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Maximal Revenue with Minimal Calculus

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A basic motivating example. If T is a right triangle and R is an inscribed rectangle with the right angle of T as one of its right angles, what is the maximum area of R? This problem can be solved by several standard methods. However, the quickest way is to treat the two triangular pieces of T outside of R as flaps and fold them on top of R. (See Figure 1.) The flaps will exactly cover R if and only if P is the midpoint of T's hypotenuse (since P is the midpoint of the hypotenuse if and only if $\Delta y = y$ and $\Delta x = x$). From Figure 1, we see also that Area $(R) \leq \frac{1}{2}$Area (T), with equality holding if and only if P is the midpoint of T's hypotenuse.

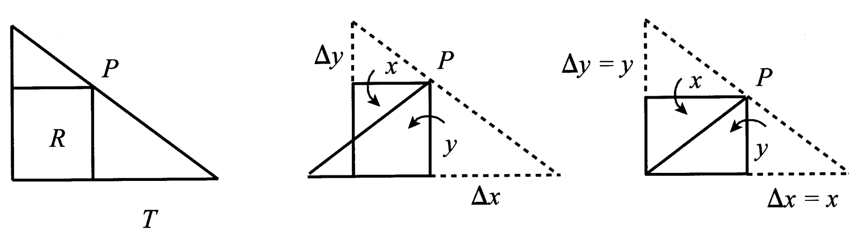

Figure 1.

The same problem in economics. Given a linear demand curve (Figure 2), what price will maximize the revenue?

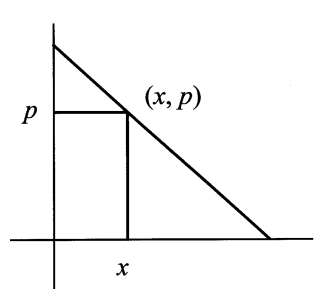

Figure 2. A linear demand curve $p = p(x)$

Since revenue is the number of products demanded (x) times the price (p), we need to maximize the area of the inscribed rectangle with vertex (x, p) on the demand curve. Reasoning as above, revenue is maximum when the price is half the price that totally dries up demand. (Equivalently, the level of production is exactly half the number of units one could give away.) If the price is higher than the revenue-maximizing price, the folded top flap intersects the p-axis above the origin on the graph of the demand curve. If we think of the origin as a target, we discover a mnemonic: the price is "high" and lowering the price in this setting will increase the revenue. (See Figure 3.) This situation is defined in economics as **elastic** demand. On the other hand, if the price is lower than the revenue-maximizing price, the folded top flap will intersect the p-axis below the origin, suggesting that the price is "low" and that raising the price will raise the revenue. This is defined as **inelastic** demand. At a point where revenue is maximized, so that a change in either direction lowers revenue, the demand is said to be **unitary**.

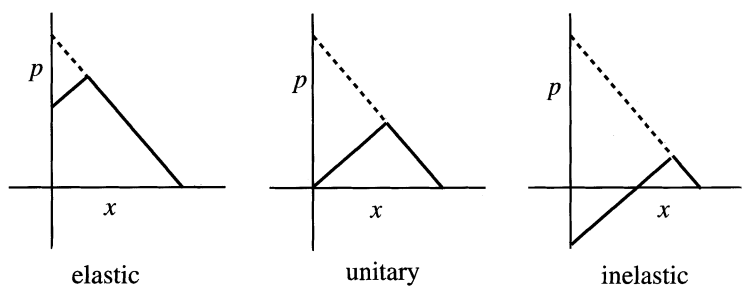

Figure 3.

General demand curves. The mnemonic also applies to nonlinear demand curves $p = p(x)$, where $p(x)$ is differentiable. In this context, our "flap" will be the left part of the tangent line at a point $P = (x_0, p_0)$ on the demand curve. (See Figure 4.) If we fold down the left part of the tangent line and it crosses the p-axis below the origin, this folded line is steeper than the line from the origin to the point P. Thus,

$$-\frac{dp}{dx} > \frac{p}{x}.$$

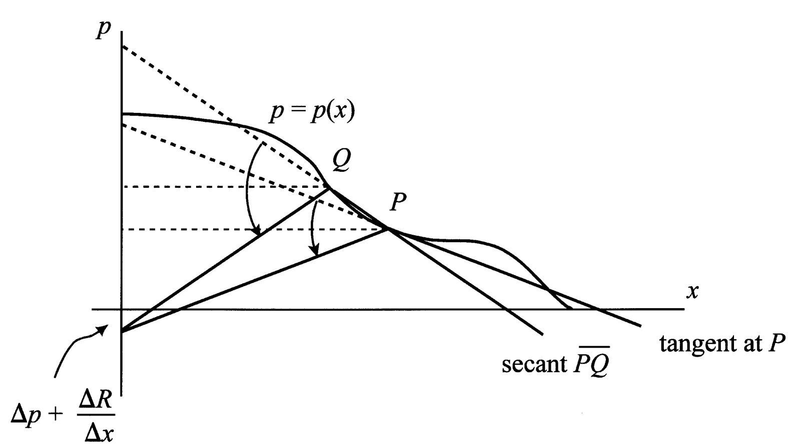

Figure 4.

To show that the point P on the demand curve is inelastic, (that is, sufficiently small price increases will increase revenues), we use a sufficiently small price increase Δp to perturb our folded tangent at P. The secant line through P and $Q = (x_0 + \Delta x, p_0 + \Delta p)$ when folded at Q has slope $-\Delta p/\Delta x$ and a negative p-intercept. A simple calculation shows that this p-intercept is

$$p = p_0 + \Delta p + \frac{\Delta p}{\Delta x}(x_0 + \Delta x) = \Delta p + \frac{p_0 \Delta x + x_0 \Delta p + \Delta p \Delta x}{\Delta x} = \Delta p + \frac{\Delta R}{\Delta x}.$$

But since Δp is positive and Δx is negative (since demand functions are decreasing), it follows that ΔR is positive. Therefore a small price increment increases revenue.

A similar argument shows that when the folded tangent intersects the p-axis above the origin,

$$-\frac{dp}{dx} < \frac{p}{x},$$

and small decrements in price will increase revenue. Thus, demand is elastic. When revenue is maximized, the folded tangent must cross the origin and thus

$$-\frac{dp}{dx} = \frac{p}{x}.$$

This yields a straightforward graphical way to estimate whether the demand curve is elastic, inelastic, or unitary at a point (x, p). (See Figure 5.) Since solving the equation $-\frac{dp}{dx} = \frac{p}{x}$ to determine critical points can be onerous or impossible, the graphical method provides a useful alternative.

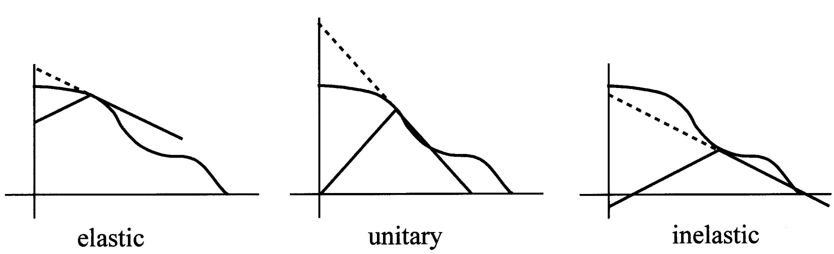

Figure 5.

We note in closing that the demand curve $p = c/x$ (c, constant) yields constant revenue, so demand is always unitary. This gives an economic interpretation to

$$-\frac{dp}{dx} = -\frac{d}{dx}\frac{c}{x} = \frac{c}{x^2} = \frac{p}{x}.$$

