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Trigonometric Identities on a Graphing Calculator

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Many calculus texts (for example [**1**], [**2**], and [**3**]) mention the power and utility of graphing calculators along with the various "pitfalls" one can experience. Figure 1 displays a misleading graph of the $\sin(190x)$ which should have 190 periods on a TI-83 graphing calculator's interval of XMIN $= 0$ to XMAX $= 2\pi$, but only appears to display two periods.

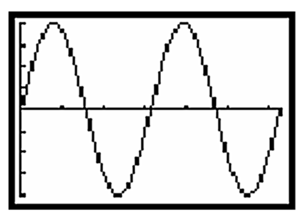


Figure 1. $Y_1 = \sin(190x)$ in the TI-83 window $[0, 2\pi]$ by $[-1, 1]$ in radian mode.

Here's a detailed mathematical explanation for this apparent discrepancy.

Understanding calculator graphics. Graphing calculators simulate the graph of a continuous function. The resolution of the graphics screen on the TI-83 has 5985 pixels, or picture elements, that can be either turned on or off. Figure 2 illustrates the 0 through 62 rows and the 0 through 94 columns of pixels. See [**4**, page 8-6].

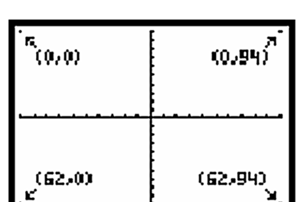


Figure 2. Pixel positions on the TI-83 graphing calculator.

Using the appropriate transformations, the graph of a continuous function is simulated by sampling the functional values for a finite number of points, actually for up to 95 equally spaced horizontal values along the domain window.

Derivation of graphing calculator trigonometric identities. Trigonometric identities are true for the common domain of the functions in the identity. Graphing calculator trigonometric "identities" hold on the specified domain on which the functions are plotted. Functions appear to be graphically equivalent if the same pixel locations are used to display the graphs of the functions. The following derivations are presented for radian mode and are similar for degree mode if 2π is replaced by $360°$.

Assume the graphing domain window is [XMIN, XMAX] and let $\Delta XN = \text{XMAX} - \text{XMIN}$. The domain values at which a function is sampled is the set $\text{XPLOT} = \{X \mid X = n(\frac{\Delta XN}{94}),\ n = 0, 1, 2, \ldots, 94\}$. If $a = k * 94 * (\frac{2\pi}{\Delta XN})$ for $k = \pm 1, \pm 2, \pm 3, \ldots$, then $a * X = k * n * 2\pi$ and hence $\sin(aX) = 0$, $\cos(aX) = 1$, and $\tan(aX) = 0$ for all values of k and n. Applying these facts to the formulae for $\sin(A \pm B)$, $\cos(A \pm B)$, and $\tan(A \pm B)$ gives:

$$\sin\big((a \pm b)X\big) = \sin(aX)\cos(bX) \pm \cos(aX)\sin(bX) = \pm\sin(bX),$$

$$\cos\big((a \pm b)X\big) = \cos(aX)\cos(bX) \mp \sin(aX)\sin(bX) = \cos(bX),$$

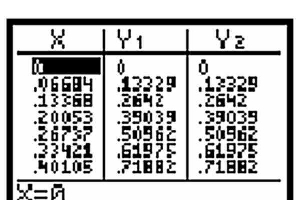
and

$$\tan\big((a \pm b)X\big) = \frac{\tan(aX) \pm \tan(bX)}{1 \pm \tan(aX)\tan(bX)} = \pm\tan(bX).$$

[XMIN, XMAX] is arbitrary, but ΔXN equal to a multiple of π simplifies the computation of a. For example, if $[\text{XMIN}, \text{XMAX}] = [0, 2\pi]$, then $a = k * 94$; and if $k = 2$, with $a = 188$ and $b = 2$, one gets Figure 1, i.e. $\sin(190X) = \sin((188 + 2)X) = \sin(2X)$.

π is the period of $\sin(2X)$ and graphically (visually) π also appears to be the period of $\sin(190X)$, whose period is actually $\frac{\pi}{95}$. To formalize this graphical (visual) equality of two functions define functions f and g to be graphing calculator equal, denoted by $f(X) \stackrel{\text{gc}}{=} g(X)$, as the pixel equality of two functions drawn on a TI-83 graphing calculator, that is, $\{(X, f(X)) \mid X \in \text{XPLOT}\} = \{(X, g(X)) \mid X \in \text{XPLOT}\}$ for $X \in [\text{XMIN}, \text{XMAX}]$.

Figure 3 displays a table of data comparing the values of $\sin(2X)$ versus $\sin(190X)$ for X starting at the value 0 and incremented by the TI-83's approximation of $\Delta x = \frac{2\pi}{94}$. As expected, the calculator's approximate numerical values for $\sin(2X)$ and $\sin(190X)$ are identical or nearly so, if one compares the twelve-digit values of Y_1 and Y_2. Because these are the values plotted for each function, their graphs appear and in fact are identical.



X	Y1	Y2
0	0	0
.06684	.13329	.13329
.13368	.2642	.2642
.20053	.39039	.39039
.26737	.50962	.50962
.33421	.61975	.61975
.40105	.71882	.71882

X=0

Figure 3. $Y_1 = \sin(2X)$ versus $Y_2 = \sin(190X)$ with $\Delta x = \frac{2\pi}{94}$.

Here is a small sample of graphing calculator trigonometric identities for $X \in [0, 2\pi]$:

$$\sin(377X) \stackrel{\text{gc}}{=} \sin(X)$$

for $a = 376$, $b = 1$ respectively, and

$$\cos(-96X) \overset{\text{gc}}{=} \cos(280X) \overset{\text{gc}}{=} \cos(2X)$$

for $a = -98$, $a = 278$, and $b = 2$.

If $\sin(2X)$ and $\sin(190X)$ are evaluated at a value that is not represented by a pixel value, in most cases they will not be equal. For example for $x = 0.12$ between 0.066842 and 0.133685, $\sin(2 * 0.12) = 0.2641954$ while $\sin(190 * 0.12) = -0.7234948$. This again illustrates that for the 95 sampled points $\sin(2X) \overset{\text{gc}}{=} \sin(190X)$, but for almost every other value of X, these functions are not equal.

Extending graphing calculator trigonometric identities. On the TI-83, $\text{XRES} = 1, 2, 3, \dots,$ or 8 is a variable that modifies pixel resolution and is listed as the last variable on the WINDOW menu. See [**4**, page 3-11]. At XRES = 1 functions are evaluated and graphed at each pixel on the x-axis, while at XRES = 3 functions are evaluated and graphed at every third pixel along the x-axis. To keep the initial presentation simple the set XPLOT implicitly assumed XRES = 1. The most general definition of graphing calculator equality should include the variable XRES. For any value of XRES, the domain values at which a function is sampled is the set

$$\text{XPLOTXRES} = \left\{ X \mid X = n\left(\frac{\Delta XN}{94/\text{XRES}}\right),\ n = 0, 1, 2, \dots, \frac{94}{\text{XRES}} \right\}.$$

Based upon this most general set of domain values, two functions f and g are graphing calculator equal, $f(X) \overset{\text{gc}}{=} g(X)$, if $\{(X, f(X)) \mid X \in \text{XPLOTXRES}\} = \{(X, g(X)) \mid X \in \text{XPLOTXRES}\}$ for $X \in [\text{XMIN}, \text{XMAX}]$ and $\text{XRES} = 1, 2, \dots,$ or 8. Upon replacing XPLOT with XPLOTXRES and noting if

$$a = k * \frac{94}{\text{XRES}} * \left(\frac{2\pi}{\Delta XN}\right) \quad \text{for} \quad k = \pm 1, \pm 2, \pm 3, \dots,$$

then $a * X = k * n * 2\pi$, the initial derivations can be easily recomputed. The previous graphing calculator trigonometric identities are true for any value of XRES since an appropriate value of k guarantees the graphing calculator trigonometric identity. There are some functions which are graphing calculator equal for only certain values of XRES. For $X \in [0, 2\pi]$ and XRES = 2, 4, 6, or 8, $\sin(2X) \overset{\text{gc}}{=} \sin(49X)$, but for XRES = 1, 3, 5, or 7, $\sin(2X)$ is not graphing calculator equal to $\sin(49X)$. The variable XRES expands the number of graphing calculator trigonometric identities.

With their periodic nature, trigonometric functions are the easiest and most obvious functions to have graphing calculator identities. Are there polynomial, rational, exponential, logarithmic, etc. functions that are graphing calculator equal?

This paper has focused on the TI-83, but with modifications to accommodate the specific resolution of the graphics display, functions can be drawn to exhibit similar behavior on most graphing calculators and computer graphics screens.

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Musical Analysis and Synthesis in Matlab

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This article presents musical examples and computer tools that could be used in an introductory pde and Fourier Series class, where students work with the heat and wave equations. The study of Fourier series from a musical perspective offers great insight into basic mathematical concepts and the physics of musical instruments. Tools available in Matlab allow students to easily analyze the wave forms and harmonics of recorded sounds and to synthesize their own. These experiments are a thought provoking way to understand how a wave form is composed of a summation of Fourier Series basis functions and how this relates to the frequency domain. Many students I have worked with have special musical interests and go on to conduct experiments with their own instruments.

The plucked string. One of the standard problems in an introductory pde course is the wave equation with Dirichlet boundary conditions,

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in (0, L), t \geq 0 \\ u(0,t) = u(L,t) = 0, & t \geq 0 \\ u(x,0) = \alpha(x),\ u_t(x,0) = \beta(x), & x \in (0, L). \end{cases} \tag{1}$$

Physically, we can think of $u(x,t)$ as the displacement of a plucked guitar string with initial displacement $\alpha(x)$ and initial velocity $\beta(x)$. Additional terms may be added to the wave equation to account for string stiffness and internal damping [**1**]. The solution, obtained by separation of variables, is

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi}{L}x \left(a_n \cos\frac{n\pi}{L}ct + b_n \sin\frac{n\pi}{L}ct\right), \tag{2}$$

where

$$a_n = \frac{2}{\pi}\int_0^L \alpha(x)\,\sin\frac{n\pi}{L}x\,dx \tag{3}$$

$$b_n = \frac{2L}{cn\pi^2}\int_0^L \beta(x)\,\sin\frac{n\pi}{L}x\,dx \tag{4}$$

are the Fourier coefficients of $\alpha(x)$ and $\beta(x)$. Using the trig identity $\cos(x-y) = \cos x\ \cos y + \sin x\ \sin y$ we may rewrite the solution as

$$u(x,t) = \sum_{n=1}^{\infty} p_n\ \sin\frac{n\pi}{L}x\ \cos\frac{n\pi}{L}c\,(t - \gamma_n). \tag{5}$$

