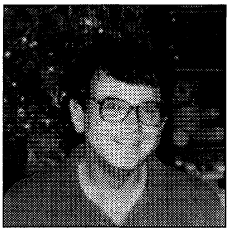
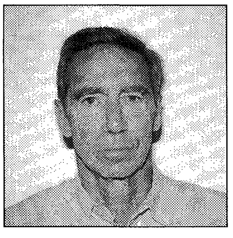
The World's Biggest Taco

David D. Bleecker and Lawrence J. Wallen



David Bleecker (bleecker@math.hawaii.edu) earned a Ph.D. at U.C. Berkeley with the gracious guidance of S. S. Chern and the inspiration of Blaine Lawson's differential geometry courses. Dick Gould coached him as an undergraduate on the Stanford tennis team, while Robert Osserman nurtured his interest in geometry. On the University of Hawaii's faculty for 24 years, he has written articles about the application of geometry to physics and with his current tennis nemesis, George Csordas, recently finished a textbook, *Basic PDEs.*



Larry Wallen (wallen@math.hawaii.edu) learned the beauty of mathematics from Everett Pitcher at Lehigh, from Witold Hurewicz at MIT (which generously gave him a doctorate), and mainly from Paul Halmos at lots of places. His research has been mostly in operator theory with forays into classical analysis and convexity. Extracurricular passions include natural history (birds, bugs, botany), basketball, badminton, and twentieth-century classical music.

Recently, a quick poll of our students indicated that, contrary to some present fashions in calculus reform, neither epidemiology nor population dynamics was on our 18-year-olds' short list of preoccupations. We did find that they thought a lot about food, especially fast food, so we decided to pose a fast-food problem. Hamburgers and pizzas being geometrically trivial, we settled on the problem of finding the volume of a taco. A taco is the solid formed by bending a circular tortilla partway around a cylinder and filling it in the obvious way—to the border, but not beyond!

A natural problem is to find shapes of cylinders that yield tacos of large volume. Better still, we sought the shape that yields the biggest taco—the taco of largest possible volume for a given tortilla. Unfortunately, we soon found that this was too tall an order for our calculus students to fill; in fact, it is a nontrivial problem in the calculus of variations. But with the help of a computer algebra system, students at all levels can grab hold of this problem. We used *Mathematica* to plot graphs, evaluate integrals, approximately solve transcendental equations, and search for the extreme values of functions.

The main lesson our students carried away is that while many problems cannot be solved *exactly* in terms of standard functions, reasonable approximations can often be found by a combination of mathematical savvy and computational power. The taco problem offers not only an attractive entry into the calculus of variations, but also a painless encounter with several special functions that students traditionally meet in more complicated circumstances.

Circular Cylinders

Take a disk of radius 1, centered at the origin in the xy-plane in xyz-space. This is the tortilla. Now, lay a long circular cylinder of radius r on the xy-plane so that it

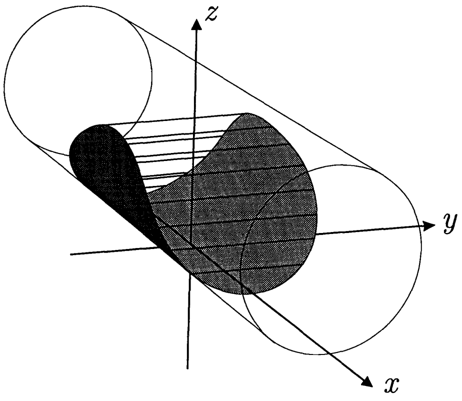


Figure 1. Wrapping the shell.

rests on the x-axis. Bend the tortilla up around the cylinder, as in Figure 1. To make sure the tortilla does not overlap, assume $\pi r > 1$.

Now fill the taco shell with beans or whatever, to form a solid (taco) whose boundary consists of the taco shell and the surface of segments parallel to the y-axis joining pairs of points on the rim of the taco shell, as in Figure 1. The cross section $\mathcal{A}_x$ perpendicular to the x-axis at x has a flat top, as in Figure 2. (Note that near the ends $x = \pm 1$, the central angle θ_x will be small, but near the center, when $x \approx 0$, this angle may exceed 180°.)

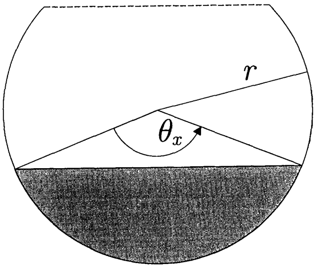


Figure 2. $\mathcal{A}_x$ is the shaded region.

We find an integral for the volume $V(r)$ of the taco as follows. The central angle θ_x satisfies $r\theta_x = 2\sqrt{1-x^2}$, so

$$\operatorname{area}(\mathcal{A}_x) = \frac{1}{2}r^2\theta_x - \frac{1}{2}r^2\sin\theta_x = r\sqrt{1-x^2} - \frac{1}{2}r^2\sin\left(\frac{2}{r}\sqrt{1-x^2}\right). \tag{1}$$

Hence, setting $x = \sin\psi$, we get

$$\begin{aligned}
V(r) &= r\int_{-1}^{1}\sqrt{1-x^2}\,dx - \frac{1}{2}r^2\int_{-1}^{1}\sin\left(\frac{2}{r}\sqrt{1-x^2}\right)dx \\
&= \frac{\pi}{2}r - \frac{1}{2}r^2\int_{-\pi/2}^{\pi/2}\sin\left(\frac{2}{r}\cos\psi\right)\cos\psi\,d\psi \\
&= \frac{\pi}{2}r - \frac{1}{2}r^2\int_{0}^{\pi}\sin\left(\frac{2}{r}\cos\psi\right)\cos\psi\,d\psi. \qquad (2)
\end{aligned}$$

The last equality uses the fact that $\sin(2/r\cos\psi)\cos\psi$ is periodic of period π in ψ.

The final integral in (2) is not elementary, and this is about as far as a competent calculus student can go without a computer. When we put it in *Mathematica*, we get the output

$$\pi\text{BesselJ}\left[1, 2\sqrt{\frac{1}{r^2}}\right]\text{Sign}[r]. \tag{3}$$

In traditional form (since $r > 0$), this is $\pi J_1(2/r)$, where J_1 is a Bessel function, one of the classical special functions [**2**] of mathematical physics. Thus,

$$V(r) = \frac{\pi r}{2}\left(1 - rJ_1\left(\frac{2}{r}\right)\right). \tag{4}$$

There is an etymological curiosity here. Bessel functions are sometimes called *cylinder* functions because of their occurrence in solutions of boundary value problems in cylindrical coordinates. Is it an accident that they appear in our cylindrical taco problem? We were a little surprised that this "higher transcendental function" emerges in such a lowly problem.

Which r gives us the most for our money? A plot of $V(r)$ leads one to conclude beyond a reasonable doubt that $V(r)$ has only one local maximum, near $x = 0.5$. Ordinarily, to maximize $V(r)$, one attempts to solve $V'(r) = 0$. *Mathematica* can differentiate the special functions, and perhaps to emphasize that these functions can be dealt with by the standard methods of calculus it is worthwhile to follow the traditional approach. *Mathematica* gives

$$V'(r) = \frac{\pi}{2}\left[1 - 2rJ_1\left(\frac{2}{r}\right) + J_0\left(\frac{2}{r}\right) - J_2\left(\frac{2}{r}\right)\right], \tag{5}$$

for $J_n(z) = 1/\pi\int_0^\pi \cos(n\psi - z\sin\psi)\,d\psi = 1/\pi\int_0^\pi \cos[z\cos\psi - (\pi n/2)]\cos(n\psi)\,d\psi$ (see [**2**]). The **FindRoot** command (Newton's method) can then be used to find the local maximum more precisely. Alternatively, one may avoid the differentiation altogether and let the computer search for the maximum by applying the **FindMinimum** command to the function $-V(r)$. The result reported is that the maximum for $V(r)$ occurs at

$$r = r_0 = 0.5727897\ldots \quad \text{with} \quad V(r_0) = 0.82713969\ldots. \tag{6}$$

When $x = 0$ (i.e., at the midsection $\mathcal{A}_0$), the angle θ_x of Figure 2 is $2/r_0 \cdot 180/\pi = 200.0586\ldots$ degrees, so the shell swings almost exactly $10°$ above a semicircle. We will refer to this best cylindrical taco as the *Bessel taco*.

General Tacos: An Upper Bound for the Volume

Instead of bending our tortilla on a circular cylinder, we could use a parabolic, elliptical, or indeed any reasonable cylinder. Note that the Bessel taco cannot give the largest possible volume, because the inward facing $\approx 10°$ arcs above the horizontal diameter can be reflected across the vertical tangents to get a "flanged" taco that strictly contains the Bessel taco.

Let $y = y(z)$, for $z \geq 0$, be an increasing function satisfying $y(0) = 0$ and $y(z) > 0$, for $z > 0$, and let C be the cylinder obtained by translating the graph of $y = y(z)$ and its reflection in the z-axis along the x-axis. Bend the tortilla up around C and

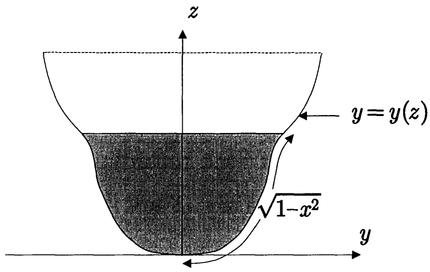


Figure 3

fill it as before. A typical cross section $\mathcal{A}_x$ is shaded in Figure 3. Again the volume of the taco is given by $V = \int_{-1}^{1} \text{area}(\mathcal{A}_x)\, dx$.

It is easy to give an upper bound for the volume of any taco. Note that the width of the flat tortilla in the plane at any x-value is $2\sqrt{1-x^2}$, and when the tortilla is bent around C this becomes the curved perimeter of the region $\mathcal{A}_x$. Let $\mathcal{S}_x$ denote the semicircular region with circular perimeter $2\sqrt{1-x^2}$, and consider the plane regions $\mathcal{A}'_x$ and $\mathcal{S}'_x$ obtained by reflecting both $\mathcal{A}_x$ and $\mathcal{S}_x$ through their boundary segments, as in Figure 4.

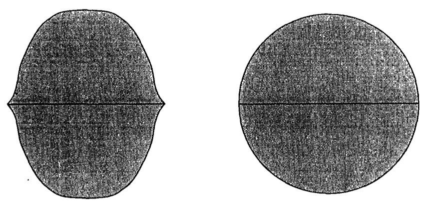


Figure 4. *Left,* $\mathcal{A}'_x$; *right,* $\mathcal{S}'_x$.

The two regions have the same perimeter so, by the isoperimetric inequality [**3**], it follows that $\text{area}(\mathcal{A}_x) \leq \text{area}(\mathcal{S}_x)$—the circle maximizes area with a given perimeter. Thus, since the radius of $\mathcal{S}_x = \left(2\sqrt{1-x^2}\right)/\pi$, we get

$$\text{area}(\mathcal{A}_x) \leq \text{area}(\mathcal{S}_x) = (2/\pi)(1-x^2),$$

so

$$V < \frac{2}{\pi}\int_{-1}^{1}(1-x^2)\, dx = \frac{8}{3\pi} = \overline{V} = 0.848826\ldots. \tag{7}$$

Using (6), we get $\overline{V}/V(r_0) = 1.02621\ldots$. That is, no taco can be more than 2.63% larger than the Bessel taco. Of course, we could form the solid having the semicircles $\mathcal{S}_x$ as cross sections, but this is not a taco by our definition: it is not formed by wrapping the tortilla on some type of cylinder. For a taco, $\mathcal{A}_x$ can be a semicircle for at most one value of $|x|$.

Some Computer Experiments

It turns out that for most computations, it is much better to use horizontal sections rather than vertical sections. Let H_z be the horizontal section of our filled taco at height z—a rectangle. Recall that the width of the flat tortilla in the xy-plane for any value of x is $2\sqrt{1-x^2}$. When the half of this strip of the tortilla with a fixed x-coordinate and $y>0$ is bent around the cylinder C, the arc length of the curve $y=y(z)$ that it covers is $s=\sqrt{1-x^2}$. Assume that the arc length along this curve between the origin and a typical point $\big(y(z),z\big)$ exists and is given by the usual integral $s(z)=\int_0^z \sqrt{1+y'(t)^2}\,dt$. The corners of the rectangular cross section $H(z)$ can then be expressed as $\left(\pm\sqrt{1-s(z)^2}, \pm y(z), z\right)$. Thus the volume of the filled taco is given by

$$V=\int_0^Z 4y(z)\sqrt{1-s(z)^2}\,dz, \tag{8}$$

where Z is the z-coordinate of the highest points $\big(0,\pm y(Z),Z\big)$ on the taco, the points where $s(Z)=1$.

Let's work out a typical example: finding the volume of the taco obtained by wrapping the unit tortilla on a cylinder with a parabolic cross section of the form $z=(1/2c)y^2$. (The parameter $c>0$ is the radius of curvature at $(0,0)$.) Then $y(z)=\sqrt{2cz}$, $y'(z)=\sqrt{c/2z}$, and

$$\begin{aligned} s(c,z) &= \int_0^z \sqrt{1+y'(t)^2}\,dt = \int_0^z \sqrt{1+\frac{c}{2t}}\,dt \\ &= \sqrt{\frac{cz}{2}+z^2}+\frac{c}{2}\operatorname{arcsinh}\left(\sqrt{\frac{2z}{c}}\right). \end{aligned} \tag{9}$$

By (8),

$$V(c)=\int_0^{Z(c)} 4\sqrt{2cz}\sqrt{1-s(c,z)^2}\,dz, \tag{10}$$

where $Z(c)$ can be found numerically via:

```
Z[c_]:=z/.FindRoot[s[c,z]==1,{z,0,1}]
```

Note that **FindRoot** returns the *list* **{z→Z[c]}**, while **z/.{z→Z[c]}** gives the *number* **Z[c]**. Because the integral in (10) cannot be carried out symbolically, (10) defines the function $V(c)$ numerically. Specifically:

```
V[c_]:=NIntegrate[4 Sqrt[2c z] Sqrt[1-s[c,z]^2],{z,0,Z[c]}]
```
(11)

We can plot the graph of $V(c)$ and see that the maximum value occurs at $c\approx 0.3$. The command **FindMinimum[-V[c],{c,0.2,0.4}]** yields $c=0.30440\ldots$ and the maximum volume $V(c)=0.80676\ldots$.

This is not nearly as good as the Bessel taco.

Exercise 1. Follow the same procedure to show that the largest volume taco whose cross section is the graph of $z=(1/4c)y^4$ has volume $V(c)=0.816108\ldots$. (Here $c>0$ is *not* the radius of curvature at the origin: that is infinite.) The arc length function $s(c,z)$ for the parabolic cylinder involved only elementary functions (e.g., an inverse hyperbolic sine), but for $z=(1/4c)y^4$, $s(c,z)$ involves a special function called a *hypergeometric* function. Actually, there are whole families of hypergeometric functions, and many special functions (e.g., the Bessel functions above and elliptic functions below) can be expressed in terms of them [**2**]. Nevertheless, *Mathematica* handles hypergeometric functions as well as elementary functions. For higher powers or more complicated functions $y(z)$, one can define $s(c, z)$ numerically.

Exercise 2. Use the procedure above to find the maximum volume taco for circular cylinders of radius $r > 1/\pi$, using *horizontal* cross sections rather than the $\mathcal{A}_x$. You will find that this time the arc length function $s(r, z)$ involves an inverse tangent function. Of course the maximum volume should be the volume of the Bessel taco. (*Hint*: Start with $y(z) = \sqrt{r^2 - (z - r)^2}$.)

Exercise 3. We can certainly do at least as well as the Bessel taco if we consider the tacos wrapped on the two-parameter family of elliptical cylinders given parametrically by $y = a \sin t$, $z = b(1 - \cos t)$, which includes the circular case. Verify that the arc length of the ellipse from $t = 0$ to $t = \hat{t} \leq \pi$ is

$$\begin{aligned} s(a, b, \hat{t}) &= a \int_0^{\hat{t}} \sqrt{1 - \left(1 - \frac{b^2}{a^2}\right) \sin^2 t} \, dt \\ &= aE\left(\hat{t} \mid 1 - \frac{b^2}{a^2}\right), \end{aligned} \tag{12}$$

where $E(\phi \mid m) := \int_0^{\phi} \sqrt{1 - m \sin^2 t} \, dt$ is the elliptic function of the second kind, a standard special function of which *Mathematica* is well aware. In fact, one can obtain (12) using *Mathematica*, but only if specific values for a and b are used. The value of $\hat{t}$, say $T(a, b)$, for which $s(a, b, \hat{t}) = 1$ can be defined in *Mathematica* by `T[a_,b_]:=t/.FindRoot[s[a,b,t]==1,{t,0,Pi}]`, where we assume that the perimeter of the ellipse is more than 2. Verify that the volume of the taco formed on the elliptical cylinder is then

$$\begin{aligned} V(a, b) &= \int_0^{b(1-\cos T(a,b))} 4y(z) \sqrt{1 - s\left[a, b, \arccos\left(1 - \frac{z}{b}\right)\right]^2} \, dz \\ &= \int_0^{T(a,b)} 4ab \sin^2 t \sqrt{1 - s(a, b, t)^2} \, dt. \end{aligned} \tag{13}$$

Figure 5 (page 8) is a *Mathematica* contour plot of the level curves of $V(a, b)$ from the level of the Bessel taco (about 0.82714) to 0.830 in 10 equal steps. If you are patient (or have a fast computer), you may wish to try reproducing this plot. The result suggests that there is a maximum around $(a, b) = (0.57, 0.68)$. (We first transposed the arguments of $V(a, b)$, so that the picture—in the ba-plane—would be wide instead of tall.) Verify that Figure 5 shows that there are elliptical tacos with b/a as high as 1.4 that yield the same volume as the Bessel taco. Is this surprising? Check that when the **`FindMinimum`** function in *Mathematica* is applied to the function $-V(a, b)$, you get the greatest volume if $a = 0.57669\ldots$ and $b = 0.68327\ldots$ (inside the smallest contour in Figure 5), giving a volume of $0.830025\ldots$. This is just about 0.35% greater than the volume $0.827139\ldots$ of the Bessel taco. Note that for this best elliptical taco $b/a \approx 1.185$, so this taco differs appreciably in shape from the Bessel taco. (Figure 7 on page 11 shows the semi-midsections of the biggest elliptical taco and the Bessel taco.)

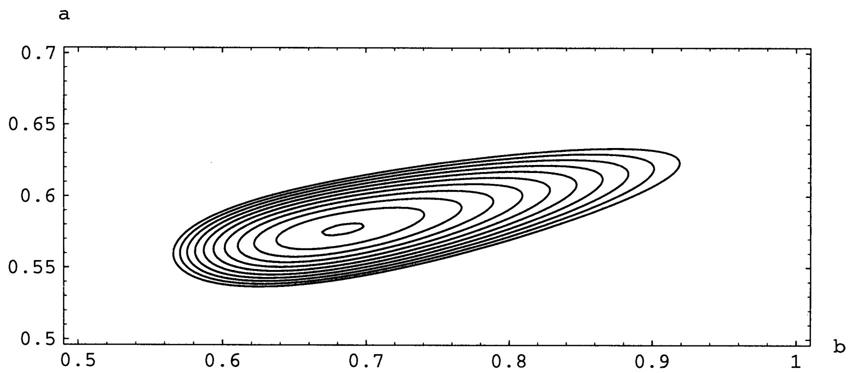


Figure 5

The World's Biggest Taco

Now let's get serious about finding the curve $y = y(z)$ that produces the biggest possible taco. We shall assume that this curve can be parametrized by arc length, so that it has parametric equations $\big(y(s), z(s)\big)$, for $0 \leq s \leq 1$. It is enough to find $y(s)$, since $y'(s)^2 + z'(s)^2 = 1$, so $z(s) = \int_0^s \sqrt{1 - y'(\sigma)^2}\, d\sigma$.

We are faced with a problem in the *calculus of variations*. Generally, in the calculus of variations one is given a function $f(s, t, u)$, and the task is to find a function $y(s)$ that maximizes (or minimizes) the integral

$$J[y] = \int_a^b f\big(s, y(s), y'(s)\big)\, ds, \tag{14}$$

where $y(a)$ and/or $y(b)$ may or may not be specified. In our case, by (8), we want to maximize $J[y] = \int_0^1 4\sqrt{1 - s^2}\, y(s) \sqrt{1 - y'(s)^2}\, ds$, since $z'(s) = \sqrt{1 - y'(s)^2}$. So we have

$$f(s, t, u) = 4\sqrt{1 - s^2}\, t \sqrt{1 - u^2}, \tag{15}$$

where $a = 0$, $b = 1$, and $y(0) = 0$, with $y(1)$ free (not specified).

Much of the elementary theory (see [**1**] or [**4**]) involves the calculation of *local* extrema for the integral $J[y]$ in the function space of sufficiently smooth candidates for $y(s)$. These are usually found to be solutions of the famed Euler-Lagrange differential equation $\frac{\partial f}{\partial t} - \frac{d}{ds}\frac{\partial f}{\partial u} = 0$, or more explicitly

$$\frac{\partial f}{\partial t}\big(s, y(s), y'(s)\big) - \frac{d}{ds}\left[\frac{\partial f}{\partial u}\big(s, y(s), y'(s)\big)\right] = 0, \tag{16}$$

which is an ordinary differential equation (typically of order 2) for $y(s)$. This equation is obtained by setting the "first derivative of J with respect to the function y" (more commonly known as the *first variation of* J) equal to zero. We are, however, looking for a *global* maximum. It turns out to be surprisingly difficult to find sufficient conditions in the elementary standard literature that guarantee the existence of a global maximum within a specified space of admissible functions.

More advanced treatments use so-called *direct* methods. In such methods, one selects a suitable class, say $\mathcal{C}$, of functions, called thereafter *admissible functions*, and first shows that the supremum of $J[y]$ for $y \in \mathcal{C}$, say $\sup_{\mathcal{C}}(J)$, is finite. (In our case, we know that $\sup_{\mathcal{C}}(J) \leq 8/3\pi$.) By the definition of supremum, a sequence of functions $y_n \in \mathcal{C}$ exists such that $J[y_n]$ converges to $\sup_{\mathcal{C}}(J)$. The hard part is to select a subsequence of the y_n that converges to an element $y_\infty \in \mathcal{C}$ with $J[y_\infty] \geq J[y_n]$ for all n. Since $\lim_{n\to\infty} J[y_n] = \sup_{\mathcal{C}}(J)$, it follows that $J[y_\infty] = \sup_{\mathcal{C}}(J)$. Often one can show that y_∞ is smooth enough that it will satisfy the Euler-Lagrange differential equation (16). Then uniqueness results for differential equations can be brought to bear as we try to prove the uniqueness of the maximizer y_∞.

This is a sketch of the procedure that we used to obtain the following theorem. We could not find a theorem in the literature that covers the case of our problem. A major difficulty is that $\frac{\partial f}{\partial u}(s, y, 1)$ does not exist. However, we managed to modify known methods to prove the existence and uniqueness of the global maximizer.

Theorem. *Let $\mathcal{C}$ be the class of functions $y(s)$ on $[0,1]$ satisfying $y(0) = 0$ and $0 \leq y(t) - y(s) \leq t - s$ for all $0 \leq s \leq t \leq 1$. Let $J : \mathcal{C} \to [0,1]$ be defined by*

$$J[y] := \int_0^1 4\sqrt{1-s^2}\, y(s)\sqrt{1-y'(s)^2}\, ds. \tag{18}$$

Then there is a unique function $y_\infty \in \mathcal{C}$ such that $J[y_\infty] = \sup_{\mathcal{C}}(J)$. Moreover, y_∞ is infinitely differentiable on $(0,1)$ with $y'_\infty(0^+) = 1$ and $y'_\infty(1^-) = 0$. For $z_\infty(s) := \int_0^s \sqrt{1-y'_\infty(\sigma)^2}\, d\sigma$, the curve $\big(y_\infty(s), z_\infty(s)\big)$, for $0 \leq s \leq 1$, is convex with a horizontal tangent at $(0,0)$ and a vertical tangent at $\big(y_\infty(1), z_\infty(1)\big)$.

The world's biggest taco made from a unit tortilla is therefore obtained by wrapping the tortilla around the cylinder with cross section given by the curve $\big(y_\infty(s), z_\infty(s)\big)$, $0 \leq s \leq 1$, and its reflection across the z-axis in the yz-plane.

Remarks. The condition $0 \leq y(t) - y(s) \leq t - s$ means that y is an increasing Lipschitz function with Lipschitz constant 1. For such y, it is known that $y'(s)$ exists and $0 \leq y'(s) \leq 1$ for all $s \in (0,1)$ outside a subset of Lebesgue measure 0 (i.e., a set that is contained in a finite union of intervals of arbitrarily small total length). Thus, $J[y]$ is defined, as a Lebesgue integral. The proof of our theorem, which occupies about 10 pages, involves nothing more advanced than Luzin's theorem; a copy can be requested by email from either author.

How Big Is the Biggest Taco?

We have no explicit expression for the function $y_\infty(s)$, so we resort to approximation methods. One such method is the Ritz method (see [**1**]), which goes as follows. Pick a sequence of functions $\{\varphi_k(s)\}_{k=0}^\infty$ such that y_∞ can be approximated by linear combinations of the φ_k. Suppose that

$$J_n := \max_{a_0,\ldots,a_n} \left\{ J\left[\sum_{k=0}^n a_k\varphi_k\right] : \sum_{k=0}^n a_k\varphi_k \in \mathcal{C} \right\} \tag{18}$$

exists. Under suitable conditions, J_n will converge to $\sup_{\mathcal{C}} J$. If J_n is attained by $y_n = \sum_{k=0}^n \alpha_k\varphi_k$, it is not at all obvious that $y_n \to y_\infty$. Nevertheless, in problems where a unique solution y_∞ is known explicitly, this Ritz method works very well—and we may hope that the convergence will happen in our case as well.

To implement the Ritz method, we must first select a family of functions $\{\varphi_k(s)\}$ such that y_∞ can be approximated by linear combinations of the φ_k. What functions would be appropriate? From the theorem, we know the general shape of the graph of $y_\infty(s)$: it is increasing with slope 1 at the start and 0 at the end. Thus, if we extend the domain of $y_\infty(s)$ to $[0,2]$ by making it symmetric about $s=1$, the extended function is continuously differentiable (C^1) and its graph might be as indicated in Figure 6.

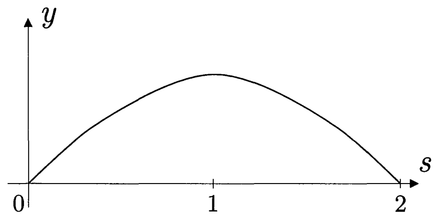


Figure 6

The extended function is continuously differentiable and can be uniformly (indeed, uniformly C^1) approximated by finite sine series of the form $\sum b_n \sin(n\pi s/2)$. (Note to the advanced reader: This series is *not* necessarily a partial Fourier sine series of the extended $y_\infty(s)$, since $y''_\infty(1)$ is not known to exist, but the integrals of partial sums of the Cesaro cosine series for $y'_\infty(s)$ will yield uniform C^1 approximations of $y_\infty(s)$.) The symmetry condition ensures that only odd terms are needed. Thus, $y_\infty(s)$ can be uniformly C^1 approximated by finite linear combinations of

$$\varphi_k(s) = \sin\left[\left(k+\frac{1}{2}\right)\pi s\right], \qquad k = 0, 1, 2, \ldots. \tag{19}$$

(Uniform C^1 approximations are needed, since J is continuous with respect to the C^1 norm, but not the C^0 norm.) Since $y'_\infty(0^+)=1$, we impose this condition on the linear combinations $\sum_{k=0}^{n} a_k\varphi_k$; thus $\sum_{k=0}^{n}(k+\frac{1}{2})\pi a_k = 1$, or

$$a_0 = \frac{2}{\pi} - \sum_{k=1}^{n}(2k+1)a_k. \tag{20}$$

In the case $n=0$ we have only a single function $y_0(s) = a_0\sin[(\pi/2)s]$, and the boundary condition (20) gives $a_0 = 2/\pi$. We calculate $J[y_0] = 0.818808245\ldots$.

Exercise 4. Show that the curve $\big((y_0(s), z_0(s)\big)$, for $0 \le s \le 1$, is a quarter-circle of radius $2/\pi$ and center $(0, 2/\pi)$, so that the midsection of the corresponding taco shell is a semicircle, unlike that of the Bessel taco which extends about 10° above a semicircle.

For $n=1$, the boundary condition (20) gives $a_0 = 2/\pi - 3a_1$, so

$$y_1(s) = y_1(s,a_1) = \left(\frac{2}{\pi} - 3a_1\right)\sin\left(\frac{1}{2}\pi s\right) + a_1\sin\left(\frac{3}{2}\pi s\right). \tag{21}$$

We must select the parameter a_1 to maximize $J[y_1(\cdot,a_1)]$. Applying **`FindMinimum`** to $-J[y_1(\cdot,a_1)]$ gives *Mathematica*'s approximation of the value α_1 of a_1 for which $J[y_1(\cdot,a_1)]$ is a maximum. The result reported is that $\alpha_1 = 0.01527299\ldots$, which gives $\alpha_0 = 2/\pi - 3\alpha_1 = 0.590800\ldots$ and $J[y(\cdot,\alpha_1)] = 0.8299928\ldots$. Continuing, *Mathematica* yields Table 1.

Table 1. Computed Coefficients $\alpha_0, \ldots, \alpha_n$ of $y_n(s)$.

n	*Volume*	α_0	α_1	α_2	α_3
0	0.8188082450...	0.63661977...	0.0000000...	0.0000000...	0.0000000...
1	0.8299928190...	0.59080079...	0.0152729...	0.0000000...	0.0000000...
2	0.8300296806...	0.59086832...	0.0141785...	$6.43181... \cdot 10^{-4}$	0.0000000...
3	0.8300305150...	0.59084004...	0.0141903...	$5.41265... \cdot 10^{-4}$	$7.17586... \cdot 10^{-5}$
4	0.8300305185...	0.59083911...	0.0141915...	$5.40593... \cdot 10^{-4}$	$7.65701... \cdot 10^{-5}$
5	0.8300305229...	0.59083835...	0.0141923...	$5.39598... \cdot 10^{-4}$	$7.72059... \cdot 10^{-5}$
6	0.8300305241...	0.59083805...	0.0141926...	$5.39599... \cdot 10^{-4}$	$7.76115... \cdot 10^{-5}$

n	α_4	α_5	α_6
4	$-3.66294... \cdot 10^{-6}$	0.0000000...	0.0000000...
5	$-7.91771... \cdot 10^{-6}$	$3.38589... \cdot 10^{-6}$	0.0000000...
6	$-8.19948... \cdot 10^{-6}$	$5.17710... \cdot 10^{-6}$	$-1.46422... \cdot 10^{-6}$

These results strongly suggest (but do not prove) that the largest possible volume is about 0.8300305.... If we have *Mathematica* plot the half cross section of the Bessel taco, the best elliptical taco, and the best six-parameter Ritz taco, we obtain Figure 7.

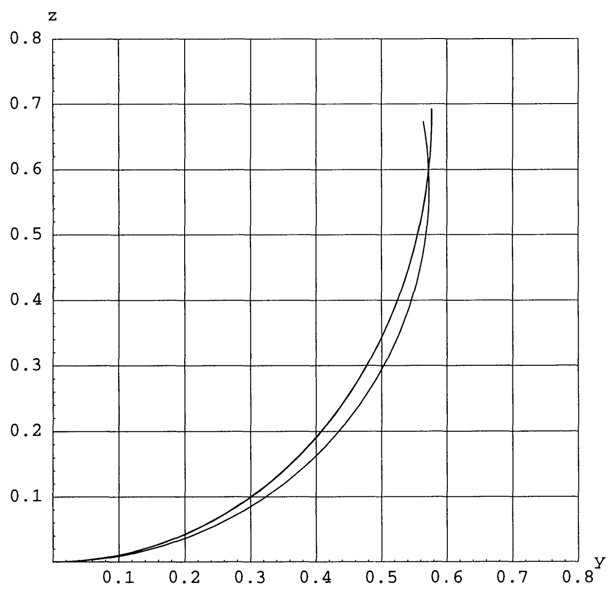


Figure 7

The best elliptical and Ritz taco curves nearly coincide. Indeed, their compound curve is just a little thicker than each would be individually. The circular arc for the Bessel taco is the rightmost (except near the top) curve in Figure 7.

Conclusions

What do we know? We know that there is a biggest taco and there is only one, and we know a couple of geometric facts about it. The only other thing we know for sure is the pretty crude upper bound $8/3\pi = 0.8488263\ldots$ from the isoperimetric inequality. Beyond a reasonable doubt (e.g., an unknown flaw in the 166-MHz Pentium chip, in *Mathematica*, or in our use of *Mathematica*), we have a lower bound of 0.8300305 from the Ritz calculations. We believe that this coincides with the volume of the biggest taco up to and including the seventh decimal place, especially since there is an unrelated iteration procedure that gives the same result. We do not know that the functions y_n that Ritz gives us converge to y_∞, but the graphical evidence is convincing. The midsection corresponding to y_6 (and even y_3) is practically indistinguishable from that of the best elliptical taco. Still, it would be nice to have proofs and error estimates.

How well do you do at your favorite taco eatery? We briefly turned experimentalists and purchased some empty taco shells from a well-known commercial establishment. We filled them with damp sand, then measured the contents and found that the volume was a mere 0.47 cubic units or about 57% of the optimal. In all fairness, however, certain anatomical constraints must be taken into account.

References

1. N. I. Akhiezer, *Calculus of Variations*, Blaisdell, New York, 1962.
2. N. N. Lebedev, *Special Functions and Their Applications,* Dover, New York, 1972.
3. R. Osserman, The isoperimetric inequality, *Bulletin of the American Mathematical Society* 84 (1978) 1182–1238.
4. H. Sagan, *Introduction to the Calculus of Variations*, Dover, New York, 1992.

———— o ————

Aha!

Perhaps I could best describe my experience of doing mathematics in terms of entering a dark mansion. One goes into the first room and it's dark, completely dark. One stumbles around bumping into the furniture, and then gradually you learn where each piece of furniture is, and finally after six months or so you find the light switch. You turn it on—suddenly it's all illuminated! You can see exactly where you were.

Andrew Wiles in "Fermat's Last Theorem " on the BBC,
January 15, 1996; rebroadcast by PBS as "The Proof."