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Cauchy's Mean Value Theorem Involving n Functions

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If $f(x)$ and $g(x)$ are two functions continuous on $[a, b]$ and differentiable on (a, b) with $g'(x) \neq 0$ for any x in (a, b), then there exists a point c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}. \tag{1}$$

The above Generalized Mean Value Theorem was discovered by Cauchy ([**1**] or [**2**]), and is very important in applications. Since Cauchy's Mean Value Theorem involves two functions, it is natural to wonder if it can be extended to three or more functions. If so, what formulas similar to (1) can we have? In this capsule we show, and then extend, the following result.

Theorem 1. *Let α, β be two real numbers such that $\alpha + \beta = 1$. If $f(x)$, $g(x)$, $h(x)$ are three functions continuous on $[a, b]$ and differentiable on (a, b) such that $g(b) \neq g(a)$ and $h(b) \neq h(a)$, then there exists a point c in (a, b) such that*

$$f'(c) = \alpha g'(c)\frac{f(b) - f(a)}{g(b) - g(a)} + \beta h'(c)\frac{f(b) - f(a)}{h(b) - h(a)}. \tag{2}$$

Observe that (2) follows by letting $\gamma = -1$ and setting $f_1 = f$, $f_2 = g$, and $f_3 = h$ in Theorem 2.

Theorem 2. *Let α, β and γ be three real numbers such that $\alpha + \beta + \gamma = 0$. If f_1, f_2 and f_3 are three functions continuous on $[a, b]$ and differentiable on (a, b) such that $f_i(a) \neq f_i(b)$ for $i = 1, 2, 3$, then there exists a point c in (a, b) such that*

$$\frac{\gamma}{f_1(b) - f_1(a)} f_1'(c) + \frac{\alpha}{f_2(b) - f_2(a)} f_2'(c) + \frac{\beta}{f_3(b) - f_3(a)} f_3'(c) = 0. \tag{3}$$

Proof. Let

$$\begin{aligned} k(x) = {} & \gamma\big(f_2(b) - f_2(a)\big)\big(f_3(b) - f_3(a)\big)\big(f_1(x) - f_1(a)\big) \\ & + \alpha\big(f_1(b) - f_1(a)\big)\big(f_3(b) - f_3(a)\big)\big(f_2(x) - f_2(a)\big) \\ & + \beta\big(f_1(b) - f_1(a)\big)\big(f_2(b) - f_2(a)\big)\big(f_3(x) - f_3(a)\big). \end{aligned}$$

It is easily checked that $k(a) = 0$ and

$$k(b) = \big(f_1(b) - f_1(a)\big)\big(f_2(b) - f_2(a)\big)\big(f_3(b) - f_3(a)\big)(\alpha + \beta + \gamma) = 0.$$

By Rolle's Theorem, there exists a point c in (a, b) such that $k'(c) = 0$. Thus, (3) follows.

For n functions, we have the following generalization.

Theorem 3. *Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be n real numbers such that $\alpha_1 + \cdots + \alpha_n = 0$. If $f_1(x), f_2(x), \dots, f_n(x)$ are n functions continuous on $[a, b]$ and differentiable on (a, b), and $f_i(b) \neq f_i(a)$ for $i = 1, 2, \dots, n$, then there exists a point c in (a, b) such that*

$$\alpha_1 \frac{\Pi_{i=1}^n (f_i(b) - f_i(a))}{f_1(b) - f_1(a)} f_1'(c) + \alpha_2 \frac{\Pi_{i=1}^n (f_i(b) - f_i(a))}{f_2(b) - f_2(a)} f_2'(c) + \cdots + \alpha_n \frac{\Pi_{i=1}^n (f_i(b) - f_i(a))}{f_n(b) - f_n(a)} f_n'(c) = 0,$$

or

$$\frac{\alpha_1}{f_1(b) - f_1(a)} f_1'(c) + \frac{\alpha_2}{f_2(b) - f_2(a)} f_2'(c) + \cdots + \frac{\alpha_n}{f_n(b) - f_n(a)} f_n'(c) = 0.$$

Theorem 2 can be useful in proving the existence of solutions of certain equations. The Corollary below is a direct consequence of Theorem 3 for $\alpha_1 = \frac{1-n}{n}$, and $\alpha_2 = \cdots = \alpha_n = \frac{1}{n}$.

Corollary. *If $f_1(x), f_2(x), \dots, f_n(x)$ are n functions continuous on $[a, b]$ and differentiable on (a, b), and $f_i(a) \neq f_i(b)$ for $i = 1, 2, \dots, n$, then the following equation has at least one solution in (a, b).*

$$\frac{1-n}{f_1(b) - f_1(a)} f_1'(x) + \frac{1}{f_2(b) - f_2(a)} f_2'(x) + \cdots + \frac{1}{f_n(b) - f_n(a)} f_n'(x) = 0.$$

Example. Let

$$F(x) = -3x + \frac{\pi}{2} \cos \frac{\pi}{2} x + \frac{e^x}{e-1} + \frac{1}{(x+1)\ln 2}.$$

Then the equation $F(x) = 0$ has at least one solution in (0, 1) because

$$F(x) = -\frac{3}{2} \frac{(x^2)'}{1^2 - 0^2} + \frac{(\sin \frac{\pi}{2} x)'}{\sin \frac{\pi}{2} \cdot 1 - \sin \frac{\pi}{2} \cdot 0} + \frac{(e^x)'}{e^1 - e^0} + \frac{[\ln(x+1)]'}{\ln(1+1) - \ln(0+1)}.$$

Remark. In the above example, since $F(0) > 0$ and $F(1) > 0$, this conclusion is not an obvious consequence of Intermediate Value Theorem for continuous functions.

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