

Again, $b(2(x + 1) - 2x) = 2b$, so differentiating again eliminates the b term:

$$2 = a(6(x + 1) - 6x) \quad (6)$$

If we let $x = 0$ in each of the three equations, we obtain the system:

$$a + b + c = 1$$

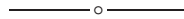
$$3a + 2b = 2$$

$$6a = 2$$

which, though it is a system in three unknowns, is trivial to solve by back-substitution.

References

1. D. Acu, Some algorithms for the sums of the integer powers, *Mathematics Magazine* **61** (1988) 189–191.
2. D. Bloom, An old algorithm for the sums of the integers powers, *Mathematics Magazine* **66** (1993) 304–305.
3. C. Kelly, An algorithm for sums of powers, *Mathematics Magazine* **57** (1984) 296–297.
4. M. Krom, On sums of powers of natural numbers, *The Two-Year College Mathematics Journal* **14** (1983) 349–351.
5. G. W. Leibniz (translated by J. M. Child), *The Early Mathematical Manuscripts of Leibniz*, Open Court, 1920.
6. L. Zia, Using the finite difference calculus to sum powers of integers, *The College Mathematics Journal* **22** (1991) 294–300.



The Pythagorean Theorem and Beyond: A Classification of Shapes of Triangles

Guanshen Ren (Gren@css.edu), College of St. Scholastica, Duluth, MN 55811

The Pythagorean theorem is well known to almost all college students. Some geometry textbooks show that if $a^2 + b^2 \neq c^2$, then the triangle could be acute or obtuse. In this note, we first ask the question: if $a^3 + b^3 = c^3$, what kinds of triangle will we have? More generally, what happens if $a^n + b^n = c^n$? To answer these questions, we present an application of the law of cosines to analyze the shape of a triangle by the equation $a^n + b^n = c^n$. We introduced this project in a geometry class for secondary math education majors and students were excited about it.

To extend our investigation beyond the Pythagorean theorem, we ask what kind of triangle has sides satisfying $a^3 + b^3 = c^3$.

Such triangles exist. For example, we use the Geometer's Sketchpad to construct a triangle ABC with $a = 2$, $b = 3$, $c = \sqrt[3]{35} \approx 3.271$, and then to measure the three angles. All are less than 90 degrees as shown in Figure 1.

This example suggests that triangle ABC is acute. To give a formal proof in general, we apply the law of cosines since it relates sides and angles in a triangle. Noting that $a^3 + b^3 = c^3$ implies that $c > a$ and $c > b$, we have

$$a^3 + b^3 = c \cdot c^2 = c(a^2 + b^2 - 2ab \cdot \cos C).$$

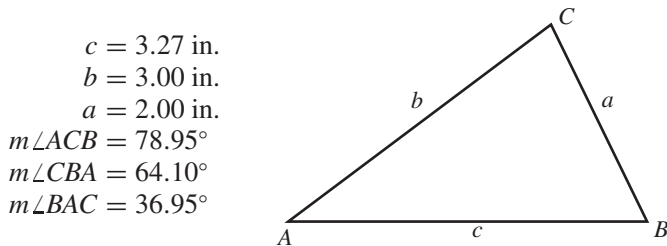


Figure 1.

This implies that

$$\cos C = \frac{a^2(c - a) + b^2(c - b)}{2abc} > 0;$$

thus angle C must be acute. From a well known theorem of geometry, $c > a$ and $c > b$ imply the measurements of angles A and B are less than the measurement of angle C , we know that a triangle is acute if $a^3 + b^3 = c^3$. Since the converse to the Pythagorean theorem says “using three sides of a triangle to construct three squares respectively, if the area of the bigger square is equal to the sum of that of two smaller ones, then the triangle is a right triangle,” we may interpret the above as “using three sides of a triangle to construct three cubes respectively, if the volume of the bigger cube is equal to the sum of that of the two smaller ones, then the triangle is an acute triangle.”

In general, we may ask what happens if $a^n + b^n = c^n$ for n an integer greater than 3. First note that $c > a$, $c > b$, then we have

$$\begin{aligned}
 a^n + b^n &= c^{n-2} \cdot c^2 = c^{n-2} \cdot (a^2 + b^2 - 2ab \cos C), \\
 \cos C &= \frac{a^2(c^{n-2} - a^{n-2}) + b^2(c^{n-2} - b^{n-2})}{2abc^{n-2}} > 0.
 \end{aligned}$$

Again this implies angle C must be acute, so triangle ABC is an acute triangle. Therefore, we conclude that if the three sides of a triangle satisfy $a^n + b^n = c^n$, $n \in \mathbb{N}$, $n \geq 3$, it is an acute triangle. Of course, not every acute triangle must satisfy $a^n + b^n = c^n$. An equilateral triangle is a counterexample.

The above arguments can be used to prove that the triangle will be acute even if n is a real number ($n > 2$). But when n is a real number between 1 and 2, we have a different result. First, such a triangle exists, for instance, $a = 1$, $b = 4$, $c = \sqrt[3]{9^2}$, $n = \frac{3}{2}$. Note that we still have, in this case, $c > a$, $c > b$. In order to apply the law of cosines, we have to come up with an expression involving c^2 . Multiplying both sides of the equation $a^n + b^n = c^n$ by c^{2-n} and applying the law of cosines, we have

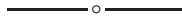
$$(a^n + b^n)c^{2-n} = c^2, \quad \cos C = \frac{a^n(a^{2-n} - c^{2-n}) + b^n(b^{2-n} - c^{2-n})}{2ab}.$$

Since $a^{2-n} - c^{2-n} < 0$, $b^{2-n} - c^{2-n} < 0$, we have $\cos C < 0$. Therefore, angle C is obtuse. In this case we have an *obtuse* triangle.

Now we may classify the shape of a triangle whose sides satisfy the equation $a^n + b^n = c^n$, where n is a real number, $n \geq 1$. If $1 < n < 2$, it is an *obtuse* triangle; If $n = 1$, the triangle degenerates to a line segment; If $n = 2$, it is a *right* triangle (the Pythagorean theorem); If $n > 2$, it is an *acute* triangle.

An interesting observation is that if we fix two sides of a triangle, say a and b , and let n become very large, then the triangle will approach an isosceles triangle. For example, $a = 2$, $b = 3$, $n = 10$, then $c = \sqrt[10]{2^{10} + 3^{10}} \approx 3.00516$; if $n = 20$, then $c = 3.00005$, which is very close to 3. In the limit case, as n approaches infinity, the triangle becomes isosceles. In fact, from calculus point of view, this exactly is the case. For fixed a and b with $0 < a < b$, the limit of $\sqrt[n]{a^n + b^n} (= c)$ is b when n goes to infinity.

Note that when $n = 1$, the triangle degenerates to a line segment. That is why we do not consider the case $0 < n < 1$. As an exercise, we leave it to students to investigate whether such a triangle exists with a, b, c as its sides satisfying the equation $a^n + b^n = c^n$ and $0 < n < 1$.



The Arithmetic of Algebraic Numbers: an Elementary Approach

Chi-Kwong Li (ckli@math.wm.edu) and David Lutzer (lutzer@math.wm.edu), College of William and Mary, Williamsburg, VA 23187

Let \mathbb{Q} and \mathbb{R} be the fields of rational and real numbers respectively. Recall that a real number r is *algebraic over the rationals* if there is a polynomial p with coefficients in \mathbb{Q} that has r as a root, i.e., that has $p(r) = 0$. Any college freshman can understand that idea, but things get more challenging when one asks about arithmetic with algebraic numbers. For example, being the roots of $x^2 - 3$ and $x^2 - 20$ respectively, the real numbers $r_1 = \sqrt{3}$ and $s_1 = 2\sqrt{5}$ are certainly algebraic over the rationals, but what about the numbers $r_1 + s_1$, r_1s_1 and $\frac{r_1}{s_1}$? As it happens, all three are algebraic over the rationals. For example, $r_1 + s_1$ is a root of $x^4 - 46x^2 + 289$. But how was that polynomial constructed, and what rational-coefficient polynomials have r_1s_1 and $\frac{r_1}{s_1}$ as roots? Students who take a second modern algebra course will learn to use field extension theory to show that the required polynomials must exist. They will learn that whenever r and $s \neq 0$ are algebraic over \mathbb{Q} , then the field $\mathbb{Q}(r, s)$ is an extension of \mathbb{Q} of finite degree with the consequence that $r + s$, rs and $\frac{r}{s}$ are indeed algebraic over \mathbb{Q} (see [2, 3, 7]). However, one would hope that students would encounter more elementary solutions for such basic arithmetic questions. Furthermore, one might want to know how to construct rational-coefficient polynomials that have $r + s$, rs and $\frac{r}{s}$ as roots and thereby obtain bounds on the minimum degrees of such polynomials.

The goal of this classroom note is to show how techniques accessible to students by the end of their first linear algebra course can answer all of these questions. Our hope is that modern algebra instructors will see such constructions as a source of student projects that tie together ideas from linear algebra and modern algebra, and as a way to study the field of algebraic numbers earlier in the usual modern algebra sequence.

We do not claim that our approach is new: the 1996 articles [5] and [1] included the same ideas. However, for some reason, these earlier articles have not led to widespread changes in the way textbook authors present the arithmetic of algebraic numbers. Therefore we believe that it is worth raising the ideas again.

Lemma. *Suppose r and s are real numbers that are algebraic over \mathbb{Q} with $s \neq 0$. Then $r + s$, rs and $\frac{r}{s}$ are also algebraic over \mathbb{Q} .*