Calculus in the Operating Room

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Here is a realistic, and potentially important, application of a familiar topic from calculus. Imagine a hospital patient about to undergo surgery. Suppose he has 5 liters (L) of blood in his body, 40% of which consists of red blood cells (this percentage is called the *hematocrit*), and, during the surgery, he will bleed 2½ liters of blood. This is a realistic estimate for certain types of hip replacement, for example. His blood volume is maintained at 5 L by controlled injection of saline solution (no blood cells), which we assume to mix instantaneously with his blood. This means that the blood lost through bleeding becomes less rich in red cells as the operation progresses.

Question 1. What is the patient's volume of red blood cells at the end of the operation?

Some of the lost blood can be recovered, washed, and returned to the patient after the operation; but there is some loss and washing is an expensive procedure. Suppose that, before the operation, some blood is removed from the patient and replaced with saline solution. This blood will be returned to the patient afterward. This procedure, called acute normovolemic hemodilution (ANH), will decrease the loss of red blood cells during the operation. However, during the transfusion the patient's total blood volume is maintained at 5 L; as with the bleeding during surgery, this affects the rate of red blood cell removal.

Question 2. If it is known that the patient's hematocrit can go as low as 20%, but no lower, how much blood should be replaced in the ANH procedure just described?

Both of these questions can be answered by a simple exponential decay model, once one makes the observation that the rate of blood cell loss during the operation is proportional to the amount of red cells present. For the numbers given, and with time measured as a fraction of the length of the operation, the proportionality constant is ½, since 2.5 of the 5 L are lost.

Now both questions can be answered. If $f(t)$ is the volume of red blood cells remaining at time t, then $f(t) = f(0)e^{-t/2}$. For Question 1, $f(1) = 2000\sqrt{e} = 1213$ milliliters.

For Question 2 we need to know what value of $f(0)$ will cause $f(0)/\sqrt{e}$ to be 1000 ml (20% of 5 L); this is $f(0) = 1000\sqrt{e}$. In order to figure out how much should be removed by ANH, we reverse the technique just discussed and solve $2000e^{-k/5000} = 1000\sqrt{e}$ to get $k = 966$ ml. This leaves $1000\sqrt{e}$, or 1649 ml of red blood cells in the patient for a hematocrit of 33%, which will become 20% after surgery.

Note that without the transfusion the red blood cell loss is 2000-1213 = 787 ml. With the transfusion the patient starts with 1649 ml of red blood cells and ends up with 1000; a loss of 649 ml. There is a net savings of 138 ml of red blood cells. This savings may in fact not be large enough to justify the procedure; it must be balanced with overall expense, risks associated with ANH, and the risk of an adverse reaction to blood the patient may have to receive from a blood bank after the operation (see M. E. Brecher and M. Rosenfeld, Mathematical and computer modeling of acute normovolemic hemodilution, *Transfusion* 1994 (34), 176-179). But it is noteworthy that a simple freshman-calculus model applies to the basic situation.

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