Continuity

The activities described here will help you become comfortable using the Continuity applet. In the first activity, we will become familiar with the applet. In the second activity, we will use the applet to explore function continuity in greater depth.

Definition 1 (Continuity at a point)**.** *A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a real number c if and only if for all $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$.*

Activity 1 (Getting comfortable with the applet)**.** Begin by launching the Continuity applet in a web browser. Once it is loaded, you are ready to begin. When you first load the applet, you will see a lot of things on the screen. Let's look at each piece in turn.

1. The default function is a parabola opening down. You will see this in the center of the screen. On the x-axis is a cyan point C. This will be the point c at which we are exploring continuity. You can adjust this point by clicking on it and sliding it to the left and the right. Do so now. As you slide C, you should notice that the point L on the y-axis is also changing. The value L is exactly $f(C)$.

2. Set C to be near 1.25. Now we are going to control ϵ. You can change ϵ by clicking on the slider to the left of the y-axis. Click on the point and move it up and down. As ϵ changes, you should see the shaded rectangle expand and contract as well. Any point within this rectangle satisfies $|f(x) - f(c)| < \epsilon$.

3. Set ϵ to be near 0.51. We can control δ with the slider beneath the x-axis. Click on this point now and move it to the left and the right. As you change δ, you will see that the red bolded part of the function expands and contracts. The red part of the function is the image of the interval $[C-\delta, C+\delta]$ under the function f. In other words, it represents those points which are in consideration if $|x - C| < \delta$.

4. The rectangle is used to determine if the conditions in the definition of function continuity have been met for a given choice of C, ϵ, and δ. Set C to be near 1.25, ϵ to be near 0.51, and δ to be near 1. The shading on the function should be red, indicating that there exists a point x which satisfies $|x-C| < \delta$ but for which $|f(x)-f(c)|$ is not less than epsilon. Now adjust δ to be near 0.25. Now the shaded portion of the function has turned green. This means that whenever $|x - C| < \delta$, we have $|f(x) - f(C)| < \epsilon$. In other words, given ϵ near 0.5, a value of δ near 0.25 works to satisfy the definition.

5. Finally you can change the function that you wish to explore by going to the bottom of the applet and typing into the Input box. Click on the Input box and try "$f(x) = 5*x$" now.

Activity 2 (Exploring Function Continuity)**.** Now that you are comfortable with the applet, let's focus more directly on function continuity. Reset the applet to its default mode by clicking on the icon located in the top-right corner of the applet.

1. Set C to be near 1.25. For each of the values of ϵ, find approximately the largest value of δ which will turn the relevant portion of the function green: ϵ near 2, ϵ near 1, ϵ near 0.5, ϵ near 0.25. Explain why it's ok that you get a different answer for δ each time.

2. Change the function to $f(x) = x^2$ by typing "$f(x) = x \wedge 2$" into the input box. Set $\epsilon = 1$. Now we are going to look at what happens as we adjust C. We will keep ϵ fixed near 1 for this problem. Set C near 0.5 and estimate the largest possible value of δ which will turn the relevant portion of the function green. Repeat this process, changing C to the following values: C near 1.0, C near 1.5, and C near 2.0. Compare the values that you got for δ at each of the points C. Do you think it's possible to choose a single value for δ which will work for all $C \in \mathbb{R}$? Why or why not?

3. Repeat the previous exercise using the function $f(x) = x^{1/2}$ and using the numbers C near $0.5, 1, 2$, and 3. Again, do you think it's possible to choose a single value for δ that works for all positive real numbers C? Why or why not?

4. Now change the function to $f(x) = 5 * \sin(1/x)$. Fix ϵ near 1 and use the applet to find a δ that works when C is near $1, 0.5, 0.4$, and 0.3. Move C even closer to 0. Explain why $f(x)$ is not continuous at $C = 0$. (Note: it will be difficult to get the applet to find an appropriate delta because the required value will be very small and obtaining precision with sliders is difficult.)

5. Explain why this applet can only provide a strong suggestion that a function f is continous at a point c and why it cannot be used to provide a definitive proof of continuity at a point.