Riemann Sums

The activities described here will help you become comfortable using the Riemann Sums applet. In the first activity, we will become familiar with the applet. In the second activity, we will use the applet to explore Riemann Sums in greater depth.

Definition 1 (Riemann sum over a partition)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined on $[a, b]$ and let $P = \{x_0, \ldots, x_n\}$ be a finite set of points such that*

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b.$$

Then the Riemann Sum *of f on $[a, b]$ with respect to the partition P is given by*

$$S = \sum_{i=1}^{n} f(y_i)(x_i - x_{i-1}),$$

where y_i is an arbitrary point in the interval $[x_{i-1}, x_i]$. If the point y_i is chosen so that $y_i = x_i$, then S is the Right Riemann Sum*.*

In this applet, we will explore the Right Riemann Sum of functions f over intervals $[a, b]$ with respect to random partitions of size N.

Activity 1 (Getting comfortable with the applet)**.** Begin by launching the Riemann Sums applet in a web browser. Once it is loaded, you are ready to begin. When you first load the applet, you will see a lot of things on the screen. Let's look at each piece in turn.

1. The default function f is a cubic with a positive coefficient of x^3. On the top-left of the applet window, you will see a lot of information. This box tells you what the current function $f(x)$ is, what interval $[a, b]$ is under consideration, how many points N are being used in the partition, what the area given by computing the Riemann Sum is, and the value of the definite integral $\displaystyle\int_a^b f(x)\, dx$.

2. You can adjust the interval $[a, b]$ under consideration by clicking and moving the red points outlined in green on the x-axis. Do this now and watch how both the picture and the information in the top-left corner change.

3. Now let's play with our partition. Click on the slider underneath "Number of rectangles:" and move it left and right. As it moves, you will see the number of points increase and decrease with a corresponding increase or decrease in the number of rectangles associated with the curve.

4. You can also ask Geogebra to randomly generate a new partition of the interval $[a, b]$. To do this, click anywhere on the main applet window and then hit F9. Slide N to $N = 5$ and then hit F9 a few times so that you can see how the partition changes.

5. You can change the function $f(x)$ that you are exploring by going to the Input box at the bottom of the applet. Click on the Input box and enter "$f(x) = e \wedge x$" and then press Enter.

6. For each pair of adjacent points in the partition, the applet will create a rectangle of height $f(x_i)$ and width $x_{i-1} - x_i$ where x_i is the rightmost of the two points. In other words, the sum of the areas of the rectangles represents the Right Riemann Sum associated to the given partition. You can see the value of this sum on the left after "Approximate Area".

Activity 2 (Exploring Riemann Sums)**.** Now that you are comfortable with the applet, let's focus more directly on Riemann Sums. Reset the applet to its default mode by clicking on the icon located in the top-right corner of the applet.

Loosely speaking, the definite integral $\int_a^b f(x)\, dx$ can be obtained as a limit of a Riemann Sum as N approaches infinity with the caveat that the largest difference $x_i - x_{i-1}$ must go to zero at the same time. In this case, the areas of the rectangles computed by the Riemann Sum will converge to the value of the definite integral.

1. Begin with the default conditions for $f(x)$, $[a, b]$ and N. Click on the applet so that hitting F9 will generate a new random partition. Now generate 10 random partitions and record the largest difference between the Approximate Area and the Actual Area.

2. Keep the default conditions for $f(x)$ and $[a, b]$ but now adjust N until it is 50. Generate 10 random paritions and record the largest difference between the Approximate Area and the Actual Area. Repeat this with $N = 100$.

3. Now change to the function $f(x) = e^{(x^2)}$ by typing "$f(x) = e \wedge (x \wedge 2)$" into the Input box. Keep the interval $[a, b]$ at its default. Notice how large the Actual Area is. Generate random partitions for $N = 11, 50$ and 100 and report the largest difference between the Approximate Area and the Actual Area in each case.

4. Explain why the Approximate Areas computed in the previous exercise are always over-estimates.

5. Explain the discrepencies between the largest errors you found using the default function and using the function $f(x) = e^{(x^2)}$.