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Math Bite: Convergence of p-series

We show the convergence of

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

for $p > 1$.

Let $p = 1 + q, q > 0$. The sequence of partial sums, whose nth term is $S_n = \sum_{k=1}^{n} 1/k^p$, is monotone increasing. It is also bounded, as follows. Let $n = 10^j - 1$, then

$$\sum_{k=1}^{10^j-1} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \cdots + \frac{1}{k^p} + \cdots + \frac{1}{(10^j-1)^p}$$

$$= \underbrace{1 + \frac{1}{2^p} + \cdots + \frac{1}{9^p}}_{9 \text{ terms}} + \underbrace{\frac{1}{10^p} + \frac{1}{11^p} + \cdots + \frac{1}{99^p}}_{90 \text{ terms}} + \underbrace{\frac{1}{100^p} + \frac{1}{101^p} + \cdots + \frac{1}{999^p}}_{900 \text{ terms}} + \cdots$$

$$< 1 + \cdots + 1 + \frac{1}{10^p} + \cdots + \frac{1}{10^p} + \frac{1}{100^p} + \cdots + \frac{1}{100^p} + \cdots + \frac{1}{(10^{j-1})^p}$$

$$= 9 + \frac{90}{10^p} + \frac{900}{100^p} + \cdots = 9\left(1 + \frac{1}{10^q} + \frac{1}{10^{2q}} + \cdots + \frac{1}{10^{(j-1)q}}\right) < \frac{9}{1 - 10^{-q}}.$$

Readers may wish to adapt the argument to show divergence in the case where $p < 1$. The first step is

$$\sum_{k=1}^{10^j} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \cdots + \frac{1}{n^p} + \cdots + \frac{1}{(10^j)^p}$$

$$= 1 + \underbrace{\frac{1}{2^p} + \cdots + \frac{1}{10^p}}_{9 \text{ terms}} + \underbrace{\frac{1}{11^p} + \frac{1}{12^p} + \cdots + \frac{1}{100^p}}_{90 \text{ terms}} + \underbrace{\frac{1}{101^p} + \frac{1}{102^p} + \cdots + \frac{1}{1000^p}}_{900 \text{ terms}} + \cdots$$

$$> \frac{10}{10^p} + \frac{90}{100^p} + \frac{900}{1000^p} + \cdots.$$

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[Editor's note: Another manuscript received at about the same time, from Eugene Boman and Richard Brazier of Penn State University, Dubois Campus, presented this

same idea using powers of 2. A referee pointed out that both methods amount to the Cauchy Condensation Test. See Konrad Knopp's classic books Infinite Sequences and Series *or* Theory and Application of Infinite Series.]

A Classification of Matrices of Finite Order over $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{Q}$

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A matrix A is said to have *finite order* $n \geq 1$ if $A^n = I$ and $A^r \neq I$ for $1 \leq r < n$. Otherwise we say that A has infinite order. An elementary exercise in abstract algebra asks for 2×2 matrices A, B over $\mathbb{R}$ each of finite order such that AB has infinite order. The matrices

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{and} \quad B_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

represent rotation about the origin through the signed angle θ and reflection in the line $y = x\tan(\theta/2)$. The matrix A_θ has finite order if and only if θ is a rational multiple of 2π, whereas every matrix B_θ has order 2. Moreover $A_\theta A_\phi = A_{\theta+\phi}$, $A_\theta B_\phi = B_{\theta+\phi}$, $B_\phi A_\theta = B_{\phi-\theta}$, and $B_\theta B_\phi = A_{\theta-\phi}$. Now let θ be an irrational multiple of 2π. Then the reflection matrices B_θ and B_0 have finite order, and their product $B_\theta B_0 = A_\theta$ has infinite order.

Are there examples other than reflections? To answer this it is natural to consider the matrices of finite order in $GL(2, \mathbb{R})$, the multiplicative group of nonsingular 2×2 matrices. The purpose of this note is to classify the matrices of finite order in $GL(k, F)$ for the fields $F = \mathbb{C}, \mathbb{R}$, and $\mathbb{Q}$, and to provide further examples of finite order matrices whose product has infinite order. The solution to this classification problem involves the factorization of $x^n - 1$ over F, and an application of the cyclic decomposition theorem of linear algebra. In this connection, we mention the paper [**3**] in which Robert Hanson determines, for a given n, the minimum k for which there is a $k \times k$ matrix A over F of order n, when F is $\mathbb{C}$, $\mathbb{R}$, or $\mathbb{Q}$.

Note that when F is a finite field with q elements then $GL(k, F)$ is a finite group of order $(q^k - 1)(q^k - q)\cdots(q^k - q^{k-1})$ [**6**, p. 178], so that each $k \times k$ matrix over F has finite order.

Minimal polynomials The text Blyth & Robertson [**1**] contains a concise account, with proofs, of the results of linear algebra stated here.

Having the same order (finite or infinite) is an equivalence relation in the multiplicative group $GL(k, F)$. We say that A is similar to B, denoted by $A \sim B$, if there exists a nonsingular matrix P such that $B = P^{-1}AP$. If A is similar to B, then A and B have the same order.

We denote the set of all $k \times k$ matrices over the field F, singular and nonsingular, by $M_k(F)$. If $A \in M_k(F)$ there is a polynomial $p \in F[x]$ for which $p(A) = 0$. One such polynomial is the *characteristic polynomial* of A defined by $\chi_A(x) = \det(xI - A)$.