Predicting Sunrise and Sunset Times

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For a given location and day of the year, can you predict the times of sunrise and sunset? Although the solution to this problem is somewhere between old and ancient, the standard solution requires the understanding of a considerable number of preliminary ideas from celestial mechanics. The following approach uses only simple tools from trigonometry and analytic geometry to produce a fairly good set of approximations for sunrise and sunset times. One can leave it at that and have a nice project for average students or, for students with stronger backgrounds, one can pursue certain corrections that make these sunrise and sunset predictions quite accurate. These corrections can be studied at any level from a simple "Here's the source of the error and here's a term that we can add on to fix it," to a more complete analysis involving Fourier sine series and a deeper understanding of celestial mechanics.

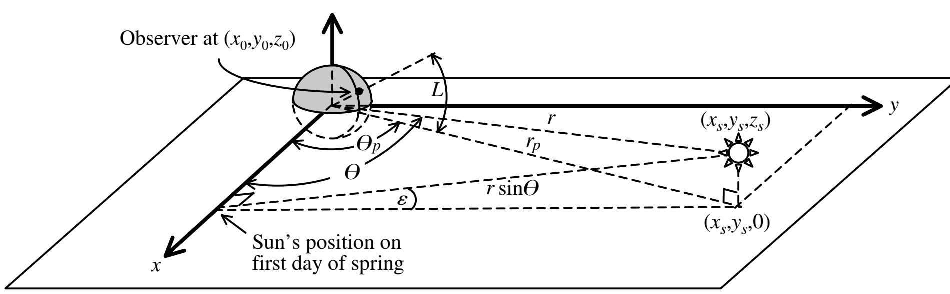

Figure 1. The earth and sun in the geocentric equatorial coordinate system.

In the *geocentric equatorial* coordinate system of Figure 1, the xy-plane contains the earth's equator, with the positive x-axis chosen so that it passes through the sun's center on the first day of spring. The angle $\epsilon = 23.45°$ (0.409 radians) is the inclination between the xy-plane and the *ecliptic* plane containing the earth's orbit about the sun. At noon on the day of our observation, the sun appears directly over the observer's meridian at $\langle x_s, y_s, z_s\rangle$. If θ_p is the angle between the projection vector $\langle x_s, y_s, 0\rangle$ and the positive x-axis, L is the observer's latitude, and $R = 6378$ km is the radius of the earth, then the observer's position (at noon) will be

$$x_0 = R\cos L\cos\theta_p, \qquad y_0 = R\cos L\sin\theta_p, \qquad z_0 = R\sin L. \tag{1}$$

(This is just the standard spherical-to-rectangular coordinate conversion, with the observation that latitude is measured *up* from the xy-plane instead of *down* from the z-axis as is usual in spherical coordinates presented in calculus texts.) Since the earth rotates 2π radians in 1440 minutes (24 hours), the observer's coordinates t minutes from noon will be

$$\begin{aligned} x_0 &= R\cos L\cos\left(\theta_p + \frac{2\pi}{1440}t\right), \\ y_0 &= R\cos L\sin\left(\theta_p + \frac{2\pi}{1440}t\right), \\ z_0 &= R\sin L, \end{aligned} \tag{2}$$

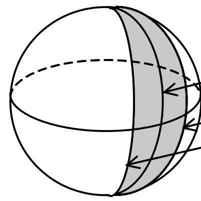

Figure 2. Standard time meridian and corresponding time zone.

where $t < 0$ before noon and $t > 0$ after noon. We also note in Figure 1 that r is the length of the earth-sun vector $\langle x_s, y_s, z_s \rangle$, θ is the angle between this vector and the positive x-axis, and r_p is the length of the projection $\langle x_s, y_s, 0 \rangle$.

Up to this point there has been no mention of the observer's location with respect to *time zones*. We shall assume throughout this discussion that our observer is on a *standard time meridian* (i.e., east or west longitude 0°, 15°, 30°, etc.) each of which corresponds roughly to the middle of a time zone. (See Figure 2.) For observers not on such a meridian, sunrise and sunset times must be adjusted by 1440 minutes/360° = 4 minutes per degree of longitude, with earlier sunrises and sunsets as one moves east, later as one moves west within a time zone.

Though the vector $\langle x_0, y_0, z_0 \rangle$ was introduced as the location of our observer, it also can be thought of as a normal vector for the plane tangent to the earth's surface at the observer's location. Hence, the equation of this tangent plane is

$$x_0(x - x_0) + y_0(y - y_0) + z_0(z - z_0) = 0.$$

Since the tangent plane provides a local approximation of the earth's surface at the observer's location, the sun lies below this plane (from the observer's perspective) before sunrise and above the plane after sunrise. At sunset, the sun moves from just above to just below the plane. Therefore, the sunrise and sunset times can be approximated as *the times at which the center of the sun lies in the tangent plane*, that is, the times at which

$$x_0(x_s - x_0) + y_0(y_s - y_0) + z_0(z_s - z_0) = 0.$$

Since $x_0^2 + y_0^2 + z_0^2 = R^2$, we obtain

$$x_0 x_s + y_0 y_s + z_0 z_s = R^2.$$

Substituting the values x_0, y_0, z_0 in (2) into this equation, we get

$$x_s R \cos L \cos\left(\theta_p + \frac{2\pi}{1440}t\right) + y_s R \cos L \sin\left(\theta_p + \frac{2\pi}{1440}t\right) + z_s R \sin L = R^2. \quad (3)$$

Using the angle addition formulas for sine and cosine and the relationships

$$r_p = \sqrt{r^2 - z_s^2}, \quad \cos\theta_p = \frac{x_s}{r_p}, \quad \sin\theta_p = \frac{y_s}{r_p}, \quad \text{and} \quad z_s = r \sin\theta \sin\epsilon,$$

we can solve equation (3) for t, obtaining

$$t_0 = \frac{1440}{2\pi} \cos^{-1}\left(\frac{R - z_s \sin L}{r_p \cos L}\right).$$

This t_0 value gives the number of minutes after noon that sunset occurs. Likewise, $-t_0$ solves equation (3) and represents the number of minutes before noon that sunrise occurs. If we assume that the earth's orbit is *circular* and use $r = 149598000$ km (the earth's mean distance from the sun), then

$$\theta = \frac{2\pi}{365.25}(d - 80),$$

where d is the day of the year. (This gives $\theta = 0$ on the 80th day of the year, which is the first day of spring.) Finally, since these computations are based on the location of the *center* of the sun, a correction is needed to account for the length of time between sunrise (or sunset) and the moment the sun's center is at the horizon. Experience suggests replacing t_0 by $\hat{t} = t_0 + 5$ minutes.

Example. June 10 is the 161st day of the year. Therefore, $d = 161$ and $\theta = 1.3934$. At $L = 40°$ (0.698 radians) north latitude, $z_s = 5.8560 \times 10^7$ and $r_p = 1.3766 \times 10^8$, and so the computed time value is $t_0 = 444$ minutes. Thus, $\hat{t} = 449$ minutes, or 7 hours 29 minutes. Therefore, sunrise occurs at 4:31 am and sunset at 7:29 pm. The sunrise time agrees exactly with that found in an almanac [**1**], and sunset time is off by 1 minute!

At this point, one can declare victory and leave it at that, but further computational examples using different days of the year and different latitudes reveal that the error is not always so small. In fact, errors as large as 28 minutes occur at some times of the year, especially at high latitudes. So what is going wrong? One source of error is fairly clear: according to Kepler's first law, the earth's orbit is elliptical, not circular, so the value for r changes from day to day. And according to Kepler's second law, the vector from sun to earth sweeps out equal areas in equal times, so θ does not change at a constant rate. But another (and, as it turns out, larger) source of error is an effect known as the *equation of time*, which we now briefly describe.

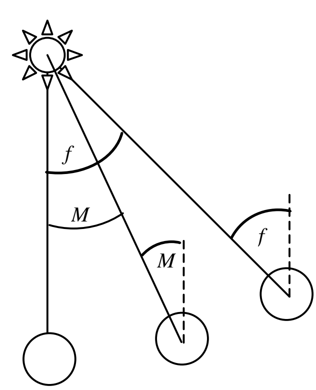

Figure 3. Solar noon on successive days.

The essential idea in the equation of time is that *the length of time between noon (i.e., sun at its highest point) one day and noon the next is not constant.* There are two reasons for this variability, the first of which is illustrated in Figure 3. Here M denotes the angle through which the earth moves from noon one day until noon the next, if we assume *constant* speed in its orbit. Note that angle M also is the angle in excess of 2π

radians that the earth must rotate in order for the sun to be directly overhead again. But in January, when the earth-sun distance is at its minimum, the earth's speed in its orbit must be greater than average to satisfy Kepler's second law. Because of the earth's higher speed in its orbit at this time of year, the angle is not M but the slightly larger f, so the earth must rotate $f - M$ radians *more* than average from noon to noon. At 1440 minutes per 2π radians, one might say that *noon is delayed*

$$\frac{1440}{2\pi}(f - M) \text{ minutes.}$$

By accurately describing f and M and expanding their difference in a Fourier sine series, one can show that for a given day d, this delay is approximated by

$$8 \sin\left(\frac{2\pi}{365.25}d\right) \text{ minutes.}$$

See [**2**, p. 87] and [**3**] for details.

The second reason that the "noon-to-noon" time varies through the year is that the earth's orbital motion and daily rotation take place in different planes, and this affects the amount in excess of 2π radians that the earth must rotate in order for the sun to to be in its directly overhead position each day. The earth's orbital motion must be projected onto the equatorial plane to correctly describe this angle. This requires that the noon-to-noon time also take into account a term whose Fourier approximation is

$$-10 \sin\left(\frac{4\pi}{365.25}(d - 80)\right).$$

Again, more details (including some clever animations) are given in [**3**].

One can achieve fairly good computational results by allowing "noon" to vary according to

$$n = 720 - 10 \sin\left(\frac{4\pi}{365.25}(d - 80)\right) + 8 \sin\left(\frac{2\pi}{365.25}d\right)$$

minutes past midnight. Using r and θ to compute $\hat{t}$ as described previously, sunrise occurs at $n - \hat{t}$ and sunset at $n + \hat{t}$ minutes past midnight. When this method is applied at latitude 40° north on all days of the year and compared to values found in an almanac, the great majority of sunrise and sunset times are correct or off by one minute. The worst error is 6 minutes, the average (absolute value) of sunrise errors is 1 minute, and the average (absolute value) of sunset errors is 1.7 minutes. The accuracy can be improved slightly by using the true, variable value of r instead of the constant value used above, and replacing θ by the true angle (which changes by a variable amount each day). Anyone wishing to pursue these changes should look up the ideas of *true anomaly, mean anomaly, eccentric anomaly* and their relationship to r in any orbital mechanics text such as [**2**].

The beauty of this application is that it can be approached at so many different levels. Use it in a freshman or sophomore class as an application of tangent planes to surfaces, or use it as a senior project complete with the celestial mechanics and Fourier series details.

References

1. W. A. McGeveran Jr., editor, *The World Almanac and Book of Facts 2002*, World Almanac Education Group, New York, 2002.
2. A. E. Roy, *Orbital Motion*, 3rd ed., Adam Hilger, Ltd., Bristol, UK, 1988.
3. R. Urschel, *Analemma*, http://www.analemma.com.

A Calculation of $\int_0^\infty e^{-x^2}\,dx$

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If you are looking for a way to tie together different strands of material your students see in their second semester of a standard three-semester calculus course, the result in the title will serve you very well. Of course, the result is

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

Students find it surprising (where does the π come from?) and meaningful (when they learn its relationship to the bell curve). You can also use this significant improper integral as an entry into a discussion of approximations of definite integrals over finite regions.

There are numerous ways to evaluate the integral in the title. In just a few minutes in the library, I located two evaluations based on Wallis' infinite product expansion of π, [**8**] and [**5**]; a calculation using contour integration in the complex plane, [**1**]; a reduction to Euler's integral of the first kind, [**2**]; two evaluations using differentiation under the integral sign, [**7**] and [**9**]; and a calculation based on solids of revolution, [**3**]. You can find the standard computation based on a double integral in polar coordinates in almost any statistics or calculus book. The integral is commonly associated with Gauss, although he credits Laplace with its discovery and publication in 1805. Euler was working with similar integrals thirty years earlier, but he seems to have missed its exact formulation. By 1813, the result appears to have been well known. See [**6**] for a more detailed history.

Few of the techniques for evaluating this integral are easily accessible to a begining student. Here is a technique that students can appreciate. Although the method may not be new (the main idea is already in [**3**] and [**4**]) our variation of it is sufficiently elementary that it can be presented to students.

Let I denote the value of the integral of interest. A comparison with e^{-x} confirms that I is finite. Let's start by finding the volume V of the solid of revolution made by revolving the graph of e^{-x^2} for $x \in [0, \infty)$ about the y-axis (Figure 1).

Using the usual technique of nested cylindrical shells, we obtain

$$V = \int_0^\infty 2\pi x e^{-x^2}\,dx = \pi.$$

Now let's compute V a second time. This time take cross sections parallel to the x-axis (Figure 2).