In most calculus books there is little effort given to showing that the cylindrical shell method and disk method give the same value when computing the volume of a solid of revolution. Indeed it is not obvious that these two distinct methods should give the same result. In some texts this is demonstrated when the trapezoid bounded by the x-axis, \( y = mx + b \), \( x = a \) and \( x = b \) is revolved about the y-axis.

In this paper we shall show that the cylindrical shell and disk methods give the same value if the region revolved about the y-axis is bounded by \( y = f(x) \), \( x = a \), \( x = b \) and the x-axis, provided \( f(x) \) is a differentiable function on \([a, b]\) and \( f(x) \) is one-to-one. The proof is simple and uses two theorems which the students have recently learned (substitution formula and integration by parts). This proof can easily be included in a calculus course.

Consider the solid of revolution \( K \) produced by revolving the region bounded by \( y = f(x) \), \( x = a \), \( x = b \) and the x-axis, about the y-axis. We use the shell method, which involves summing the volumes of cylindrical shells, to define the volume of \( K \) to be \( \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi x_i f(x_i) \Delta x \). If \( f(x) \) is differentiable on \([a, b]\) and hence continuous there, this limit exists and is equal to \( \int_{a}^{b} 2\pi xf(x) \, dx \).

Suppose the region is bounded by the function \( x = g(y) \), \( y = c \), \( y = d \) and the y-axis. In the disk method, which involves summing the volumes of disks, we consider
\[
\lim_{\Delta y \to 0} \sum_{i=1}^{n} \pi [g(y_i)]^2 \Delta y_i.
\]
If \( g(y) \) is continuous on \([c, d]\), this limit exists and is equal to \( \int_{c}^{d} \pi [g(y)]^2 \, dy \).

**Theorem.** Let \( 0 \leq a < b \), and let \( y = f(x) \) be differentiable, nonnegative and \( 1 - 1 \) on \([a, b]\), with \( f(a) = c \) and \( f(b) = d \) where \( c < d \). Also let \( f'(x) \) be continuous on \([a, b]\) and let \( x = g(y) \) iff \( y = f(x) \). If \( R \) is the region bounded by \( y = f(x) \), the x-axis, \( x = a \) and \( x = b \) and \( R \) is revolved about the y-axis, then the value obtained by using the disk method is equal to the value obtained by using the cylindrical shell method. Equivalently \( \pi [b^2 - a^2c - \int_{c}^{d} \pi [g(y)]^2 \, dy] = \int_{a}^{b} 2\pi xf(x) \, dx \).

**Proof.** The region \( R \) can also be described as the region bounded by \( x = m(y) \), \( x = b \), \( y = 0 \) and \( y = d \), where
\[
m(y) = \begin{cases} 
    a & \text{if } 0 \leq y < c, \\
    g(y) & \text{if } c \leq y \leq d.
\end{cases}
\]
We observe from the way that \( m(y) \) is defined that it is continuous on \([0, d]\). If we evaluate the volume obtained by revolving the region \( R \) about the y-axis by using the disk method, we find this to be \( \int_{0}^{d} \pi b^2 \, dy - \int_{0}^{d} \pi m(y) \, dy \). This is equal to
\[
\int_{0}^{d} \pi b^2 \, dy - \int_{0}^{c} \pi a^2 \, dy - \int_{c}^{d} \pi [g(y)]^2 \, dy = \pi \left[ b^2d - a^2c \right] - \int_{c}^{d} \pi [g(y)]^2 \, dy.
\]
By the substitution formula, the latter expression is equal to \( \pi \left[ b^2d - a^2c \right] - \int_{c}^{d} \pi x^2 f'(x) \, dx \). A straightforward application of the integration by parts formula and algebraic simplification shows that
\[
\int_{a}^{b} 2\pi xf(x) \, dx = \pi \left[ b^2d - a^2c \right] - \int \pi x^2 f'(x) \, dx,
\]
and the argument is complete.

---

*The result is true in case \( d < c \), but a slight alteration is needed in the argument.*