implies that

$$\sin 2\theta_1 + \sin 2\theta_2 + \sin 2\theta_3 = 4\sin\theta_1\sin\theta_2\sin\theta_3. \tag{12}$$

We return to the three equations in (5). Multiplying the ith equation by $(a_j^2 - a_k^2)$ and summing, we obtain

$$\left(a_2^2 - a_3^2\right)a_1\cos\theta_1 + \left(a_3^2 - a_1^2\right)a_2\cos\theta_2 + \left(a_1^2 - a_2^2\right)a_3\cos\theta_3 = 0.$$

This has a very interesting interpretation if we identify the vectors v_1, v_2, v_3 with complex numbers. We find, after a little calculation, that

$$\operatorname{Re}\left[\bar{v}_1\bar{v}_2\bar{v}_3(v_1 - v_2)(v_2 - v_3)(v_3 - v_1)\right] = 0. \tag{13}$$

Hence,

$$\begin{aligned}|v_1 - v_2|\,|v_2 - v_3|\,|v_3 - v_1| &= \frac{\left|\operatorname{Im}\left[\bar{v}_1\bar{v}_2\bar{v}_3(v_1 - v_2)(v_2 - v_3)(v_3 - v_1)\right]\right|}{|v_1|\,|v_2|\,|v_3|}\\ &= \left|\left(a_2^2 + a_3^2\right)a_1\sin\theta_1 + \left(a_1^2 + a_3^2\right)a_2\sin\theta_2\right.\\ &\qquad \left. + \left(a_1^2 + a_2^2\right)a_3\sin\theta_3\right|, \end{aligned} \tag{14}$$

where the latter equation follows from a computation which essentially reverses the procedure for deriving (13.) It will turn out that when $\triangle C_1C_2C_3$ is acute the expression inside the absolute values in (14) is always positive. However, since we do not know that yet, we will denote its sign by ε. By adding the equations (5) two at a time, we find that

$$a_i^2 + a_j^2 = 2r^2 - 2c^2 + 2ca_i \cos\theta_i + 2ca_j \cos\theta_j.$$

Plugging these into (14),

$$|v_1 - v_2|\,|v_2 - v_3|\,|v_3 - v_1| = 2\varepsilon\big[(r^2 - c^2)(a_1 \sin\theta_1 + a_2 \sin\theta_2 + a_3 \sin\theta_3) \\ + c(a_1 a_2 \sin\theta_3 + a_1 a_3 \sin\theta_2 + a_2 a_3 \sin\theta_1)\big].$$

(Note that we have used equation (11) at this step, so that, for example, $\sin\theta_3 = \sin\theta_1 \cos\theta_2 + \sin\theta_2 \cos\theta_1$.) Let

$$X = a_1 a_2 \sin\theta_3 + a_1 a_3 \sin\theta_2 + a_2 a_3 \sin\theta_1, \\ Y = a_1 \sin\theta_1 + a_2 \sin\theta_2 + a_3 \sin\theta_3. \tag{15}$$

Then, from (9), (10) and (15),

$$a = \varepsilon \frac{(r^2 - c^2)Y + cX}{X} \quad \text{and} \quad b = \delta\varepsilon \frac{(r^2 - c^2)Y + cX}{a_1 a_2 a_3 Y}(r^2 - c^2),$$

hence

$$(a - \varepsilon c)\left(b - \delta\varepsilon \frac{(r^2 - c^2)^2}{a_1 a_2 a_3}\right) = \delta \frac{c(r^2 - c^2)^2}{a_1 a_2 a_3},$$

or

$$ab - \varepsilon cb - \delta\varepsilon \frac{(r^2 - c^2)^2}{a_1 a_2 a_3} a = 0. \tag{16}$$

This nearly gives us a relation between a, b, c, and r, but not quite. We still need to eliminate $a_1 a_2 a_3$, and it is not so obvious that this can be done!

The starting point is identity (7), which we have not yet used explicitly. This gives us

$$b_1 \sin\theta_1 + b_2 \sin\theta_2 + b_3 \sin\theta_3 \\ = -(a_1 \sin\theta_1 + a_2 \sin\theta_2 + a_3 \sin\theta_3) + c(\sin 2\theta_1 + \sin 2\theta_2 + \sin 2\theta_3).$$

But from (6) we also have

$$b_1 \sin\theta_1 + b_2 \sin\theta_2 + b_3 \sin\theta_3 \\ = -\left(\frac{r^2 - c^2}{a_1 a_2 a_3}\right)(a_1 a_2 \sin\theta_3 + a_1 a_3 \sin\theta_2 + a_2 a_3 \sin\theta_1).$$

Thus

$$-\left(\frac{r^2 - c^2}{a_1 a_2 a_3}\right)X + Y = c(\sin 2\theta_1 + \sin 2\theta_2 + \sin 2\theta_3). \tag{17}$$

Continuing in the same vein, we have from (7):

$$b_1 b_2 \sin\theta_3 + b_1 b_3 \sin\theta_2 + b_2 b_3 \sin\theta_1 = a_1 a_2 \sin\theta_3 + a_1 a_3 \sin\theta_2 + a_2 a_3 \sin\theta_1$$
$$-2c(a_1 \sin\theta_1 + a_2 \sin\theta_2 + a_3 \sin\theta_3) + 4c^2 \sin\theta_1 \sin\theta_2 \sin\theta_3,$$

while from (6),

$$b_1 b_2 \sin\theta_3 + b_1 b_3 \sin\theta_2 + b_2 b_3 \sin\theta_1 = \frac{(r^2-c^2)^2}{a_1 a_2 a_3}(a_1 \sin\theta_1 + a_2 \sin\theta_2 + a_3 \sin\theta_3).$$

Thus

$$-X + \left[2c + \frac{(r^2-c^2)^2}{a_1 a_2 a_3}\right] Y = 4c^2 \sin\theta_1 \sin\theta_2 \sin\theta_3. \tag{18}$$

At this point we could solve for X and Y, but it is more convenient to work directly with equations (17) and (18). By equations (9), (10), (12) and (17), we obtain

$$a - \delta b = \frac{|v_1 - v_2|\,|v_2 - v_3|\,|v_3 - v_1|}{XY}(2c \sin\theta_1 \sin\theta_2 \sin\theta_3). \tag{19}$$

Similarly, from (18) we obtain

$$\left[2c + \frac{(r^2-c^2)^2}{a_1 a_2 a_3}\right] a - \delta\left(\frac{a_1 a_2 a_3}{r^2 - c^2}\right) b$$
$$= \frac{|v_1 - v_2|\,|v_2 - v_3|\,|v_3 - v_1|}{XY}\left(2c^2 \sin\theta_1 \sin\theta_2 \sin\theta_3\right). \tag{20}$$

From equations (16), (19) and (20) we conclude that

$$ab + \delta\varepsilon ac - \varepsilon \frac{a_1 a_2 a_3}{r^2 - c^2} b = 0. \tag{21}$$

This allows us to solve for $a_1 a_2 a_3$:

$$a_1 a_2 a_3 = \varepsilon(r^2 - c^2)(b + \delta\varepsilon c)a/b. \tag{22}$$

Substituting into (16) and simplifying, we obtain the equation

$$(b + \delta\varepsilon c)(a - \varepsilon c) = \delta(r^2 - c^2) = |r^2 - c^2| \tag{23}$$

or

$$\delta ab + \varepsilon ac - \delta\varepsilon bc = r^2. \tag{24}$$

Since δ and ε are both equal to ± 1, this equation has the desired form $\pm|ab| \pm |ac| \pm |bc| = r^2$. We introduce absolute values here because the radii in the statement of Theorem 2 are assumed nonnegative, while the radii we have used in the proof are signed. Moreover, we note that the product of the three summands in equation (24) is negative, hence either one or three of the $\pm$ signs are negative. However, they could not all be negative, since their sum, r^2, is positive. Hence, exactly one of the $\pm$ signs is negative, as claimed. This finishes the proof in the case where $\triangle C_1C_2C_3$ is acute.

If $\triangle C_1C_2C_3$ is obtuse, let us assume, without loss of generality, that $\angle C_1$ is obtuse. Then every numbered formula in the above proof still holds, provided we *reverse* the sign of a_1 and b_1. That is, we define a_1 to be positive if A_1 lies on the ray *opposite* $\overrightarrow{OM_1}$, and similarly for B_1. The details are left to the reader. □

Proof of Theorem 3. Most of the work for this theorem has already been done; we need only show that the number ε in formula (23) is always $+1$. This is clearly true if all the signed distances a_i are positive (because every term in the expression on the right-hand side of equation (14) is positive). Thus if $\delta = -1$ (i.e. $r < c$), we are already done (see the remarks in the first paragraph of the proof of Theorem 2). Hence we may assume $\delta = 1$. By hypothesis, at most one of the a_i's is negative; without loss of generality, we may assume $a_1 < 0$, a_2 and $a_3 > 0$. We claim that, in this case, *a and b cannot both be positive*. Indeed, if a and b are both positive, then, from equations (9) and (10) we have

$$\frac{a_2 a_3 \sin\theta_1}{a_2\sin\theta_3 + a_3\sin\theta_1} > |a_1| > \frac{a_2\sin\theta_2 + a_3\sin\theta_3}{\sin\theta_1},$$

hence

$$a_2a_3\sin^2\theta_1 > (a_2\sin\theta_3 + a_3\sin\theta_2)(a_2\sin\theta_2 + a_3\sin\theta_3) > a_2a_3(\sin^2\theta_2 + \sin^2\theta_3).$$

But this implies $\sin^2\theta_1 > \sin^2\theta_2 + \sin^2\theta_3$, which is impossible in an acute triangle. By contradiction, a or b must be negative.

To finish the argument, we turn to equation (22). Suppose $\varepsilon = -1$. Since $a_1a_2a_3 < 0$ and $r^2 - c^2 > 0$, we conclude that $(b-c)a/b > 0$. By equation (23), $\operatorname{sign}(a+c) = \operatorname{sign}(b-c)$. By the equations leading up to formula (16), $\operatorname{sign}(ab) = \operatorname{sign}(a_1a_2a_3XY) = -\operatorname{sign}(XY)$. Then by equation (19), $\operatorname{sign}(a-b) = -\operatorname{sign}(ab)$. If $ab < 0$, then we have $a - b > 0$, hence $a > 0 > b$. Thus $a + c > 0$, contradicting the assertion that $\operatorname{sign}(a+c) = \operatorname{sign}(b-c)$. If $ab > 0$, then $b - c > 0$, thus $b > 0$, and $a > 0$ as well. But this contradicts the assertion proven above, that a and b are not both positive. □

Proof of Theorem 4. If all of the points A_i are outer intersection points, a and b are automatically positive because the distances a_i are.

If $\delta = 1$, then $c < r$ by definition. By equation (19), $a > b$. If $a \le r$, then by (2),

$$r^2 = ab + c(a-b) < r(b + a - b) \le r^2,$$

a contradiction. Hence $a > r$. Again by (2), $(b+c)(a-c) = (r+c)(r-c)$; since $a - c > r - c$, it follows that $b < r$.

The fact that $a \le b + c$ is an easy corollary of Theorem 5. (To avoid circular reasoning, note that this inequality is not used in the proof of that theorem.)

The arguments in the case $\delta = -1$ are very similar, and are left to the reader. In particular, the inequality $c \le a + b$ also follows from Theorem 5. □

Exercises. Several more inequalities concerning a, b and c can be derived under the hypotheses of Theorem 4 ($\triangle C_1C_2C_3$ is acute, A_i are the outer intersection points). All of these are more or less straightforward applications of formulas (2) and (3) (one does not need any more to go into the details of how these were derived). We leave these as exercises for the interested reader:

$$\begin{array}{lll}\text{If } \delta > 0: & \text{(a)} & a^2 - ac + c^2 \le r^2, \\ & \text{(b)} & a^2 - ab + b^2 \le r^2, \\ & \text{(c)} & b^2 + bc + c^2 \ge r^2. \\ \text{If } \delta < 0: & \text{(a)} & a^2 - ac + c^2 \le r^2, \\ & \text{(b)} & b^2 - bc + c^2 \le r^2, \\ & \text{(c)} & a^2 + ab + b^2 \ge r^2. \end{array}$$

If $\delta > 0$, it is an easy exercise, using equation (2) and inequality (c) above, to show that $a \le 2r/\sqrt{3}$. Thus three discs of radius r can cover a disc of radius at most $2r/\sqrt{3}$. (This is not a new result; see **[8]** for a more general theorem.)

Remark. It is also possible to establish a converse to inequality (a) above: *Given a, c, r such that $a^2 - ac + c^2 \le r^2$, then there exists a triquetra of circles of radius r, whose centers lie on a circle of radius c and whose outer intersection points lie on a circle of radius a. In fact, the triquetra may be chosen to be "isosceles."* The proof is not very interesting, so we only sketch it here. First, it is easy to see that the hypothesis implies $a < r < c$ or $c > r > a$ (the case $c = r = a$ is obvious by Theorem 1). In the terminology of the proof of Theorem 2, set $a_1 = a_2 = x$ and $a_3 = y$. The angles θ_i may be found from equation (7). The condition (9) gives one equation relating x and y, and another can be obtained from (22) by substituting in the value of b found from equation (2). These equations are fairly easy to solve for x^2 and y. For instance, in the case $c < r < a$, we get

$$x^2 = \frac{r^2 - c^2}{a - c}\left[a \pm \sqrt{ac(r^2 - a^2 + ac - c^2)/(r^2 - ac)}\,\right],$$
$$y = a \mp \sqrt{ac(r^2 - a^2 + ac - c^2)/(r^2 - ac)}\,.$$

Using the inequalities

$$r^2 - ac > r^2 - a^2 + ac - c^2 \ge 0,$$

it is easily seen that the expression under the radical is nonnegative and less than a^2, so positive solutions for x and y exist. From x, y and θ_i the triquetra can be reconstructed.

Proof of Theorem 5. If $r > c$, then equations (6) imply that B_1, B_2, and B_3 are obtained by inverting A_1, A_2, and A_3 with respect to the circle of radius $(r^2 - c^2)^{1/2}$ centered at O, then performing a half-turn around O. If $r < c$, then B_1, B_2 and B_3 are obtained instead by inverting A_1, A_2 and A_3 in a circle of radius $(c^2 - r^2)^{1/2}$ centered at O. No half-turn is required. In either case, A and B are collinear with O.

To find the distance OA, we apply the same inversion. We have seen that $\odot B_1B_2B_3$ is the inverse, with respect to a circle of radius $|r^2 - c^2|^{1/2}$, of a circle of radius a whose center is a distance OA from the center of inversion. By inversive geometry, the radius b of this circle must be equal to

$$b = \pm|c^2 - r^2|a/\left|a^2 - (OA)^2\right|. \tag{25}$$

The hypothesis that A_i are outer intersection points ensures that a and b are nonnegative (Theorem 4). If $\triangle C_1C_2C_3$ is acute, we may use formula (2) for $|c^2 - r^2|$. Finally, one can also use the hypothesis that $\triangle C_1C_2C_3$ is acute to show that $a > OA$, as follows. Because A_i are the outer intersection points, $\angle A_iOA_j = \angle M_iOM_j$ is obtuse for each i, j (see the beginning of the proof of Theorem 2), and hence O lies in the interior of $\triangle A_1A_2A_3$. It follows that O lies in the interior of $\odot A_1A_2A_3$, and hence the distance OA is less than the radius of $\odot A_1A_2A_3$, which is a. With these facts in hand, it is easy to solve for OA in equation (25) and arrive at equation (4). To obtain the formula for OB, note that $\odot B_1B_2B_3$ can be obtained from $\odot A_1A_2A_3$ by a dilation with center O and magnification factor b/a, followed by a half-turn about O. □

Geometric Consequences of the Triquetra Theorem

In this section we will continue to let a denote the circumradius of the *outer* intersection points of a triquetra and b denote the circumradius of the *inner* intersection points. We will call the four parameters a, b, c and r which describe the geometry of a triquetra its *fundamental radii*. In Theorems 2 and 3 we discovered that there is an algebraic relationship among a, b, c, and r. Why is this surprising? We may think of a triquetra as being determined by a triangle ($\triangle C_1C_2C_3$) and a radius (r). Up to isometry, a triangle is determined by three parameters (for example, its side lengths); hence the space of all triquetras is 4-dimensional, and we would expect four arbitrarily chosen parameters to be algebraically independent. The fact that a, b, c and r are dependent means that something geometrically nontrivial is going on. In fact, by Sard's theorem, for almost all "qualifying" values of a, b, c and r (e.g. those satisfying equation (2) and $a^2 - ac + c^2 \le r^2$, by the remark before the proof of Theorem 5) there must be a 1-parameter family of triquetras (or, to put it another way, infinitely many triquetras) with those fundamental radii. The goal of this section is to understand this 1-parameter family.

A *porism* is usually defined to be a problem with either no solutions or infinitely many solutions (see, for example, [**6**, footnote on p. 113]). Thus the Triquetra Theorem may be rephrased as a porism: given a, b, c, r, find a triquetra with this set of fundamental radii. From Section 2, we know the problem has no solutions if $(a-c)(b \pm c) \neq |r^2 - c^2|$ or if $a^2 - ac + c^2 > r^2$; otherwise, we know it does have a solution and, by the heuristic argument above, it should have infinitely many.

One of the best-known and prettiest porisms in Euclidean geometry is *Poncelet's theorem*, illustrated in Figure 5. This states (see [**1**, Sec. 16.6]):

If two conics are positioned so that there is an n-sided polygon inscribed in one and circumscribed about the other, then there exist infinitely many such polygons. In fact, we can be even more precise: if such a polygon exists and we are given one line segment with endpoints on the first conic and tangent to the second, then we can complete an "inscribed-circumscribed" n-gon containing that segment. In many cases—for example, when the conics are ellipses, with the second in the interior of the first (as in Figure 5)—it follows that any point on the first conic can be used as the "starting point" for drawing such an n-gon.

The general case of Poncelet's theorem is a deep result in projective geometry. However, the case $n = 3$, which is the one relevant to our discussion, is more elementary. If the two conics are circles, it is a converse of Euler's formula relating the circumradius R and inradius r of a triangle $\triangle ABC$ to the distance d between

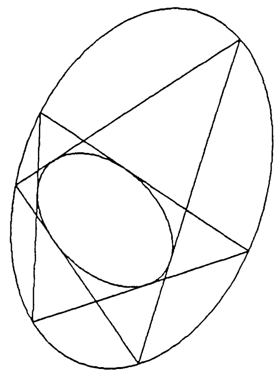


Figure 5
Poncelet's Theorem

the incenter and circumcenter:

$$R^2 - 2rR = d^2.$$

(See [**1**, exercise 10.13.3], and its solution in [**2**, p. 191].) The case $n = 3$ for arbitrary conics follows from Desargues' involution theorem ([**4**, exercise 9.4.3]). Ultimately, we will trace the surprising existence of a 1-parameter family of triquetras with given fundamental radii back to the case $n = 3$ of Poncelet's theorem. We will also discover some attractive new "Poncelet-like" porisms.

By Theorem 5, the infinitely many triquetras that have the same fundamental radii a, b, c, r have yet two more parameters in common, namely the distances OA and OB. Moreover, since the points O, A, B are collinear (Theorem 5), the distance AB is also the same for each of these triquetras. Since the radii and distance between the centers of $\odot A_1A_2A_3$ and $\odot B_1B_2B_3$ are the same for each triquetra, *we may consider these circles themselves as being fixed*, and hence:

Theorem 6. *Given two circles, one in the interior of the other. If there exists a triquetra of circles of radius r, with outer intersection points on the outer circle and inner intersection points on the inner circle, there exist infinitely many such triquetras.*

(See Figure 6. One may imagine these as frames from a movie.) In fact, reasoning by analogy with Poncelet's theorem, we may suspect a stronger statement is true: *Every point on the outer circle and every point on the inner circle is an intersection point of some such triquetra.*

Theorem 6 has the slight drawback that it is comprehensible only to someone who knows what a triquetra is. Our last porism has no such drawback.

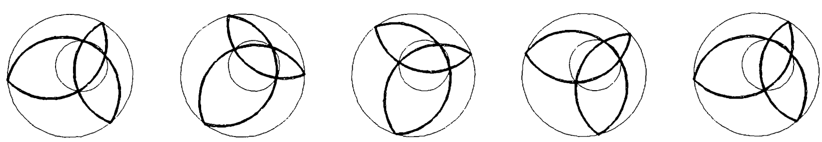


Figure 6
A family of triquetras

Theorem 7. *Given two circles γ and γ', and a number r, such that for every point A in either of the circles, there exist two points in the other circle at distance r from A. Suppose there exists an equilateral hexagon with sides of length r, whose vertices lie alternately on γ and γ'. Then there are infinitely many such hexagons; in fact, every point of γ and γ' lies on some hexagon of this type.*

(See Figure 7.) The heuristic argument for Theorem 7 is to take the circles $\odot C_1C_2C_3$ and $\odot A_1A_2A_3$ as being fixed. Then the hexagon $A_1C_3A_2C_1A_3C_2$ alternates between these two circles and has side lengths r, as described. Our infinite family of triquetras will then provide us with an infinite family of hexagons with this property.

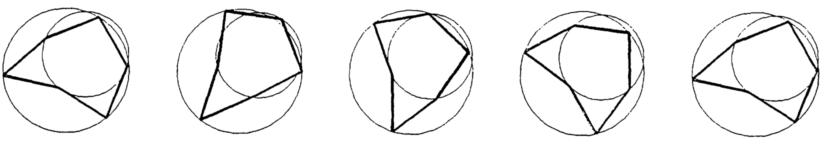


Figure 7
A family of equilateral hexagons
Note that Theorem 7 may hold even if neither circle is inside the other.

The heuristic arguments I have given for Theorems 6 and 7 fall a little short of being actual proofs. To begin with, our argument by counting parameters does not actually guarantee that for *every* value of a, b, c, r there are infinitely many triquetras; Sard's theorem allows for a measure-zero set of exceptions. Secondly, even when we do have a 1-parameter family of triquetras with given values of a, b, c, OA, OB, and r, it is still a big step to get to the stronger claims in Theorem 6 and 7, that *any* point on the outer circle is a vertex of such a triquetra.

Instead of trying to make our heuristic arguments more rigorous, we can more easily use Poncelet's theorem itself to prove our "Poncelet-like" conjectures. Indeed, to obtain Theorem 7 it is sufficient to prove the following lemma:

Lemma 8. *Given two circles, γ and γ', and a fixed number r, satisfying the first hypothesis of Theorem 7. Consider the set of all triangles $\triangle ABA'$ such that A, A' lie on γ, B lies on γ', and $AB = BA' = r$. Then the set of all the lines $\overleftrightarrow{AA'}$ is the dual to a conic Γ.*

To deduce Theorem 7 from this, begin with any point $A \in \gamma$. By hypothesis, there exists a point $B \in \gamma'$ at distance r from A, and a second point $A' \in \gamma$ at distance r from B. By Lemma 8, $\overleftrightarrow{AA'}$ is tangent to Γ. By Poncelet's theorem, $\overline{AA'}$ is one side of a triangle inscribed in γ and circumscribed about Γ. By Lemma 8, this triangle gives rise to the desired equilateral hexagon.

Proof of Lemma 8. Let a be the radius of γ and b the radius of γ'. Choose coordinates so that γ' is centered at the origin, and γ is centered at the point $C = (c, 0)$, $(c \geq 0)$. If $B \in \gamma'$, the line $l(B) = \overleftrightarrow{AA'}$ has the following equation:

$$2X \cdot (C - B) = r^2 + c^2 - a^2 - b^2.$$

For each $B \in \gamma'$, let

$$f(B) = \frac{r^2 + c^2 - a^2 - b^2}{2(B \cdot C - b^2)} B, \tag{26}$$

and let Γ denote the curve $f(\gamma')$. Note that $f(B) \in l(B)$. A routine computation shows that the tangent to Γ at $f(B)$ is perpendicular to $C - B$, which is also the normal to the line $l(B)$. Hence in fact the tangent line to Γ at $f(B)$ is $l(B)$. If we let $B = (b\cos\theta,\ b\sin\theta)$, then the equation for Γ in polar coordinates follows immediately from (26):

$$f(\theta) = \frac{a^2 + b^2 - r^2 - c^2}{2(b - c\cos\theta)},$$

and hence Γ is a conic, one of whose foci is the center of γ'. □

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