A Codeword Proof of the Binomial Theorem

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The standard proof of the Binomial Theorem for positive integer exponent n treats the x in $(1+x)^n$ as a variable and the nth power as a product of n factors of $(1+x)$. (See, for example, [**1**], [**2**], [**3**], [**5**].) The proof below relies more on counting and less on polynomial algebra. (See [**4**] for other identities involving binomial coefficients that use counting arguments.)

Theorem 1. *For any positive integers n and r,*

$$(1+r)^n = \sum_{k=0}^{n} \binom{n}{k} r^k.$$

Proof. Consider an alphabet consisting of r ordinary letters and one special letter, say $*$. We count the number of possible codewords of length n in two ways. First, since there are a total of $1+r$ symbols available and there are n slots to fill, the Multiplication Principle shows that the number of codewords is $(1+r)^n$.

On the other hand, we can partition the codewords according to the number of non-$*$'s that appear. Let this number be k, so that $0 \le k \le n$. Constructing a codeword with k non-$*$'s consists of selecting $n-k$ of the n slots for the $*$'s, and then selecting a codeword of length k containing no $*$'s. The Multiplication Rule says that the number of ways of constructing such codewords is the product of the number of ways of making each of the two selections. The first selection is made in $\binom{n}{n-k} = \binom{n}{k}$ ways. The second, using the Multiplication Rule again, is made in r^k ways. Summing on k from $k=0$ to $k=r$ by the Sum Rule completes the calculation of the number of codewords of length n. Since both sides of the Binomial Theorem count the same quantity, we therefore have equality and have proved the theorem.

We can now proceed to extend the result to all real numbers r by using the fact that polynomials over $\Re$ have only finitely many zeros. Let $F(x) = (1+x)^n - \sum_{k=0}^{n} \binom{n}{k} x^k$. By the theorem above, $F(r) = 0$ for all positive integers r. Now F is a polynomial of degree n, and has more than n zeros (in fact, it has infinitely many zeros). Hence, F is identically zero. Therefore, $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ for all $x \in \Re$.

References

1. Kenneth P. Bogart, *Introductory Combinatorics*, 3rd ed., Harcourt/Academic Press, 2000.
2. Richard A. Brualdi, *Introductory Combinatorics*, 3rd ed., Prentice-Hall, 1999.
3. Fred S. Roberts, *Applied Combinatorics*, Prentice-Hall, 1984.
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Column Integration and Series Representations

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In the movie *Stand and Deliver*, which premiered in the 1980s, the calculus teacher Jaime Escalante demonstrated a short-cut scheme for integration. This method, some-

