Inverse Problems and Torricelli's Law

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The undergraduate curriculum is dominated by *direct problems*, that is, problems in which just enough information is provided for the student to carry out a well-defined stable process leading to a unique solution. Such problems may be represented schematically as in Figure 1, where K is an operator modeling some process, u is some input in the domain of K, and v is the required output. Because K is a function, a unique output v exists for each input u in the domain of K and if K is in some sense *continuous*, as is very often the case, the output v depends continuously on u. Putting it another way, the output v is stable with respect to appropriate small changes in the input u.

Two *inverse problems* are associated with every such direct problem. One, the *causation* problem, involves determining the cause u given the effect v and the model K. When K is a finite dimensional linear operator, the causation problem is at the heart of linear algebra: it is the problem of solving the linear system

$$Ku = v$$

for u. The other inverse problem, the *model identification* problem, consists of determining the operator K (from a given class of models) given cause-effect pairs (u, v). (See [**1**, **2**, **6**] for an introduction to some physical and mathematical aspects of inverse problems.)

Three issues relating to these inverse problems come to mind. The first is *existence*. Given a model K, is there a cause u (in some class) that can account for an observed effect v? For the model identification problem the corresponding question is: given an observed collection of cause-effect relationships (u, v), is there a model (in some class) that can explain it? *Uniqueness* is also an issue that presents itself. Might two distinct causes account for the same effect? Can distinct models determine the same cause-effect observations? Finally, since in most practical inverse problems the data of the problem are measured quantities, or are represented in a computer in a truncated form, the *stability* of solutions is an issue of some relevance. Specifically, is the solution continuous with respect to the data?

Early in this century Jacques Hadamard adopted the attributes of existence, uniqueness, and stability as the hallmarks of a well-set mathematical problem. He defined a problem to be *well posed* (actually, he used the term *correctly set*) if it has a unique solution that is continuous with respect to the data and he expressed the view that any mathematical problem representing physical reality must be well

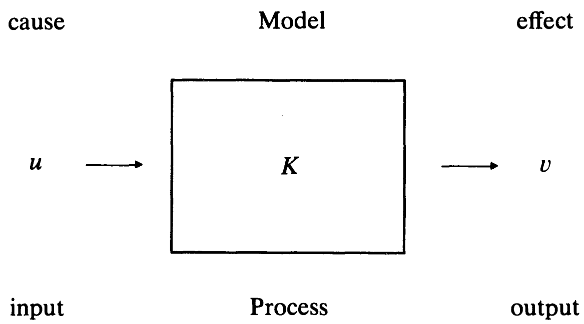

Figure 1

posed. A problem that is not well posed is called *ill posed*. Today Hadamard's belief that physical problems must be well posed has been shown to be mistaken. A myriad of important problems in remote sensing, medical imaging, nondestructive testing, environmental monitoring, system identification, and other fields are inverse problems of the types discussed above and are ill posed. Modern science and technology will confront our students with such inverse problems when they leave the university, yet the existence-uniqueness-stability trinity holds sway in the current curriculum and scant attention is given to inverse problems. The aim of this note is to call attention to three elementary inverse problems, all stemming from Torricelli's law, that are interesting in themselves and can be used to introduce undergraduates to notions related to ill-posed problems.

The first problem is algebraic in nature and illustrates how a physical problem can have a non-unique solution. In this problem instability is not at issue, but in the second problem, which is a model identification problem, instability arises as a consequence of the inherent instability of the differentiation process (see **[5]** for more on this). The third problem is a causation problem involving a simple integral equation, something not often encountered by undergraduates. This problem illustrates that solving such integral equations can be an unstable process. Before taking up the problems, we have a few words on Torricelli.

Torricelli and His Law

Evangelista Torricelli (1608–1647) was an important figure in science whose contributions do not usually receive the degree of recognition they deserve. He served as secretary to the blind Galileo in Florence during his last year and succeeded him as the Tuscan court mathematician and professor at the University of Florence. Torricelli's invention of the mercury barometer (1643), which established the weight of the atmosphere and the reality of the vacuum, was widely publicized to the scientific community by Père Mersenne. On getting the news from Mersenne many leading European natural philosophers of the day replicated and refined Torricelli's experiments. The ensuing debate on the vacuum was, after Galileo's planetary observations, the second great front of the scientific revolution. As observed by Redondi **[7]**, "Now the scientific revolution was taking place on earth."

Torricelli's law gives the horizontal velocity of the effluent through a hole in a water tank as a function of the depth of the water above the hole, that is, the *hydraulic head*. Its utility as a physical demonstration in the calculus classroom was noted recently by Farmer and Gass [4].

Referring to Figure 2, Torricelli's law states that the velocity v of the effluent satisfies

$$v = \sqrt{2g(D-h)}$$

where g is the gravitational acceleration constant.

The law follows from the principle of conservation of energy. As water drains through the hole, the potential energy of an infinitesimal disk of mass m at the surface, $mg(D-h)$, is converted into kinetic energy, $\frac{1}{2}mv^2$, of an equal volume of effluent, that is,

$$\tfrac{1}{2}mv^2 = mg(D-h)$$

giving Torricelli's law.

An Algebraic Inverse Problem: Height of a Spurt

A common direct problem posed in calculus texts is that of finding the range R of the initial spurt in Figure 2, given the depth of water D and the height h of the opening (air resistance is neglected). Of course this problem has a unique solution R for each $h \in [0, D]$. The corresponding inverse problem of determining h from R is a useful device for illustrating the notions of existence and uniqueness of solutions. Specifically, given the depth D and the range R, what is the height h of the hole?

On being presented with the problem, most students guess, correctly, that a solution exists if the specified value of R is not "too large" (in relation to D). More often than not, they also guess, incorrectly, that the solution is unique. Frequently, even though they find two solutions, they tend to eliminate one as "extraneous."

The analysis is quite simple and illuminating. Choosing a coordinate system as indicated in Figure 2, the velocity of the leading drop at time t is, according to the

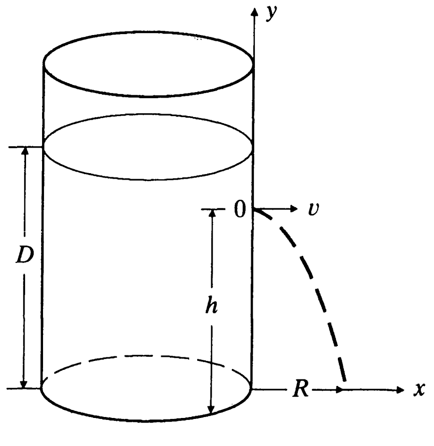

Figure 2

laws of Torricelli and Newton,

$$\mathbf{v}(t) = \sqrt{2g(D-h)}\,\mathbf{i} - gt\mathbf{j}.$$

Using the appropriate initial conditions, the position vector is found to be

$$\mathbf{r}(t) = \sqrt{2g(D-h)}\,t\mathbf{i} + (h - gt^2/2)\mathbf{j}.$$

The time of descent is therefore given by

$$h - gt^2/2 = 0 \text{ or } t = \sqrt{2h/g}\,.$$

Substituting this into the first component, we find that

$$R = 2\sqrt{h(D-h)}\,.$$

The right hand side of this expression ranges in value from 0 to D and therefore a solution of the problem *exists* if and only if $0 \leq R \leq D$. However, the solution

$$h = \left(D \pm \sqrt{D^2 - R^2}\right)/2$$

is *unique* if and only if $R = D$. Students realize that the additional root of the equation is in no way "extraneous" when they appreciate the *physical* basis for the lack of uniqueness: the smaller root is associated with a larger hydraulic head giving a larger velocity to the effluent, while the smaller velocity at the larger root is compensated with a longer time of descent.

A Coefficient Determination Problem: Cross Section of an Irregular Vessel

Visitors to the Mayan ruins at Chichén Itzá in Yucatan are guided to a deep sacrificial well of very irregular cross-section, the sacred *cenote*, which the Mayans believed to be a kind of gate to paradise through which victims were conducted. The cross sections of the cenote are quite inaccessible. The Mayan sacrificial well inspired the problem we take up in this section, namely an indirect approach to determine the cross-sectional areas of an irregular vessel from observations of its drain rate.

How does the shape of a vessel affect its drain rate? Consider an irregular water vessel as illustrated in Figure 3. Suppose the cross-sectional area of the vessel at

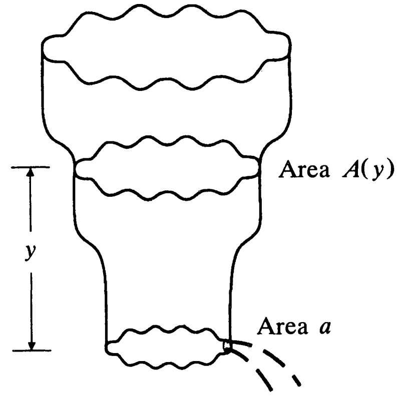

Figure 3

height y along a vertical axis is $A(y)$ (assumed to be a continuous function of y) and that the water in the vessel is draining through a horizontal spigot of cross-sectional area a in its base.

A standard direct problem in an elementary differential equations course is to determine the water level $y(t)$, given the cross-sectional profile $A(y)$. The differential equation governing this relationship is derived by equating the volume lost in time Δt due to the drop in the water level Δy with the volume of water drained through the base. According to Torricelli's law,

$$-A(y)\,\Delta y \approx a\sqrt{2gy}\,\Delta t$$

and in the limit one obtains the differential equation

$$A(y)\frac{dy}{dt} = -a\sqrt{2gy}\,. \tag{1}$$

The corresponding model identification problem is the inverse problem of determining the profile $A(y)$ from observations of y. Since the unknown A is a function of the solution y of the differential equation (1), in this problem we are said to be "identifying a distributed coefficient in a differential equation." For any realistic vessel dy/dt is never zero and therefore the inverse problem has the unique solution

$$A(y) = -a\sqrt{2gy}\bigg/\frac{dy}{dt}. \tag{2}$$

So existence and uniqueness of the solution of the inverse problem is not an issue. But what about stability? Do small errors in observations of y result in small changes in the calculated profile $A(y)$? Not necessarily. In fact, the differentiation process is notoriously unstable (e.g., **[5]**), a fact that we do not always point out to our students.

A concrete example of the instability of $A(y)$ with respect to small changes in y is easy to construct. For example, suppose the vessel is a cylinder of height 1 and constant cross-sectional area $A(y) = C$. It is then easy to solve (1) for the water level:

$$y(t) = (1 - bt)^2, \qquad 0 \le t \le 1/b$$

where $b = a\sqrt{2g}\,/2C$. Let ϵ be an arbitrarily small positive number and for a given (large) positive number M, let $\eta(t)$ be the continuous piecewise linear function that satisfies

$$\eta(t) = \begin{cases} 0, & t \in \left[0, \dfrac{1}{2b} - \dfrac{\epsilon}{2M}\right] \\[2ex] \epsilon, & t \in \left[\dfrac{1}{2b} + \dfrac{\epsilon}{2M}, \dfrac{1}{b}\right]. \end{cases}$$

Then η has slope M on

$$\left(\frac{1}{2b} - \frac{\epsilon}{2M}, \frac{1}{2b} + \frac{\epsilon}{2M}\right)$$

and satisfies $0 \le \eta(t) \le \epsilon$. Consider the perturbed observation

$$y^{\epsilon}(t) = y(t) + \eta(t)$$

of the water level. According to equation (1), the corresponding cross section then satisfies

$$A(y^{\epsilon})(-2b(1-bt) + \eta'(t)) = -a\sqrt{2g}\left((1-bt)^2 + \eta(t)\right)^{1/2}$$

on

$$\left(\frac{1}{2b} - \frac{\epsilon}{2M}, \frac{1}{2b} + \frac{\epsilon}{2M}\right)$$

and in particular at the level corresponding to $t = 1/2b$ we find, using (2), that the cross-sectional area computed from the observation y^{ϵ} satisfies

$$A(y^{\epsilon}) = \frac{-a\sqrt{2g}\left(\frac{1}{4} + \epsilon/2\right)^{1/2}}{-b + M}.$$

By choosing M large enough, we can guarantee an arbitrarily small computed cross-sectional area $A(y^{\epsilon})$ using data containing an arbitrarily small error, while the true cross-sectional area is the constant C. This type of instability with respect to small data errors is a signature of inverse problems involving identification of distributed coefficients in differential equations.

Geometry Via Flow Rates: An Integral Equation

Integral equations are seldom treated in the undergraduate curriculum. To some extent this accounts for the neglect of inverse problems in undergraduate courses because many inverse problems, particularly causation problems, can be phrased as integral equations. In this section we present an inverse problem arising from Torricelli's law that is expressed as a particularly simple integral equation which can be solved by Laplace transforms. A special case of the equation was treated long ago in [**3**] by converting it to an Abel integral equation. Our only goal is to develop the model and show that the solution of the inverse problem is unstable.

Consider an irrigation canal of depth h. In the wall of the canal is a weir notch fitted with a sluice gate that is symmetric with respect to a central vertical axis through the gate, as in Figure 4. The shape of the notch is given by a function $x = f(y)$ as indicated in the figure and it is clear that this shape will determine the total rate of flow through the notch.

The relationship between the notch shape and the flow rate is easily obtained from Torricelli's law. If we consider a horizontal slab of water of thickness Δy at height y, then the volume of this slab passing through the notch per unit time is

$$2f(y)\,\Delta y\sqrt{2g(h-y)}\,.$$

Summing over all slabs and taking a limit, we find that the total volume of flow per unit time through the notch, $V(h)$, when the water level is h, is given by

$$V(h) = 2\int_0^h \sqrt{2g(h-y)}\,f(y)\,dy. \tag{3}$$

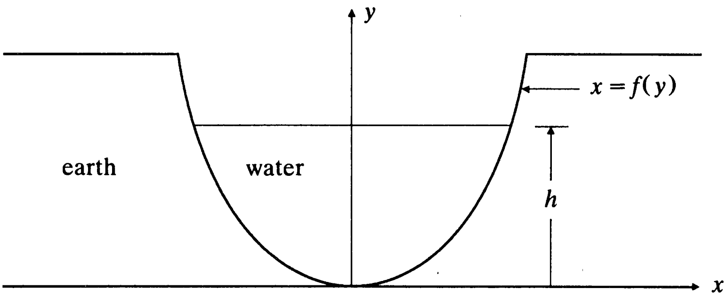

Figure 4

The direct problem of determining V, given f, has a unique stable solution V. The inverse problem, which is of the type we called a causation problem in the introduction, is to determine the geometry f from knowledge of the flow rate V. This problem involves the solution of an integral equation for f. The integral equation is a particularly simple convolution equation and its solution for various simple functions $V(h)$ is a nice elementary exercise on the use of Laplace transforms. For example, if the volume rate of flow is given by $V(h) = 2\sqrt{2g}\,h^2$, then substituting into (3) results in the integral equation

$$h^2 = \int_0^h \sqrt{h-y}\, f(y)\, dy \tag{4}$$

to be solved for notch profile $f(y)$. Applying Laplace transforms to (4) and invoking the convolution theorem results in

$$2s^{-3} = H(s) \cdot F(s) \tag{5}$$

where $H(s) = \mathscr{L}\{h^{1/2}\} = (\sqrt{\pi}/2)s^{-3/2}$ and $F(s)$ is the Laplace transform of $f(y)$. We then find from (5) that

$$F(s) = \frac{4}{\sqrt{\pi}} s^{-3/2}$$

and hence the desired notch profile is given by

$$f(y) = \frac{4}{\sqrt{\pi}} \mathscr{L}^{-1}\{s^{-3/2}\}$$
$$= \frac{8}{\pi}\sqrt{y}\,.$$

Our interest in this section is not solution methods, but simply to point out that, unlike the direct problem, the process of solving the inverse problem is unstable. One way to see this is to note that for a given $H > 0$, the function

$$V(h) = \epsilon\pi\sqrt{2g}\,h,\ 0 \le h \le H$$

has an arbitrarily small amplitude, for suitably small $\epsilon > 0$. But the corresponding solution to the inverse problem

$$f(y) = \epsilon/\sqrt{y}$$

is unbounded. (This solution can be verified by substitution into (3) and using $\int_0^\infty \frac{u^2}{(u^2+1)^2} du = \frac{\pi}{4}$, a good exercise for students.)

Conclusion

Ill-posed inverse problems are becoming increasingly important in modern science and technology. Such problems and the issues of existence, uniqueness, and stability that they suggest deserve more than lip service in the undergraduate mathematics curriculum, but simple models of ill-posed inverse problems are hard to come by. We hope that we have succeeded in this note in promoting Torricelli's law as a useful and simple physical basis for a number of "undergraduate-friendly" inverse problems.

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Poetry and Hidden Meanings in Mathematics

Gel'fand amazed me by talking of mathematics as though it were poetry. He once said about a long paper bristling with formulas that it contained the vague beginnings of an idea which he could only hint at and which he had never managed to bring out more clearly. I had always thought of mathematics as being much more straightfoward: a formula is a formula, and an algebra is an algebra, but Gel'fand found hedgehogs lurking in the rows of his spectral sequences!

Dusa McDuff's response to her award of the 1991 Ruth Lyttle Satter Prize in *Mathematical Notices*, vol. 38, No. 3, March 1991, pp. 185–7.