(iii) The standard argument yields an analogue of the Proposition for the Integral Mean Value Theorem: *For continuous f on $[a, b]$, there exists $c \in (a, b)$ such that*

$$f(c)\left[\int_c^b f(t)\,dt - \int_a^c f(t)dt\right] = (c-a) - (b-c).$$

(iv) Other standard arguments now lead to expected analogues of the Proposition and of (iii), for the Cauchy Mean Value Theorems. We leave the interested reader to fill in the details.

References

1. J. Tong, The Mean Value Theorems of Lagrange and Cauchy, *Int. J. Math. Ed. Sci. Tech.* **30** (1999) 456–458.
2. Media Highlights, *College Math. Journal*, **31** (2000) 153–154.

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Symmetric or Skewed?

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Can the symmetry or skewness of a random variable's distribution be determined solely by inspecting its measures of center? On the other hand, does the direction of skewness indicate the ordering among measures of central tendency? Although there appears to be some confusion in both textbooks and periodicals on these issues, the present note suggests that the answer to both questions—at least with respect to discrete distributions—is "no".

In the past decade, several writers, including Chambers [**1**] and Lee [**2**], have discussed the relationships among measures of central tendency in continuous probability distributions. But as MacGillivray [**3**, p. 366] notes, "the relationship between the mean, medians [sic], and modes for discrete distributions is of course a more difficult problem." Indeed, when discussing discrete distributions, textbooks often make assertions such as, "If the data set is unimodal, but not symmetrical, the mean, mode, and median will be located at different points in the distribution [**6**, p. 47]." This is typically followed by an explicit ordering of the three measures; remarkably, Mogull [**5**] found such presentations in about eighty percent of the textbooks he sampled. Noting that such statements are incorrect, Mogull [**5**, p. 745] argued that "with a positively (negatively) skewed *sample* distribution, both the median and mean lie to the right (left) of the mode but in *unpredictable order*" (emphasis in original). In fact, however, even this weaker claim is invalid. An inequality between the mean and other measures of central tendency may suggest asymmetry in a discrete unimodal distribution, but the reverse is not true. Skewness does not necessarily imply that the mean, median, and mode are unequal. Nor does equality among the measures of central tendency guarantee symmetry in either a discrete probability distribution or a sample distribution.

To demonstrate these propositions, it is sufficient to produce examples of skewed distributions in which the mean, median, and mode are all equal. Among discrete probability distributions, the binomial with ten trials and a ten percent probability of success is one obvious example: it is unambiguously skewed to the right, but the mean, median, and mode are all equal to one. The distribution is depicted in Figure 1.

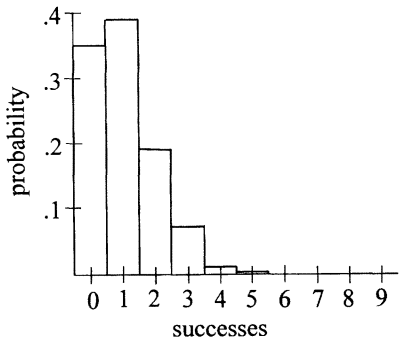

Figure 1. Binomial distribution with 10 trials and .10 probability of success.

Likewise, constructing a sample distribution with these properties is elementary. If we begin with any discrete, skewed distribution and add a sufficient number of observations at the mean, the mean will inevitably become both the mode and the median, but the distribution will still be skewed. An interesting example is the distribution generated by the function $f(x) = x$, where x takes positive integer values from 1 through 10, and $f(x)$ denotes the frequency with which x occurs. The distribution is clearly asymmetric, and sports the long left tail characteristic of a negative skew. Yet the mean and median are both 7, and by simply adding four more observations at the mean, the mode becomes 7 as well; all three measures of central tendency are then equal.

As a final example, consider a discrete distribution such that

$$f(\bar{x}-1) = \sum kf(\bar{x}+k) \quad \text{and} \quad f(\bar{x}) > f(\bar{x}-1)$$

where k takes positive integer values, and the only non-zero values of $f(x)$ occur at $\bar{x}$, $\bar{x}-1$, and all $\bar{x}+k$. The equality $f(\bar{x}-1) = \sum kf(\bar{x}+k)$ indicates that positive and negative deviations from $\bar{x}$ offset each other, so that $\bar{x}$ is the sample mean. The inequalities $f(\bar{x}) > f(\bar{x}-1)$ and $f(\bar{x}) > f(\bar{x}+k)$ for all k indicate that $\bar{x}$ is the modal observation. And because $f(\bar{x}) > f(\bar{x}-1) - \sum f(\bar{x}+k)$, $\bar{x}$ is also the median. Thus, all three measures of central tendency are equal, yet the distribution will only be symmetric if $\max(k) = 1$. For $\max(k) > 1$, the distribution is positively skewed. Indeed, the extent of the skewness increases with k, while the measures of central tendency remain equal. As a concrete illustration, consider the following hypothetical set of 44 observations:

x:	6	7	8	9	10
$f(x)$:	10	28	3	2	1.

This data set is clearly skewed to the right, yet it has a mean of 7, a median of 7, and a mode of 7. Indeed, by virtue of having enough observations at the mean, its first and third quartiles are both equal to 7 as well. Moreover, adding an observation at 11, four more observations at 6, and eleven more observations at 7 would increase the extent of the asymmetry without distorting either the equality among the measures of center or the equality among the quartiles. Of course, any number of such examples, including positive and negative skews, can be generated in this manner.

It follows from this simple demonstration that measures of skewness which rely on differences in measures of central tendency cannot provide a completely unambiguous distinction between symmetric and skewed distributions, when applied to discrete variables. As noted in [**3**] and [**5**], a century ago Karl Pearson developed one measure of skewness for continuous distributions based on the difference between the mean and the median, $Sk = 3(\text{Mean} - \text{Median})/\sigma$, and another based on the difference between the mean and the mode, $Sk' = (\text{Mean} - \text{Mode})/\sigma$, where σ is the standard deviation. The equivalence of these measures is the subject of the debate between Chambers [**1**] and Lee [**2**]; at issue here, however, is whether either measure is valid for discrete distributions. Mogull [**5**], for example, claims that Sk' is superior to Sk for determining the skewness of sample data, and Wei and Mingshu [**7**] likewise judge a frequency distribution's skewness by the difference between the mean and the mode. But for each of the examples presented here, $Sk = Sk' = 0$; both measures suggest symmetry, even though the distributions are all clearly skewed to one side or the other.

A third measure, the quartile coefficient of skewness (see for example [**4**]) is based on differences in quartiles, rather than central tendency:

$$Skq = \frac{(Q_3 - Q_2) - (Q_2 - Q_1)}{Q_3 - Q_1} = \frac{Q_3 - 2Q_2 + Q_1}{Q_3 - Q_1}$$

where Q_i represents the ith quartile. But here again, for each of the distributions given above, $Skq = 0$ despite the obvious asymmetries; in the last example, this outcome results simply from the large number of observations occurring at the mean. Clearly, neither Sk, Sk', nor Skq provides an adequate indication of skewness for discrete distributions.

The asymmetry can, however, be captured more readily by an alternative coefficient of skewness also attributed [**4**] to Pearson, which is measured as $\alpha_3 = m_3/(m_2)^{3/2}$ where m_2 and m_3 are the second and third moments around the mean, respectively. Because it uses every observation rather than a few summary statistics, this measure gives a more accurate description of the shape of the distribution. For any symmetric distribution, $\alpha_3 = 0$. In contrast, the binomial example given above yields $\alpha_3 = .85$; for the negatively skewed sample above, $\alpha_3 = -0.6$; and in the final example, α_3 equals 1.5 initially and rises to 2.1 when the right-hand tail is extended. And like Sk, Sk', and Skq, α_3 is independent of the units of measurement, and is thus invariant with respect to linear transformations of data; a sample of golf scores, for instance, would yield the same value of α_3 whether measured in total strokes or relative to par. Compared to the alternatives, then, this 'momentary' coefficient of skewness provides an equally robust yet more accurate method of distinguishing between symmetric and skewed distributions of discrete random variables.

References

1. Ll. G. Chambers, A spurious relation between measures of central tendency, *Mathematical Gazette*, **74** (1990) 54–55.
2. P. M. Lee, Not so spurious, *Mathematical Gazette* **75** (1991) 200–201.

3. H. L. MacGillivray, Mean, median, mode, and skewness, in S. Kotz and N. L. Johnson, eds., *Encyclopedia of Statistical Sciences*, Vol. 5, Wiley (1985), 364–367.
4. H. L. MacGillivray, Measures of skewness, in S. Kotz and N. L. Johnson, eds., *Encyclopedia of Statistical Sciences*, Vol. 5, Wiley (1985), 146.
5. R. G. Mogull, Popular measures of central tendency, *Mathematics Teacher*, **83** (1990) 744–746.
6. R. Sandy, *Statistics for Business and Economics*, McGraw-Hill, 1990.
7. S. Wei, and Y. Mingshu, "Mean > median > mode for positively skewed distribution?, *Teaching Statistics*, **15** (1993) 92.

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Elementary Linear Algebra and the Division Algorithm

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The set of polynomials is perhaps the first meaningful example of a vector space, appearing in all college textbooks in linear algebra, whose elements are familiar objects to any high school student. However, the old well known properties (of the ring) of polynomials, thoroughly explored in high school, are in general not related, in any convincing way, to the new structure of vector space. This suggests to the student the erroneous idea that no relevant link does exist between those two structures.

The purpose of this note is to exhibit an elementary example of such a link. Namely, we shall describe how to derive, from the simplest properties of the finite dimensional vector spaces, the

Division Algorithm. Let $\mathbb{K}$ be a field and $\mathbb{K}[x]$ be the ring of polynomials, in one indeterminate, with coefficients in $\mathbb{K}$. If $F \in \mathbb{K}[x]$ is not the zero polynomial, then, given $G \in \mathbb{K}[x]$ there exist unique polynomials $Q, R \in \mathbb{K}[x]$ satisfying:

(i) $G = Q \cdot F + R$
(ii) either $R = 0$ or degree $(R) <$ degree (F).

Towards the establishment of this result we shall only make use of the following facts:

(1) In an n-dimensional vector space E, any linearly independent set of n vectors constitutes a basis of E.
(2) The set $\mathcal{P}_n$ of polynomials in $\mathbb{K}[x]$ of degree $d \leq n$ is a vector space of dimension $n+1$ over $\mathbb{K}$.
(3) Any set of nonzero polynomials in $\mathcal{P}_n$ of distinct degrees is linearly independent.

Now, the proof is carried out as follows:

Let $m =$ degree(F) and $n =$ degree(G). If $n < m$ there is nothing to prove since, $G = O \cdot F + G$ is the only possible way to write down G satisfying (i) and (ii) above. Hence, we shall suppose in the sequel that $n \geq m$.

The set $\mathcal{B} = \{1, x, \ldots, x^{m-1}, F, xF, \ldots, x^{n-m}F\}$ of $n+1$ polynomials in $\mathcal{P}_n$ is a basis of $\mathcal{P}_n$. In fact, $\mathcal{B}$ is a linearly independent set because of (3) and consequently a basis in view of (2) and (1). Since $G \in \mathcal{P}_n$ there exist unique scalars $a_0, a_1, \ldots, a_{m-1}$ and $b_0, b_1, \ldots, b_{n-m}$ such that