and it then follows from the Pythagorean Theorem that

$$|AB| = \sqrt{|AD|}.$$

Thus the ratios of the lengths of the sides of triangle ABD are

$$|AD| : |AB| : |BD| = |AD| : \sqrt{|AD|} : \sqrt{|AD|(1 - |AD|)}.$$

When $|AD| = 1/4$ this becomes $1 : 2 : \sqrt{3}$ from which we conclude that triangle ABD is a $30° : 60°$ right triangle, whence $\theta = \pi/6$. A similar argument gives $\theta = \{\pi/4, \pi/3, \pi/2\}$ when $|AD| = \{1/2, 3/4, 1\}$.

The mnemonic is then verified by observing that

$$\sin\theta = \frac{|AD|}{|AB|} = \sqrt{|AD|}.$$

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Introducing the Sums of Powers

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Introduction. Integration is invariably introduced by approximating the area under a curve using the sum of the areas of inscribed or circumscribed rectangles. But unless the number of rectangles is trivially small, the actual summation must be done using a calculator, computer, or by introducing summation formulas such as

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

which, so far as students are concerned, are arbitrary formulas that come from nowhere, formulas that must be memorized or given to them. Proofs of these formulas may be found using induction ([**4**]) but this has the drawback of requiring knowledge of the formula before the validity of the formula can be shown. Methods of deriving the formulas for the sums of the nth powers usually require knowledge of the formulas for some or all of the sums of the kth powers where $k < n$ (see [**1**], [**2**], [**6**]) or identities that are no less obvious to the students than the summation formulas themselves ([**3**]). Here's a genetic way to introduce these summation formulas that gives students a quick lesson in the history of mathematics and prepares them to deal with Taylor series and partial fractions in later courses.

Leibniz's algebraic method. Leibniz's method, which he described in ([**5**], p. 51 and after), is probably the simplest of the historical methods. We'll illustrate it using the sum of the squares of the whole numbers. Let

$$S(x) = 0^2 + 1^2 + 2^2 + 3^2 + \cdots + x^2 \tag{1}$$

Consider the difference $S(x+1) - S(x)$. It should be obvious that this is just the square of the next whole number, or

$$S(x+1) - S(x) = (x+1)^2$$

Leibniz proceeded as follows: If we wish to find the sum of the nth powers, assume $S(x)$ to be a $n+1$st degree polynomial. In this case, since we are seeking the sum of the squares, assume $S(x) = ax^3 + bx^2 + cx + d$. It remains to find the coefficients a, b, c, and d. Obviously $S(0) = 0$, so $d = 0$ (and in general, the constant term will be zero). Moreover:

$$(x+1)^2 = S(x+1) - S(x) \tag{2}$$

$$x^2 + 2x + 1 = a\left((x+1)^3 - x^3\right) + b\left((x+1)^2 - x^2\right) + c\left((x+1) - x\right) \tag{3}$$

Expanding the right hand side and collecting the like terms gives:

$$x^2 + 2x + 1 = 3ax^2 + (2b + 3a)x + (a + b + c) \tag{4}$$

By comparing coefficients, we find $a = \frac{1}{3}$, $b = \frac{1}{2}$, and $c = \frac{1}{6}$.

In the most general case, we can assume the sums of the nth powers of the whole numbers from 0 to x to be given by $S(x)$, an arbitrary kth degree polynomial, and by comparing coefficients, come to the conclusion that $S(x)$ must be a polynomial of degree $n+1$; a proof that the sum of the kth powers of the whole numbers can be expressed as a $k+1$st degree polynomial is straightforward.

Another algebraic method. Leibniz's method requires the expansion of the nth power of a binomial, but it is easy to avoid this step. In any class where solving multilinear systems of equations is discussed (such as precalculus, discrete math, or college algebra), we might proceed as follow: If the left hand side of Equation 3 is truly equal to the right hand side, then the two must be equal for any and all values of x (this is, after all, the definition of the equality of two polynomials). Thus we may let x take on arbitrary values. Since we have three unknowns, we need three equations. If $x = 0$, Equation 3 becomes:

$$a + b + c = 1$$

If we let $x = 1$, Equation 3 becomes:

$$7a + 3b + c = 4$$

If we let $x = 2$, in which case Equation 3 becomes:

$$19a + 5b + c = 9$$

This gives us a system of three equations in three unknowns, and the ordinary rules of solving multilinear equations will suffice to find the coefficients a, b, and c. Alternatively, we may use Equation 1 directly and find the values of $S(0)$, $S(1)$, and $S(2)$.

A calculus-based method. I introduced these first two in my calculus class as examples of other methods, but since we had already covered differentiation and the students who were continuing on to second semester calculus would deal with Taylor series, I used the following method. Suppose we differentiate both sides of Equation 3. Note that $c((x+1) - x) = c$, so upon differentiation, the c term will vanish. Differentiating Equation 3 once:

$$2x + 2 = a\left(3(x+1)^2 - 3x^2\right) + b\left(2(x+1) - 2x\right) \tag{5}$$

Again, $b(2(x+1)-2x)=2b$, so differentiating again eliminates the b term:

$$2 = a\,(6(x+1)-6x) \tag{6}$$

If we let $x=0$ in each of the three equations, we obtain the system:

$$\begin{aligned} a+b+c &= 1 \\ 3a+2b &= 2 \\ 6a &= 2 \end{aligned}$$

which, though it is a system in three unknowns, is trivial to solve by back-substitution.

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The Pythagorean Theorem and Beyond: A Classification of Shapes of Triangles

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The Pythagorean theorem is well known to almost all college students. Some geometry textbooks show that if $a^2+b^2 \neq c^2$, then the triangle could be acute or obtuse. In this note, we first ask the question: if $a^3+b^3=c^3$, what kinds of triangle will we have? More generally, what happens if $a^n+b^n=c^n$? To answer these questions, we present an application of the law of cosines to analyze the shape of a triangle by the equation $a^n+b^n=c^n$. We introduced this project in a geometry class for secondary math education majors and students were excited about it.

To extend our investigation beyond the Pythagorean theorem, we ask what kind of triangle has sides satisfying $a^3+b^3=c^3$.

Such triangles exist. For example, we use the Geometer's Sketchpad to construct a triangle ABC with $a=2$, $b=3$, $c=\sqrt[3]{35}\approx 3.271$, and then to measure the three angles. All are less than 90 degrees as shown in Figure 1.

This example suggests that triangle ABC is acute. To give a formal proof in general, we apply the law of cosines since it relates sides and angles in a triangle. Noting that $a^3+b^3=c^3$ implies that $c>a$ and $c>b$, we have

$$a^3+b^3 = c\cdot c^2 = c(a^2+b^2-2ab\cdot\cos C).$$